Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2008, Article ID 237937, 21 pages
doi:10.1155/2008/237937

Research Article

Commutators of the Hardy-Littlewood Maximal Operator with BMO Symbols on Spaces of Homogeneous Type

Guoen Hu,[1] Haibo Lin,[2] and Dachun Yang[2]

[1] *Department of Applied Mathematics, University of Information Engineering, P.O. Box 1001-747 Zhengzhou 450002, China*
[2] *School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, China*

Correspondence should be addressed to Dachun Yang, dcyang@bnu.edu.cn

Received 20 August 2007; Revised 19 November 2007; Accepted 5 January 2008

Recommended by Yong Zhou

Weighted L^p for $p \in (1, \infty)$ and weak-type endpoint estimates with general weights are established for commutators of the Hardy-Littlewood maximal operator with BMO symbols on spaces of homogeneous type. As an application, a weighted weak-type endpoint estimate is proved for maximal operators associated with commutators of singular integral operators with BMO symbols on spaces of homogeneous type. All results with no weight on spaces of homogeneous type are also new.

1. Introduction

We will be working on a space of homogeneous type. Let $\mathcal{X}$ be a set endowed with a positive Borel regular measure μ and a symmetric quasimetric d satisfying that there exists a constant $\kappa \geq 1$ such that for all $x, y, z \in \mathcal{X}$, $d(x,y) \leq \kappa[d(x,z) + d(z,y)]$. The triple $(\mathcal{X}, d, \mu)$ is said to be a space of homogeneous type in the sense of Coifman and Weiss [1] if μ satisfies the following doubling condition: there exists a constant $C \geq 1$ such that for all $x \in \mathcal{X}$ and $r > 0$,

$$\mu(B(x,2r)) \leq C\mu(B(x,r)). \tag{1.1}$$

It is easy to see that the above doubling property implies the following strong homogeneity: there exist positive constants C and n such that for all $\lambda \geq 1$, $r > 0$, and $x \in \mathcal{X}$,

$$\mu(B(x,\lambda r)) \leq C\lambda^n \mu(B(x,r)). \tag{1.2}$$

Moreover, there also exist constants $C > 0$ and $N \in [0, n]$ such that for all $x, y \in \mathcal{X}$ and $r > 0$,

$$\mu(B(y,r)) \leq C\left(1 + \frac{d(x,y)}{r}\right)^N \mu(B(x,r)). \tag{1.3}$$

We remark that although all balls defined by d satisfy the axioms of complete system of neighborhoods in $\mathcal{X}$, and therefore induce a (separated) topology in $\mathcal{X}$, the balls $B(x,r)$ for $x \in \mathcal{X}$ and $r > 0$ need not be open with respect to this topology. However, by a remarkable result of Macías and Segovia in [2], we know that there exists another quasimetric $\widetilde{d}$ such that

(i) there exists a constant $C \geq 1$ such that for all $x, y \in \mathcal{X}$, $C^{-1}\widetilde{d}(x,y) \leq d(x,y) \leq C\widetilde{d}(x,y)$;

(ii) there exist constants $C > 0$ and $\gamma \in (0,1]$ such that for all $x, x', y \in \mathcal{X}$,

$$\left|\widetilde{d}(x,y) - \widetilde{d}(x',y)\right| \leq C\left(\widetilde{d}(x,x')\right)^{\gamma}\left(\widetilde{d}(x,y) + \widetilde{d}(x',y)\right)^{1-\gamma}. \tag{1.4}$$

The balls corresponding to $\widetilde{d}$ are open in the topology induced by $\widetilde{d}$. Thus, throughout this paper, we always assume that there exist constants $C > 0$ and $\gamma \in (0,1]$ such that for all $x, x', y \in \mathcal{X}$,

$$\left|d(x,y) - d(x',y)\right| \leq C\left(d(x,x')\right)^{\gamma}\left(d(x,y) + d(x',y)\right)^{1-\gamma}, \tag{1.5}$$

and that the balls $B(x,r)$ for all $x \in \mathcal{X}$ and $r > 0$ are open.

Now let k be a positive integer and $b \in \mathrm{BMO}(\mathcal{X})$, define the kth-order commutator $M_{b,k}$ of the Hardy-Littlewood maximal operator with b by

$$M_{b,k}f(x) = \sup_{B \ni x} \frac{1}{\mu(B)} \int_B \left|b(x) - b(y)\right|^k \left|f(y)\right| d\mu(y) \tag{1.6}$$

for all $x \in \mathcal{X}$. For the case that $(\mathcal{X}, d, \mu)$ is the Euclidean space, García-Cuerva et al. [3] proved that $M_{b,k}$ is bounded on $L^p(\mathbb{R}^n)$ for any $p \in (1,\infty)$, and Alphonse [4] proved that $M_{b,1}$ enjoys a weak-type $L(\log L)$ estimate, that is, there exists a positive constant C, depending on $\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}$, such that for all suitable functions f,

$$\left|\left\{x \in \mathbb{R}^n : M_{b,1}f(x) > \lambda\right\}\right| \leq C\int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \log\left(\mathrm{e} + \frac{|f(x)|}{\lambda}\right) dx. \tag{1.7}$$

Li et al. [5] established a weighted estimate with any general weight for $M_{b,1}$ in $\mathbb{R}^n$. As it was shown in [3–5] for the setting of Euclidean spaces, the operator $M_{b,k}$ plays an important role in the study of commutators of singular integral operators with BMO symbols. In this paper, we establish weighted estimates with general weights for $M_{b,k}$ in spaces of homogeneous type. To state our results, we first give some notation.

Let E be a measurable set with $\mu(E) < \infty$. For any fixed $p \in [1,\infty)$, $\delta > 0$, and suitable function f, set

$$\|f\|_{L^p(\log L)^{\delta},E} = \inf\left\{\lambda > 0 : \frac{1}{\mu(E)}\int_E \left(\frac{|f(x)|}{\lambda}\right)^p \log^{\delta}\left(\mathrm{e} + \frac{|f(x)|}{\lambda}\right) d\mu(x) \leq 1\right\}. \tag{1.8}$$

The maximal operator $M_{L^p(\log L)^{\delta}}$ is defined by $M_{L^p(\log L)^{\delta}}f(x) = \sup_{B \ni x}\|f\|_{L^p(\log L)^{\delta},B}$, where the supremum is taken over all balls containing x. In the following, we denote $M_{L^1(\log L)^{\delta}}$ by $M_{L(\log L)^{\delta}}$ for simplicity, and denote by $L_b^{\infty}(\mathcal{X})$ the set of bounded functions with bounded support.

With the notation above, we now formulate our main results as follows.

Theorem 1.1. *Let k be a positive integer, $p \in (1,\infty)$, and $b \in \mathrm{BMO}(\mathcal{X})$. Then for any $\delta > 0$, there exists a positive constant C, depending only on p, k, and δ, such that for all nonnegative weights w and $f \in L_b^\infty(\mathcal{X})$,*

$$\int_{\mathcal{X}} (M_{b,k}f(x))^p w(x)d\mu(x) \le C\|b\|_{\mathrm{BMO}(\mathcal{X})}^{kp}\int_{\mathcal{X}} |f(x)|^p M_{L(\log L)^{kp+\delta}}w(x)d\mu(x). \tag{1.9}$$

Theorem 1.2. *Let k be a positive integer, $b \in \mathrm{BMO}(\mathcal{X})$, and $\delta > 0$. There exists a positive constant $C = C_{k,\|b\|_{\mathrm{BMO}(\mathcal{X})}}$ such that for all nonnegative weights w, $f \in L_b^\infty(\mu)$ and $\lambda > 0$,*

$$w(\{x \in \mathcal{X} : M_{b,k}f(x) > \lambda\}) \le C\int_{\mathcal{X}} \frac{|f(x)|}{\lambda}\log^k\left(e + \frac{|f(x)|}{\lambda}\right)M_{L(\log L)^{\tilde{k}+\delta}}w(x)d\mu(x), \tag{1.10}$$

where, and in the following, $\tilde{k} = k$ if k is even and $\tilde{k} = k+1$ if k is odd.

As a corollary of Theorem 1.2, we establish a weighted endpoint estimate for the maximal commutator of singular integral operators with $\mathrm{BMO}(\mathcal{X})$ symbols. Let T be a Calderón-Zygmund operator, that is, T is a linear $L^2(\mathcal{X})$-bounded operator and satisfies that for all $f \in L^2(\mathcal{X})$ with bounded support and almost all $x \notin \operatorname{supp} f$,

$$Tf(x) = \int_{\mathcal{X}} K(x,y)f(y)d\mu(y), \tag{1.11}$$

where K is a locally integrable function on $\mathcal{X} \times \mathcal{X} \setminus \{(x,y) : x = y\}$ and satisfies that for all $x \neq y$,

$$|K(x,y)| \le \frac{C}{\mu(B(x,d(x,y)))}, \tag{1.12}$$

and that for all $x, y, y' \in \mathcal{X}$ with $d(x,y) \ge 2d(y,y')$,

$$|K(x,y) - K(x,y')| + |K(y,x) - K(y',x)| \le C\frac{(d(y,y'))^\tau}{\mu(B(x,d(x,y)))(d(x,y))^\tau} \tag{1.13}$$

with positive constants C and $\tau \le 1$. For any $\epsilon > 0$, suitable function f and $x \in \mathcal{X}$, define the truncated operator T_ϵ by

$$T_\epsilon f(x) = \int_{d(x,y)>\epsilon} K(x,y)f(y)d\mu(y). \tag{1.14}$$

Let $b \in \mathrm{BMO}(\mathcal{X})$ and let k be a positive integer. Define the commutator $T_{\epsilon;b,k}$ by

$$T_{\epsilon;b,0}f(x) = T_\epsilon f(x), T_{\epsilon;b,k}f(x) = b(x)T_{\epsilon;b,k-1}f(x) - T_{\epsilon;b,k-1}(bf)(x) \tag{1.15}$$

for all $x \in \mathcal{X}$ and $f \in L_b^\infty(\mathcal{X})$. The maximal operator associated with the commutator $T_{b,k}$ is defined by

$$T_{b,k}^* f(x) = \sup_{\epsilon>0}|T_{\epsilon;b,k}f(x)| \tag{1.16}$$

for all $x \in \mathcal{X}$. In [6], it was proved that if T is a Calderón-Zygmund operator, then for any $p \in (1,\infty)$, there exists a positive constant C such that for all $f \in L_b^\infty(\mathcal{X})$ and all nonnegative weights w,

$$\int_{\mathcal{X}} \left|T_{b,k}^* f(x)\right|^p w(x)d\mu(x) \leq C\|b\|_{\mathrm{BMO}\,(\mu)}^{kp} \int_{\mathcal{X}} \left|f(x)\right|^p M_{L(\log L)^{(k+1)p+\delta}} w(x)d\mu(x). \tag{1.17}$$

In [5], it was proved that in $\mathbb{R}^n$, $T_{b,1}^*$ enjoys the following weighted weak-type endpoint estimate: for any $\delta > 0$, there exists a positive constant C, depending on n, δ, and $\|b\|_{\mathrm{BMO}\,(\mathbb{R}^n)}$, such that

$$w\left(\left\{x \in \mathbb{R}^n : T_{b,1}^* f(x) > \lambda\right\}\right) \leq C\int_{\mathbb{R}^n} \frac{\left|f(x)\right|}{\lambda} \log\left(\mathrm{e} + \frac{\left|f(x)\right|}{\lambda}\right) M_{L(\log)^{2+\delta}} w(x)dx. \tag{1.18}$$

Using Theorem 1.2, we will prove the following result.

Theorem 1.3. *Let T be a Calderón-Zygmund operator. Then for any $b \in \mathrm{BMO}\,(\mathcal{X})$, nonnegative integer k and $\delta > 0$, there exists a positive constant C, depending on k, δ, and $\|b\|_{\mathrm{BMO}\,(\mathcal{X})}$, such that for all $\lambda > 0$, $f \in L_b^\infty(\mathcal{X})$ and nonnegative weights w,*

$$\int_{\{x\in\mathcal{X}:T_{b,k}^* f(x)>\lambda\}} w(x)d\mu(x) \leq C\int_{\mathcal{X}} \frac{\left|f(x)\right|}{\lambda} \log^k\left(e + \frac{\left|f(x)\right|}{\lambda}\right) M_{L(\log L)^{k+1+\delta}} w(x)d\mu(x). \tag{1.19}$$

We mention that Theorems 1.1, 1.2, and 1.3 are also new even when $w(x) \equiv 1$ for all $x \in \mathcal{X}$.

We now make some conventions. Throughout the paper, we always denote by C a positive constant which is independent of main parameters, but it may vary from line to line. We denote $f \leq Cg$ and $f \geq Cg$ simply by $f \lesssim g$ and $f \gtrsim g$, respectively. If $f \lesssim g \lesssim f$, we then write $f \sim g$. Constant, with subscript such as C_1, does not change in different occurrences. A weight w always means a nonnegative locally integrable function. For a measurable set E and a weight w, χ_E denotes the characteristic function of E, $w(E) = \int_E w(x)\mathrm{d}\mu(x)$. Given $\lambda > 0$ and a ball B, λB denotes the ball with the same center as B and whose radius is λ times that of B. For a fixed p with $p \in [1,\infty)$, p' denotes the dual exponent of p, namely, $p' = p/(p-1)$. For any measurable set E and any integrable function f on E, we denote by $m_E(f)$ the mean value of f over E, that is, $m_E(f) = (1/\mu(E))\int_E f(x)d\mu(x)$. For any locally integrable function f and $x \in \mathcal{X}$, the Fefferman-Stein sharp maximal function $M^\# f(x)$ is defined by

$$M^\# f(x) = \sup_{B\ni x} \frac{1}{\mu(B)} \int_B \left|f(y) - m_B(f)\right| d\mu(y), \tag{1.20}$$

where the supremum is taken over all balls B containing x. For any fixed $q \in (0,1)$, the sharp maximal function $M_q^\# f$ of the function f is defined by $M_q^\# f = (M^\#(|f|^q))^{1/q}$.

A generalization of Hölder's inequality will be used in the proofs of our theorems. For any measurable set E with $\mu(E) < \infty$, positive integer l, and suitable function f, set

$$\|f\|_{\exp L^{1/l},E} = \inf\left\{\lambda > 0 : \frac{1}{\mu(E)} \int_E \exp\left(\frac{\left|f(x)\right|}{\lambda}\right)^{1/l} d\mu(x) \leq 2\right\}. \tag{1.21}$$

Then the following generalization of Hölder's inequality:

$$\frac{1}{\mu(E)} \int_E \left|f(x)h(x)\right| d\mu(x) \leq C\|f\|_{L(\log L)^l,E} \|h\|_{\exp L^{1/l},E} \tag{1.22}$$

holds for any suitable functions f and h; see [7] for details.

2. Proof of Theorem 1.1

To prove Theorem 1.1, we need some technical lemmas. In what follows, we denote by M the Hardy-Littlewood maximal function. Moreover, for any $s > 0$ and suitable function f, we set $M_s(f) = [M(|f|^s)]^{1/s}$.

Lemma 2.1 (see [8]). *There exists a positive constant C such that for all weights w and all nonnegative functions f satisfying $\mu(\{x \in \mathcal{X} : f(x) > \lambda\}) < \infty$ for all $\lambda > 0$, then*

(i) *if $\mu(\mathcal{X}) = \infty$,*

$$\int_{\mathcal{X}} f(x)w(x)d\mu(x) \le C\int_{\mathcal{X}} M^{\#}f(x)Mw(x)d\mu(x); \tag{2.1}$$

(ii) *if $\mu(\mathcal{X}) < \infty$,*

$$\int_{\mathcal{X}} f(x)w(x)d\mu(x) \le C\int_{\mathcal{X}} M^{\#}f(x)Mw(x)d\mu(x) + Cw(\mathcal{X})m_{\mathcal{X}}(f). \tag{2.2}$$

Lemma 2.2. *For any $q \in (0,1)$, there exists a positive constant C such that for all $f \in L^p(\mathcal{X})$ with $p \in [1,\infty)$ and all $x \in \mathcal{X}$, $M^{\#}_q(Mf)(x) \le CM^{\#}f(x)$.*

For the case that $(\mathcal{X}, d, \mu)$ is the Euclidean space, this lemma was proved in [9]. For spaces of homogeneous type, the proof is similar to the case of Euclidean spaces; see [6].

Lemma 2.3. *Let $p \in (1,\infty)$ and let k be a positive integer.*

(a) *There exists a positive constant C, depending only on k and p, such that for all $f \in L^{\infty}_b(\mathcal{X})$ and all weights w,*

$$\int_{\mathcal{X}} [M_{L(\log L)^k}f(x)]^p w(x)d\mu(x) \le C\int_{\mathcal{X}} |f(x)|^p Mw(x)d\mu(x). \tag{2.3}$$

(b) *For any $\delta_1 > 0$, there exists a positive constant C, depending only on k, p, and δ_1, such that for all $f \in L^{\infty}_b(\mathcal{X})$ and all weights w,*

$$\int_{\mathcal{X}} (M^k f(x))^p [M_{L(\log L)^{kp'-1+\delta_1}}w(x)]^{1-p} d\mu(x) \le C\int_{\mathcal{X}} |f(x)|^p [w(x)]^{1-p} d\mu(x). \tag{2.4}$$

For Euclidean spaces, Lemma 2.3(a) is just Corollary 1.8 in [10] and Lemma 2.3(b) is included in the proof of Theorem 2 in [11] together with (4.11) in [12]. For spaces of homogeneous type, Lemma 2.3(a) is a simple corollary of Theorem 1.4 in [13]. On the other hand, by Theorem 1.4 in [13], and the estimate that for all weights w, $M_{L(\log L)^k}w \approx M^{k+1}w$ (see [12]), we can prove Lemma 2.3(b) by the ideas used in [11, page 751]. For details, see [6].

By a similar argument that was used in the proof of Theorem 2.1 in [14], we can verify the existence of the following approximation of the identity of order γ with bounded support on $\mathcal{X}$. We omit the details here.

For any $x \in \mathcal{X}$ and $r > 0$, set $V_r(x) = \mu(B(x,r))$.

Lemma 2.4. *Let γ be as in* (1.5)*. Then there exists an approximation of the identity $\{S_k\}_{k\in\mathbb{Z}}$ of order γ with bounded support on $\mathcal{X}$. Namely, $\{S_k\}_{k\in\mathbb{Z}}$ is a sequence of bounded linear integral operators on $L^2(\mathcal{X})$, and there exist constants C_0, $\widetilde{C} > 0$ such that for all $k \in \mathbb{Z}$ and all x, x', y, and $y' \in \mathcal{X}$, $S_k(x,y)$, the integral kernel of S_k is a measurable function from $\mathcal{X} \times \mathcal{X}$ into $\mathbb{C}$ satisfying*

(i) $S_k(x,y) = 0$ *if* $d(x,y) \geq \widetilde{C}2^{-k}$ *and* $0 \leq S_k(x,y) \leq C_0(1/(V_{2^{-k}}(x) + V_{2^{-k}}(y)))$;

(ii) $S_k(x,y) = S_k(y,x)$ *for all* $x, y \in \mathcal{X}$;

(iii) $|S_k(x,y) - S_k(x',y)| \leq C_0 2^{k\gamma} d(x,x')^\gamma (1/(V_{2^{-k}}(x) + V_{2^{-k}}(y)))$ *for* $d(x,x') \leq \max\{\widetilde{C}/\kappa, 1/\kappa\}2^{1-k}$;

(iv) $C_0 V_{2^{-k}}(x) S_k(x,x) > 1$ *for all* $x \in \mathcal{X}$ *and* $k \in \mathbb{Z}$;

(v) $\int_{\mathcal{X}} S_k(x,y)\,d\mu(y) = 1 = \int_{\mathcal{X}} S_k(x,y)\,d\mu(x)$.

For any $\epsilon > 0$ and $x, y \in \mathcal{X}$, let

$$S_\epsilon(x,y) \equiv S_k(x,y)\chi_{(2^{-k-1},2^{-k}]}(\epsilon). \tag{2.5}$$

Obviously, S_ϵ satisfies (i) through (v) of Lemma 2.4 with 2^{-k} replaced by ϵ. From (iii) and (iv) of Lemma 2.4, it follows that there exist constants $C' \in (0, \min\{\widetilde{C}/\kappa, 1/\kappa, (C_0)^{-2/\gamma}\})$ and $C > 1$ such that for all $\epsilon > 0$ and all $x, y \in \mathcal{X}$ satisfying $d(x,y) < C'\epsilon$,

$$CV_\epsilon(x) S_\epsilon(x,y) > 1. \tag{2.6}$$

For a positive integer k and a function $b \in \mathrm{BMO}\,(\mathcal{X})$, let $\widetilde{M}_{b,k}$ be the operator defined by

$$\widetilde{M}_{b,k} f(x) = \sup_{\epsilon>0} \widetilde{M}_{\epsilon;b,k} f(x) \tag{2.7}$$

for all $f \in L_b^\infty(\mathcal{X})$ and $x \in \mathcal{X}$, where for $\epsilon > 0$,

$$\widetilde{M}_{\epsilon;b,k} f(x) = \int_{\mathcal{X}} S_\epsilon(x,y) \left|b(x) - b(y)\right|^k \left|f(y)\right| d\mu(y). \tag{2.8}$$

If $k = 0$, we denote $\widetilde{M}_{b,k}$ and $\widetilde{M}_{\epsilon;b,k}$ simply by $\widetilde{M}$ and $\widetilde{M}_\epsilon$, respectively. From (i) of Lemma 2.4 together with (1.1), it follows that $S_\epsilon(x,y) \lesssim 1/V_\epsilon(x) \lesssim 1/V_{2\widetilde{C}\epsilon}(x)$. Notice that if $d(x,y) \geq 2\widetilde{C}\epsilon$, then $S_\epsilon(x,y) = 0$. Thus,

$$\widetilde{M}_{b,k} f(x) = \sup_{\epsilon>0} \widetilde{M}_{\epsilon;b,k} f(x) \lesssim \sup_{\epsilon>0} \frac{1}{V_{2\widetilde{C}\epsilon}(x)} \int_{B(x,2\widetilde{C}\epsilon)} \left|b(x) - b(y)\right|^k \left|f(y)\right| d\mu(y) \lesssim M_{b,k} f(x). \tag{2.9}$$

On the other hand, for each fixed $\epsilon > 0$, by (2.6) and $V_{C'\epsilon}(x) \sim V_\epsilon(x)$, we have

$$\begin{aligned} \frac{1}{V_{C'\epsilon}(x)} \int_{B(x,C'\epsilon)} \left|b(x) - b(y)\right|^k \left|f(y)\right| d\mu(y) &\lesssim \int_{\mathcal{X}} S_\epsilon(x,y) \left|b(x) - b(y)\right|^k \left|f(y)\right| d\mu(y) \\ &\lesssim \widetilde{M}_{b,k} f(x). \end{aligned} \tag{2.10}$$

By the definition of $M_{b,k}$, we further obtain $M_{b,k} f(x) \lesssim \widetilde{M}_{b,k} f(x)$. Thus, there exists some constant $C \geq 1$ such that for all $x \in \mathcal{X}$ and $f \in L_b^\infty(\mathcal{X})$,

$$C^{-1} \widetilde{M}_{b,k} f(x) \leq M_{b,k} f(x) \leq C \widetilde{M}_{b,k} f(x). \tag{2.11}$$

For the sharp function estimate of $\widetilde{M}_{b,k}$, we have the following estimate.

Lemma 2.5. *Let k be a positive integer and $b \in \mathrm{BMO}(\mathcal{X})$. For any q and s with $0 < q < s < 1$, there exists a positive constant C such that for all $f \in L^{\infty}_{b}(\mathcal{X})$ and all $x \in \mathcal{X}$,*

$$M^{\#}_{q}(\widetilde{M}_{b,k} f)(x) \le C \sum_{j=0}^{k-1} \|b\|^{k-j}_{\mathrm{BMO}(\mathcal{X})} M_s(\widetilde{M}_{b,j} f)(x) + C\|b\|^{k}_{\mathrm{BMO}(\mathcal{X})} M_{L(\log L)^k} f(x). \tag{2.12}$$

Proof. By (i), (ii), and (iii) of Lemma 2.4, we obtain that for all $x, y \in \mathcal{X}$,

$$|S_\epsilon(x,y)| \lesssim \frac{1}{\mu(B(x, d(x,y)))}, \tag{2.13}$$

and that for all $\epsilon > 0$ and all $x, y, y' \in \mathcal{X}$ with $d(x,y) \ge 2\kappa d(y,y')$,

$$|S_\epsilon(x,y) - S_\epsilon(x,y')| + |S_\epsilon(y,x) - S_\epsilon(y',x)| \lesssim \frac{1}{\mu(B(x,d(x,y)))}\left(\frac{d(y,y')}{d(x,y)}\right)^{\gamma}. \tag{2.14}$$

To verify (2.12), by homogeneity, we may assume that $\|b\|_{\mathrm{BMO}(\mathcal{X})} = 1$. For all $f \in L^{\infty}_{b}(\mathcal{X})$, $x \in \mathcal{X}$, and balls B containing x, it suffices to prove that

$$\inf_{c\in\mathbb{C}} \left(\frac{1}{\mu(B)}\int_B |\widetilde{M}_{b,k} f(y) - c|^q d\mu(y)\right)^{1/q} \lesssim \sum_{j=0}^{k-1} M_s(\widetilde{M}_{b,j} f)(x) + M_{L(\log L)^k} f(x). \tag{2.15}$$

We consider the following three cases.
Case 1 ($\mu(\mathcal{X} \setminus C_1 B) = 0$). Where and in what follows $C_1 = \kappa(4\kappa + 1)$. In this case, we have that for all $x \in \mathcal{X}$,

$$\widetilde{M}_{b,k} f(x) \lesssim \sum_{j=0}^{k-1} |m_B(b) - b(x)|^{k-j} \widetilde{M}_{b,j} f(x) + \widetilde{M}\left((b - m_B(b))^k f\right)(x). \tag{2.16}$$

The Kolmogorov inequality (see [15, page 102]), along with the fact that M (and so $\widetilde{M}$) is bounded from $L^1(\mathcal{X})$ to $L^{1,\infty}(\mathcal{X})$ and the inequality (1.22) gives us that

$$\begin{aligned}
&\left\{\frac{1}{\mu(B)}\int_B \left[\widetilde{M}\left((b - m_B(b))^k f\right)(y)\right]^q d\mu(y)\right\}^{1/q} \\
&\quad\lesssim \frac{1}{\mu(B)}\int_{C_1 B} |b(y) - m_B(b)|^k |f(y)| d\mu(y) \\
&\quad\lesssim \frac{1}{\mu(B)}\int_{C_1 B} |b(y) - m_{C_1 B}(b)|^k |f(y)| d\mu(y) + |m_{C_1 B}(b) - m_B(b)| \frac{1}{\mu(B)}\int_{C_1 B} |f(y)| d\mu(y) \\
&\quad\lesssim M_{L(\log L)^k} f(x),
\end{aligned} \tag{2.17}$$

where the last inequality follows from the John-Nirenberg inequality, which states that for any ball Q,

$$\left\| |m_Q(b) - b|^k \right\|_{\exp L^{1/k}, Q} \lesssim \|b\|^k_{\mathrm{BMO}(\mathcal{X})}. \tag{2.18}$$

On the other hand, if $0 < q < s < 1$, an application of Hölder's inequality implies that

$$\begin{aligned}
&\left\{\frac{1}{\mu(B)}\int_B\left[\widetilde{M}_{b,k}f(x)\right]^q d\mu(x)\right\}^{1/q}\\
&\quad\lesssim\sum_{j=0}^{k-1}M_s\left(\widetilde{M}_{b,j}f\right)(x)+\left\{\frac{1}{\mu(B)}\int_B\left[\widetilde{M}\left((b-m_B(b))^k f\right)(y)\right]^q d\mu(y)\right\}^{1/q}.
\end{aligned}\tag{2.19}$$

We then get (2.15).

Case 2 ($\mu(\mathcal{X}\setminus C_1B)\neq 0$ and $\mu(C_1B\setminus B)>0$). In this case, decompose f into $f = f\chi_{C_1B}+f\chi_{\mathcal{X}\setminus C_1B}\equiv f_1+f_2$, recalling that χ_E denotes the characteristic function of the set E. Let y_0 be a point in B such that

$$C_B\equiv\sup_{\epsilon>0}\int_{\mathcal{X}}S_\epsilon(y_0,z)\left|m_B(b)-b(z)\right|^k\left|f_2(z)\right|dz<\infty.\tag{2.20}$$

With the aid of the formula

$$\left(m_B(b)-b(z)\right)^k=\left(b(y)-b(z)\right)^k+\sum_{j=0}^{k-1}C_k^j\left(b(y)-b(z)\right)^j\left(m_B(b)-b(y)\right)^{k-j},\tag{2.21}$$

where C_k^j is the constant from Newton's formula, we have

$$\begin{aligned}
&\left|\left|b(y)-b(z)\right|^k\left|f(z)\right|-\left|m_B(b)-b(z)\right|^k\left|f_2(z)\right|\right|\\
&\quad\lesssim\sum_{j=0}^{k-1}\left|b(y)-b(z)\right|^j\left|m_B(b)-b(y)\right|^{k-j}\left|f(z)\right|+\left|m_B(b)-b(z)\right|^k\left|f_1(z)\right|.
\end{aligned}\tag{2.22}$$

Thus for any $y\in B$,

$$\begin{aligned}
\left|\widetilde{M}_{b,k}f(y)-C_B\right|&=\left|\sup_{\epsilon>0}\left|\widetilde{M}_{\epsilon;b,k}f(y)\right|-\sup_{\epsilon>0}\left|\widetilde{M}_\epsilon\left(\left(m_B(b)-b\right)^k f_2\right)(y_0)\right|\right|\\
&\le\sup_{\epsilon>0}\left|\widetilde{M}_{\epsilon;b,k}f(y)-\widetilde{M}_\epsilon\left(\left(m_B(b)-b\right)^k f_2\right)(y_0)\right|\\
&\lesssim\widetilde{M}\left(\left(m_B(b)-b\right)^k f_1\right)(y)+\sum_{j=0}^{k-1}\left|m_B(b)-b(y)\right|^{k-j}\widetilde{M}_{b,j}f(y)\\
&\quad+\sup_{\epsilon>0}\left|\widetilde{M}_\epsilon\left(\left(m_B(b)-b\right)^k f_2\right)(y)-\widetilde{M}_\epsilon\left(\left(m_B(b)-b\right)^k f_2\right)(y_0)\right|\\
&\equiv\mathrm{I}\,(y)+\mathrm{II}\,(y)+\mathrm{III}\,(y).
\end{aligned}\tag{2.23}$$

As in Case 1, we have that

$$\begin{aligned}
\left\{\frac{1}{\mu(B)}\int_B\left|\mathrm{I}(y)\right|^q d\mu(y)\right\}^{1/q}&\lesssim M_{L(\log L)^k}f(x),\\
\left\{\frac{1}{\mu(B)}\int_B\left|\mathrm{II}(y)\right|^q d\mu(y)\right\}^{1/q}&\lesssim\sum_{j=0}^{k-1}M_s\left(\widetilde{M}_{b,j}f\right)(x).
\end{aligned}\tag{2.24}$$

As for III (y), by (2.14) and (1.22), it is easy to get

$$\begin{aligned}
\mathrm{III}\,(y) &\le \sup_{\epsilon>0}\int_{\mathcal{X}}\left|S_\epsilon(y,z)-S_\epsilon(y_0,z)\right|\left|m_B(b)-b(z)\right|^k\left|f_2(z)\right|d\mu(z)\\
&\lesssim \int_{\mathcal{X}\setminus C_1B}\frac{(d(y,y_0))^\gamma}{\mu(B(y,d(y,z)))(d(y,z))^\gamma}\left|m_B(b)-b(z)\right|^k\left|f(z)\right|dz\\
&\lesssim \sum_{l=1}^{\infty}2^{-l\gamma}\frac{1}{\mu(2^lC_1B)}\int_{2^lC_1B}\left|m_{2^lC_1B}(b)-b(z)\right|^k\left|f(z)\right|d\mu(z)\\
&\quad+\sum_{l=1}^{\infty}2^{-l\gamma}\left|m_{2^lC_1B}(b)-m_B(b)\right|^k\frac{1}{\mu(2^lC_1B)}\int_{2^lC_1B}\left|f(z)\right|d\mu(z)\\
&\lesssim \sum_{l=1}^{\infty}2^{-l\gamma}\left\|\left|m_{2^lC_1B}(b)-b\right|^k\right\|_{\exp L^{1/k},2^lC_1B}\|f\|_{L(\log L)^k,2^lC_1B}\\
&\quad+\sum_{l=1}^{\infty}2^{-l\gamma}\left|m_{2^lC_1B}(b)-m_B(b)\right|^kMf(x)\lesssim M_{L(\log L)^k}f(x),
\end{aligned}\tag{2.25}$$

where the last inequality follows from (2.18) and $|m_{2^lQ}(b)-m_Q(b)|\lesssim l$. This leads to our desired estimate (2.15).

Case 3 ($\mu(\mathcal{X}\setminus C_1B)\neq 0$ and $\mu(C_1B\setminus B)=0$). In this case, we take B' such that $B\subset B'$, $\mu(B')=\mu(B)$, and $\mu(C_1B'\setminus B')>0$. We then have that

$$\inf_{c\in\mathbb{C}}\frac{1}{\mu(B)}\int_B\left|\widetilde{M}_{b,k}f(x)-c\right|^q d\mu(x)\le\inf_{c\in\mathbb{C}}\frac{1}{\mu(B')}\int_{B'}\left|\widetilde{M}_{b,k}f(x)-c\right|^q d\mu(x).\tag{2.26}$$

With the ball B replaced by B' in Case 2, we also obtain the result that for any $y\in B'$,

$$\sup_{\epsilon>0}\int_{\mathcal{X}}\left|S_\epsilon(y,z)-S_\epsilon(y_0,z)\right|\left|m_{B'}(b)-b(z)\right|^k\left|f_2(z)\right|dz\le CM_{L(\log L)^k}f(x),\tag{2.27}$$

which completes the proof of Lemma 2.5. □

Lemma 2.6. *Let $\alpha,\beta\in[0,\infty)$. There exists a positive constant C, depending only on α and β, such that for all weights w,*

$$M_{L(\log L)^\alpha}\left(M_{L(\log L)^\beta}w\right)(x)\le CM_{L(\log L)^{\alpha+\beta+1}}w(x).\tag{2.28}$$

For Euclidean spaces, a generalization of Lemma 2.6 was proved in [16]. For spaces of homogeneous type, by a standard argument involving a covering lemma in [17, page 138], we have that for any $\lambda>0$ and suitable function f,

$$\mu\left(\left\{x\in\mathcal{X}:M_{L(\log L)^\alpha}f(x)>\lambda\right\}\right)\lesssim\int_{\mathcal{X}}\frac{|f(x)|}{\lambda}\log^\alpha\left(\mathrm{e}+\frac{|f(x)|}{\lambda}\right)d\mu(x).\tag{2.29}$$

Using this, Lemma 2.6 can be proved by applying the ideas used in [16]. For details, see [6, Lemma 7].

Proof of Theorem 1.1. We assume again that $\|b\|_{\mathrm{BMO}(\mathcal{X})} = 1$. At first, we claim that when $\mu(\mathcal{X}) = \infty$, for all $\lambda > 0$ and $f \in L_b^\infty(\mathcal{X})$, $\mu(\{x \in \mathcal{X} : \widetilde{M}_{b,k}f(x) > \lambda\}) < \infty$. In fact, for any $f \in L_b^\infty(\mathcal{X})$, let R be large enough such that $\operatorname{supp} f \subset B(x_0, R)$ for some $x_0 \in \mathcal{X}$. Notice that for all $x \in \mathcal{X} \setminus B(x_0, 3R)$,

$$Mf(x) \lesssim \frac{\|f\|_{L^1(\mathcal{X})}}{\mu(B(x, d(x, x_0)))}. \tag{2.30}$$

It then follows that for $p \in (1, \infty)$,

$$\begin{aligned}
&\mu\left(\left\{x \in \mathcal{X} \setminus B(x_0, 3R) : \left|b(x) - m_{B(x_0,R)}(b)\right|^k Mf(x) > \lambda\right\}\right) \\
&\quad \le \lambda^{-p} \int_{\mathcal{X} \setminus B(x_0,3R)} \left|b(x) - m_{B(x_0,R)}(b)\right|^{kp} (Mf(x))^p d\mu(x) \\
&\quad \le \lambda^{-p} \|f\|_{L^1(\mathcal{X})}^p \int_{\mathcal{X} \setminus B(x_0,3R)} \frac{\left|b(x) - m_{B(x_0,d(x,x_0))}(b)\right|^{kp}}{\{\mu(B(x, d(x, x_0)))\}^p} d\mu(x) < \infty.
\end{aligned} \tag{2.31}$$

This, together with the estimate that

$$\mu\left(\left\{x \in \mathcal{X} : M\left(\left(b - m_{B(x_0,R)}(b)\right)^k f\right)(x) > \lambda\right\}\right) \lesssim \lambda^{-p} \left\| \left|b - m_{B(x_0,R)}(b)\right|^k f \right\|_{L^p(\mathcal{X})}^p < \infty, \tag{2.32}$$

leads to our claim.

By (2.11), to prove Theorem 1.1, it suffices to prove that for all weights w,

$$\int_{\mathcal{X}} \left(\widetilde{M}_{b,k} f(x)\right)^p w(x) d\mu(x) \lesssim \int_{\mathcal{X}} |f(x)|^p M_{L(\log L)^{kp+\delta}} w(x) d\mu(x). \tag{2.33}$$

We proceed our proof by an inductive argument on k. When $k = 0$, (2.33) is implied by the fact that $Mw(x) \le M_{L(\log L)^\delta} w(x)$ for all $x \in \mathcal{X}$ and the following known inequality:

$$\int_{\mathcal{X}} \left[\widetilde{M} f(x)\right]^p w(x) d\mu(x) \lesssim \int_{\mathcal{X}} |f(x)|^p Mw(x) d\mu(x). \tag{2.34}$$

See [18, pages 150-151], for a proof of the last inequality when $\mathcal{X} = \mathbb{R}^n$. The same ideas also work for $\mathcal{X}$. Now we assume that k is a positive integer and (2.33) holds for any integer l with $0 \le l \le k-1$. Then $\widetilde{M}_{b,l}$ $(0 \le l \le k-1)$ can extend to a bounded operator on $L^p(\mathcal{X})$ for $p \in (1, \infty)$ and so for any $\lambda > 0$ and $\sigma \in (0, 1)$,

$$\mu\left(\left\{x \in \mathcal{X} : M\left(\left(\widetilde{M}_{b,l} f\right)^\sigma\right)(x) > \lambda\right\}\right) < \infty. \tag{2.35}$$

We now prove (2.33). To begin with, we prove that for any given $q \in (0, 1)$ and $k \in \mathbb{N}$, and for all weights h and all $f \in L_b^\infty(\mathcal{X})$,

$$\int_{\mathcal{X}} \left(\widetilde{M}_{b,k} f(x)\right)^q h(x) d\mu(x) \lesssim \int_{\mathcal{X}} \left(M_{L(\log L)^k} f(x)\right)^q M^k h(x) d\mu(x). \tag{2.36}$$

We first consider the case that $\mu(\mathcal{X}) = \infty$. Choose $r_1, \dots, r_{k-1}, r_k$ such that $0 < q = r_0 < r_1 < \cdots < r_{k-1} < r_k < 1$. By Lemma 2.5, we obtain that for any $1 \le m \le k-1$ and any weight h,

$$\begin{aligned} &\int_{\mathcal{X}} \left(M^{\#}_{r_j}(\widetilde{M}_{b,m}f)(x)\right)^q h(x)\,d\mu(x) \\ &\quad\lesssim \sum_{l=0}^{m-1} \int_{\mathcal{X}} \left(M_{r_{j+1}}(\widetilde{M}_{b,l}f)(x)\right)^q h(x)\,d\mu(x) + \int_{\mathcal{X}} \left(M_{L(\log L)^m} f(x)\right)^q h(x)\,d\mu(x). \end{aligned} \tag{2.37}$$

Therefore, applying Lemmas 2.1 and 2.2, and the estimate (2.35), we have

$$\begin{aligned} \int_{\mathcal{X}} \left(M_{r_{j+1}}(\widetilde{M}_{b,l}f)(x)\right)^q h(x)\,d\mu(x) &= \int_{\mathcal{X}} \left(M\left((\widetilde{M}_{b,l}f)^{r_{j+1}}\right)(x)\right)^{q/r_{j+1}} h(x)\,d\mu(x) \\ &\lesssim \int_{\mathcal{X}} \left(M^{\#}_{q/r_{j+1}}\left(M\left((\widetilde{M}_{b,l}f)^{r_{j+1}}\right)(x)\right)\right)^{q/r_{j+1}} Mh(x)\,d\mu(x) \\ &\lesssim \int_{\mathcal{X}} \left(M^{\#}\left((\widetilde{M}_{b,l}f)^{r_{j+1}}(x)\right)\right)^{q/r_{j+1}} Mh(x)\,d\mu(x) \\ &= \int_{\mathcal{X}} \left(M^{\#}_{r_{j+1}}(\widetilde{M}_{b,l}f)(x)\right)^q Mh(x)\,d\mu(x), \end{aligned} \tag{2.38}$$

which leads to

$$\begin{aligned} &\int_{\mathcal{X}} \left(M^{\#}_{r_j}(\widetilde{M}_{b,m}f)(x)\right)^q h(x)\,d\mu(x) \\ &\quad\lesssim \sum_{l=0}^{m-1} \int_{\mathcal{X}} \left(M^{\#}_{r_{j+1}}(\widetilde{M}_{b,l}f)(x)\right)^q Mh(x)\,d\mu(x) + \int_{\mathcal{X}} \left(M_{L(\log L)^m} f(x)\right)^q h(x)\,d\mu(x). \end{aligned} \tag{2.39}$$

Repeating the argument above $k-1$ times, we then have that for all weights h,

$$\begin{aligned} &\int_{\mathcal{X}} \left(M^{\#}_{q}(\widetilde{M}_{b,k}f)(x)\right)^q h(x)\,d\mu(x) \\ &\quad\lesssim \sum_{j=0}^{k-1} \int_{\mathcal{X}} \left(M^{\#}_{r_1}(\widetilde{M}_{b,j}f)(x)\right)^q Mh(x)\,d\mu(x) + \int_{\mathcal{X}} \left(M_{L(\log L)^k} f(x)\right)^q h(x)\,d\mu(x) \\ &\quad\lesssim \sum_{j=0}^{k-2} \int_{\mathcal{X}} \left(M^{\#}_{r_2}(\widetilde{M}_{b,j}f)(x)\right)^q M^2h(x)\,d\mu(x) + \sum_{j=0}^{k-1} \int_{\mathcal{X}} \left(M_{L(\log L)^j} f(x)\right)^q Mh(x)\,d\mu(x) \\ &\qquad + \int_{\mathcal{X}} \left(M_{L(\log L)^k} f(x)\right)^q h(x)\,d\mu(x) \\ &\quad\lesssim \sum_{j=0}^{1} \int_{\mathcal{X}} \left(M^{\#}_{r_{k-1}}(\widetilde{M}_{b,j}f)(x)\right)^q M^{k-1}h(x)\,d\mu(x) + \sum_{j=0}^{k} \int_{\mathcal{X}} \left(M_{L(\log L)^j} f(x)\right)^q M^{k-1}h(x)\,d\mu(x). \end{aligned} \tag{2.40}$$

On the other hand, notice that for all $x \in \mathcal{X}$, $M^{\#}_{r_{k-1}}(\widetilde{M}_{b,0}f)(x) \lesssim M^2 f(x)$, and that, by (2.12) and the fact that $M^2 f(x) \sim M_{L(\log L)} f(x)$ for all $x \in \mathcal{X}$ (see [12, (4.11)]), we then have that for all

$x \in \mathcal{X}$, $M^{\#}_{r_{k-1}}(\widetilde{M}_{b,1}f)(x) \lesssim M_{r_k}(\widetilde{M}_{b,0}f)(x) + M_{L(\log L)}f(x) \lesssim M^2 f(x)$. From these inequalities, it then follows that

$$\begin{aligned}
&\int_{\mathcal{X}} \left(M^{\#}_q(\widetilde{M}_{b,k}f)(x)\right)^q h(x)d\mu(x) \\
&\quad \lesssim \int_{\mathcal{X}} \left(M^2 f(x)\right)^q M^{k-1}h(x)d\mu(x) + \sum_{j=0}^{k}\int_{\mathcal{X}} \left(M_{L(\log L)^j} f(x)\right)^q M^{k-1}h(x)d\mu(x) \\
&\quad \lesssim \int_{\mathcal{X}} \left(M_{L(\log L)^k} f(x)\right)^q M^{k-1}h(x)d\mu(x),
\end{aligned} \tag{2.41}$$

which together with (i) of Lemma 2.1 gives (2.36).

We turn our attention to (2.36) for the case of $\mu(\mathcal{X}) < \infty$. For all $x \in \mathcal{X}$,

$$\widetilde{M}_{b,l}f(x) \lesssim \sum_{j=0}^{l} \left|b(x) - m_{\mathcal{X}}(b)\right|^j \widetilde{M}\left(\left(m_{\mathcal{X}}(b) - b\right)^{l-j} f\right)(x). \tag{2.42}$$

Moreover, the Kolmogorov inequality, together with Hölder's inequality, the inequalities (1.22), and (2.18), tells us that for any $0 \le j \le k$, $r \in (0,1)$, and $t \in (r,1)$,

$$\begin{aligned}
&\frac{1}{\mu(\mathcal{X})}\int_{\mathcal{X}} \left|b(x) - m_{\mathcal{X}}(b)\right|^{rj} \left(\widetilde{M}\left(\left(m_{\mathcal{X}}(b) - b\right)^{l-j} f\right)(x)\right)^r d\mu(x) \\
&\quad \lesssim \left\{\frac{1}{\mu(\mathcal{X})}\int_{\mathcal{X}} \left[\widetilde{M}\left(\left(m_{\mathcal{X}}(b) - b\right)^{l-j}\right) f(x)\right]^t d\mu(x)\right\}^{r/t} \\
&\quad \lesssim \left\{\frac{1}{\mu(\mathcal{X})}\int_{\mathcal{X}} |f(x)|\left|m_{\mathcal{X}}(b) - b(x)\right|^{l-j} d\mu(x)\right\}^r \lesssim \left(\|f\|_{L(\log L)^{l-j},\mathcal{X}}\right)^r.
\end{aligned} \tag{2.43}$$

Combining the above estimates, we obtain

$$m_{\mathcal{X}}\left(\left(\widetilde{M}_{b,l}f\right)^r\right) \lesssim \left(\inf_{x\in\mathcal{X}} M_{L(\log L)^l} f(x)\right)^r. \tag{2.44}$$

Let $q, r_1, r_2, \ldots, r_k$ be as in the case of $\mu(\mathcal{X}) = \infty$. Another application of Kolmogorov inequality and the fact that M is bounded from $L^1(\mathcal{X})$ to $L^{1,\infty}(\mathcal{X})$ leads to

$$\left\{m_{\mathcal{X}}\left(M\left(\widetilde{M}_{b,l}f\right)^{r_j}\right)^{q/r_j}\right\}^{r_j/q} \lesssim m_{\mathcal{X}}\left(\left(\widetilde{M}_{b,l}f\right)^{r_j}\right) \lesssim \left(\inf_{x\in\mathcal{X}} M_{L(\log L)^l} f(x)\right)^{r_j}. \tag{2.45}$$

As in the case of $\mu(\mathcal{X}) = \infty$, by Lemmas 2.1, 2.2, and 2.5, we have that for any $q \in (0,1)$,

$$\begin{aligned}
\int_{\mathcal{X}} \left(\widetilde{M}_{b,k}f\right)^q h(x)d\mu(x) &\lesssim \int_{\mathcal{X}} \left(M^{\#}_q(\widetilde{M}_{b,k}f)(x)\right)^q Mh(x)d\mu(x) + h(\mathcal{X})m_{\mathcal{X}}\left(\left(\widetilde{M}_{b,k}f\right)^q\right) \\
&\lesssim \sum_{j=0}^{k-1}\int_{\mathcal{X}} \left(M^{\#}_{r_1}(\widetilde{M}_{b,j}f)(x)\right)^q M^2 h(x)d\mu(x) \\
&\quad + \int_{\mathcal{X}} \left(M_{L(\log L)^k} f(x)\right)^q Mh(x)d\mu(x) \\
&\quad + \sum_{j=0}^{k-1}(Mh)(\mathcal{X})m_{\mathcal{X}}\left(M\left(\widetilde{M}_{b,j}f\right)^{r_1}\right)^{q/r_1} + h(\mathcal{X})m_{\mathcal{X}}\left(\left(\widetilde{M}_{b,k}f\right)^q\right) \\
&\lesssim \int_{\mathcal{X}} \left(M_{L(\log L)^k} f(x)\right)^q M^k h(x)d\mu(x).
\end{aligned} \tag{2.46}$$

Combining the two cases yields(2.36).

For any fixed $p \in (1,\infty)$ and $\delta > 0$, choose $q \in (0,1)$ and $\delta_1 > 0$ such that $kp/q+\delta_1 < kp+\delta$. This, via a duality argument, (2.36), and Lemma 2.3, leads to

$$\begin{aligned}
&\left\|(\widetilde{M}_{b,k}f)^q\right\|_{L^{p/q}(w)}\\
&\quad= \sup_{\|h\|_{L^{(p/q)'}\left(w^{1-(p/q)'}\right)}\le 1}\left|\int_{\mathcal{X}}\left[\widetilde{M}_{b,k}f(x)\right]^q h(x)\,d\mu(x)\right|\\
&\quad\lesssim \sup_{\|h\|_{L^{(p/q)'}\left(w^{1-(p/q)'}\right)}\le 1}\int_{\mathcal{X}}\left[M_{L(\log L)^k}f(x)\right]^q M^k h(x)\,d\mu(x)\\
&\quad\lesssim \sup_{\|h\|_{L^{(p/q)'}\left(w^{1-(p/q)'}\right)}\le 1}\left\{\int_{\mathcal{X}}\left[M_{L(\log L)^k}f(x)\right]^p M_{L(\log L)^{kp/q-1+\delta_1}}w(x)\,d\mu(x)\right\}^{q/p}\\
&\qquad\times\left\{\int_{\mathcal{X}}\left[M^k h(x)\right]^{(p/q)'}\left[M_{L(\log L)^{kp/q-1+\delta_1}}w(x)\right]^{1-(p/q)'}d\mu(x)\right\}^{1/(p/q)'}\\
&\quad\lesssim \left\{\int_{\mathcal{X}}\left[M_{L(\log L)^k}f(x)\right]^p M_{L(\log L)^{kp/q-1+\delta_1}}w(x)\,d\mu(x)\right\}^{q/p}\\
&\quad\lesssim \left\{\int_{\mathcal{X}}|f(x)|^p M_{L(\log L)^{kp+\delta}}w(x)\,d\mu(x)\right\}^{q/p},
\end{aligned}\tag{2.47}$$

where in the last inequality we have used Lemma 2.6. This completes the proof of Theorem 1.1. □

3. Proof of Theorem 1.2

We begin with some preliminary lemmas.

Lemma 3.1 (see [17]). *Let $(\mathcal{X}, d, \mu)$ be a space of homogeneous type and let f be a nonnegative integrable function. Then for every $\lambda > m_{\mathcal{X}}(f)(m_{\mathcal{X}}(f) = 0$ if $\mu(\mathcal{X}) = \infty)$, there exist a sequence of pairwise disjoint balls $\{B_j\}_{j\ge1}$ and a constant $C_4 \ge 1$ such that*

$$m_{C_4B_j}(f) \le \lambda < m_{B_j}(f) \tag{3.1}$$

and $m_B(f) \le \lambda$ for every ball B centered at $x \in \mathcal{X} \setminus (\cup_j C_4B_j)$.

Lemma 3.2. *Let d and l be two nonnegative integers. Then for all $t_1, t_2 \ge 0$,*

$$t_1t_2^d \log\left(e + t_1t_2^d\right) \le C\left(t_1\log^{d+l}(e + t_1) + \exp t_2\right). \tag{3.2}$$

Proof. We may assume that $d \ge 1$, otherwise the conclusion holds obviously. Set $\Phi_1(t) = t\log^l(\mathrm{e}+t)$, $\Phi_2(t) = t\log^{l+d}(\mathrm{e}+t)$, and $\Phi_3(t) = \exp(t^{1/d})$. Let $j = 1, 2, 3$. Denote by Φ_j^{-1} the inverse of Φ_j, that is, $\Phi_j^{-1}(t) = \inf\{s > 0 : \Phi_j(s) > t\}$. It is well known that $\Phi_1^{-1}(t) \approx t\log^{-l}(\mathrm{e} + t)$ and $\Phi_2^{-1}(t) \approx t\log^{-(d+l)}(\mathrm{e} + t)$ (see [19]). On the other hand, it is easy to verify that $\Phi_3^{-1}(t) = 0$ when $t \in [0, 1)$ and $\Phi_3^{-1}(t) = \log^d t$ when $t \in [1,\infty)$. Therefore, for all $t \in [0,\infty)$, $\Phi_2^{-1}(t)\Phi_3^{-1}(t) \lesssim \Phi_1^{-1}(t)$. This via [7, Lemma 6, page 63] tells us that $\Phi_1(t_1t_2^d) \lesssim \Phi_2(t_1) + \Phi_3(t_2^d)$. Our desired conclusion then follows directly. □

Proof of Theorem 1.2. With the notation $\widetilde{M}_{b,k}$ as in (2.7), by (2.11), it suffices to prove that for $\|b\|_{\mathrm{BMO}(\mathcal{X})} = 1$ and all $f \in L_b^\infty(\mathcal{X})$ and $\lambda > 0$,

$$w\left(\left\{x \in \mathcal{X} : \widetilde{M}_{b,k} f(x) > \lambda\right\}\right) \lesssim \int_{\mathcal{X}} \frac{|f(x)|}{\lambda} \log^k\left(\mathrm{e} + \frac{|f(x)|}{\lambda}\right) M_{L(\log L)^{\tilde{k}+\delta}} w(x) d\mu(x), \tag{3.3}$$

where $\tilde{k} = k$ when k is even and $\tilde{k} = k+1$ when k is odd.

Recall that for all $f \in L_b^\infty(\mathcal{X})$,

$$w(\{x \in \mathcal{X} : \widetilde{M} f(x) > \lambda\}) \lesssim \frac{1}{\lambda} \int_{\mathcal{X}} |f(x)| M w(x) d\mu(x). \tag{3.4}$$

(See [18, page 151] for a proof when $\mathcal{X} = \mathbb{R}^n$. The same idea also works for $\mathcal{X}$.) By Hölder's inequality, it follows that for all $x \in \mathcal{X}$,

$$\widetilde{M}_{b,k} f(x) \le \left(\widetilde{M}_{b,k+1} f(x)\right)^{k/(k+1)} \left(\widetilde{M} f(x)\right)^{1/(k+1)} \tag{3.5}$$

and so when k is odd,

$$\begin{aligned} w\left(\left\{x \in \mathcal{X} : \widetilde{M}_{b,k} f(x) > \lambda\right\}\right) &\le w\left(\left\{x \in \mathcal{X} : \widetilde{M}_{b,k+1} f(x) > \lambda\right\}\right) + w\left(\left\{x \in \mathcal{X} : \widetilde{M} f(x) > \lambda\right\}\right) \\ &\lesssim w\left(\left\{x \in \mathcal{X} : \widetilde{M}_{b,k+1} f(x) > \lambda\right\}\right) + \frac{1}{\lambda} \int_{\mathcal{X}} |f(x)| M w(x) d\mu(x). \end{aligned} \tag{3.6}$$

Thus, it suffices to prove (3.3) for the case that k is even. We employ some ideas from [20], and proceed our proof of (3.3) by an inductive argument. When $k = 0$, (3.3) is implied by the fact that $Mw(x) \le M_{L(\log L)^\delta} w(x)$ for all $x \in \mathcal{X}$ and (3.4). Now let k be a positive integer. We may assume that $M_{L(\log L)^{k+\delta}} w$ is finite almost everywhere, otherwise there is nothing to be proved. For any fixed $\delta > 0$, we assume that for any nonnegative integer l with $0 \le l \le k-1$, there exists a constant $C = C_{l,\delta}$ such that for all $\lambda > 0$,

$$w\left(\left\{x \in \mathcal{X} : \widetilde{M}_{b,l} f(x) > \lambda\right\}\right) \lesssim \int_{\mathcal{X}} \frac{|f(x)|}{\lambda} \log^l\left(\mathrm{e} + \frac{|f(x)|}{\lambda}\right) M_{L(\log L)^{\tilde{l}+\delta}} w(x) d\mu(x), \tag{3.7}$$

where and in what follows, $\tilde{l} = l$ when l is even and $\tilde{l} = l+1$ when l is odd. If $\mu(\mathcal{X}) < \infty$ and $\lambda \le \|f\|_{L^1(\mathcal{X})}[\mu(\mathcal{X})]^{-1}$, the inequality (3.3) is trivial. So it remains to consider the case that $\lambda > \|f\|_{L^1(\mathcal{X})}[\mu(\mathcal{X})]^{-1}$. For each fixed bounded function f with bounded support and $\lambda > \|f\|_{L^1(\mathcal{X})}[\mu(\mathcal{X})]^{-1}$, applying Lemma 3.1 to $|f|$ at level λ, we obtain a sequence of balls $\{B_j\}_{j\ge1}$ with pairwise disjoint interiors. As in the proof of Lemma 2.10 in [17], set $V_1 = (C_4 B_1) \setminus (\cup_{n\ge2} B_n)$ and $V_j = (C_4 B_j) \setminus [\cup_{n=1}^{j-1} V_n \cup \cup_{l\ge j+1} B_l]$, it then follows that $B_j \subset V_j \subset (C_4 B_j)$ and $\cup_j V_j = \cup_j (C_4 B_j)$. Define the functions g and h, respectively, by $g \equiv |f| \chi_{\mathcal{X} \setminus \cup_j V_j} + \sum_j m_{V_j}(|f|) \chi_{V_j}$ and $h \equiv \sum_j h_j$ with $h_j \equiv (|f| - m_{V_j}(|f|)) \chi_{V_j}$. Recall that μ is regular and the set of continuous function is dense in $L^p(\mathcal{X})$ for any $p \in [1, \infty)$. Lemma 3.1 implies that for any fixed j,

$$C_6^{-1} \lambda \le \frac{1}{\mu(V_j)} \int_{V_j} |f(y)| d\mu(y) \le C_6 \lambda \tag{3.8}$$

with $C_6 > 1$ a constant independent of f and j, which together with the Lebesgue differentiation theorem and Lemma 3.1 again yields that

$$\|g\|_{L^\infty(\mathcal{X})} \le C_6\lambda. \tag{3.9}$$

Let $\Omega = \cup_j(C_7B_j)$ with $C_7 = C_1C_4$. The doubling property of μ and (3.1) now state that

$$w(\Omega) \lesssim \sum_j \frac{w(C_7B_j)}{\mu(C_7B_j)}\mu(B_j) \lesssim \frac{1}{\lambda}\sum_j \inf_{x\in B_j} Mw(x)\int_{B_j}|f(y)|d\mu(y) \lesssim \frac{1}{\lambda}\int_{\mathcal{X}}|f(y)|Mw(y)d\mu(y). \tag{3.10}$$

Following an argument similar to the case of Euclidean spaces (see [18, page 159]), we have that for any $\gamma \ge 0$, there exists a positive constant C, depending only on γ, such that for all $x \in C_4B_j$,

$$M_{L(\log L)^\gamma}(w\chi_{\mathcal{X}\setminus\Omega})(x) \lesssim \inf_{y\in C_4B_j} M_{L(\log L)^\gamma}(w\chi_{\mathcal{X}\setminus\Omega})(y). \tag{3.11}$$

Thus,

$$\begin{aligned}
|m_{V_j}(f)|\int_{V_j} M_{L(\log L)^{k+\delta}}(w\chi_{\mathcal{X}\setminus\Omega})(x)d\mu(x) &\lesssim \lambda\mu(V_j)\inf_{y\in B_j} M_{L(\log L)^{k+\delta}}(w\chi_{\mathcal{X}\setminus\Omega})(y) \\
&\lesssim \lambda\mu(B_j)\inf_{y\in B_j} M_{L(\log L)^{k+\delta}}(w\chi_{\mathcal{X}\setminus\Omega})(y) \\
&\lesssim \int_{B_j}|f(x)|M_{L(\log L)^{k+\delta}}(w\chi_{\mathcal{X}\setminus\Omega})(x)d\mu(x).
\end{aligned} \tag{3.12}$$

For each fixed $\delta > 0$, choose $p_0 \in (1,\infty)$ and $\delta_1 > 0$ such that $kp_0 + \delta_1 < k + \delta$. From the last estimate, (2.11), Theorem 1.1, and (3.9), it follows that

$$\begin{aligned}
w(\{x \in \mathcal{X}\setminus\Omega : \widetilde{M}_{b,k}g(x) > \lambda/2\}) &\lesssim \lambda^{-p_0}\int_{\mathcal{X}}|g(x)|^{p_0} M_{L(\log L)^{kp_0+\delta_1}}(w\chi_{\mathcal{X}\setminus\Omega})(x)d\mu(x) \\
&\lesssim \lambda^{-1}\int_{\mathcal{X}}|g(x)|M_{L(\log L)^{kp_0+\delta_1}}(w\chi_{\mathcal{X}\setminus\Omega})(x)d\mu(x) \\
&\lesssim \lambda^{-1}\Bigg(\int_{\mathcal{X}\setminus\cup_j V_j}|f(x)|M_{L(\log L)^{k+\delta}}(w\chi_{\mathcal{X}\setminus\Omega})(x)d\mu(x) \\
&\qquad + \sum_j |m_{V_j}(f)|\int_{V_j} M_{L(\log L)^{k+\delta}}(w\chi_{\mathcal{X}\setminus\Omega})(x)d\mu(x)\Bigg) \\
&\lesssim \lambda^{-1}\int_{\mathcal{X}}|f(x)|M_{L(\log L)^{k+\delta}}w(x)d\mu(x).
\end{aligned} \tag{3.13}$$

Thus, our proof is now reduced to proving

$$w\left(\left\{x \in \mathcal{X}\setminus\Omega : \widetilde{M}^*_{b,k}h(x) > \frac{\lambda}{2}\right\}\right) \lesssim \int_{\mathcal{X}} \frac{|f(x)|}{\lambda}\log^k\left(\mathrm{e} + \frac{|f(x)|}{\lambda}\right) M_{L(\log L)^{k+\delta}}w(x)d\mu(x), \tag{3.14}$$

where

$$\widetilde{M}^{*}_{b,k}h(x)=\sup_{\epsilon>0}\left|\int_{\mathcal{X}}S_{\epsilon}(x,y)\left|b(x)-b(y)\right|^{k}h(y)d\mu(y)\right|. \tag{3.15}$$

For any $j\geq 1$, let

$$\widetilde{M}^{*}h_j(x)=\sup_{\epsilon>0}\left|\int_{\mathcal{X}}S_{\epsilon}(x,y)h_j(y)d\mu(y)\right|. \tag{3.16}$$

We now prove (3.14). With the aid of the formula that for all $x,y\in\mathcal{X}$,

$$(b(x)-b(y))^{k}=\sum_{l=0}^{k}C_k^l\left(b(x)-m_{B_j}(b)\right)^{l}\left(m_{B_j}(b)-b(y)\right)^{k-l}, \tag{3.17}$$

since k is even, for $x\in\mathcal{X}\setminus\Omega$, we write

$$\begin{aligned}\widetilde{M}^{*}_{b,k}h(x)&\lesssim\sum_j\left|b(x)-m_{B_j}(b)\right|^{k}\widetilde{M}^{*}h_j(x)+\sum_{l=0}^{k-1}\widetilde{M}_{b,l}\left(\sum_j\left(b(\cdot)-m_{B_j}(b)\right)^{k-l}h_j\right)(x)\\&\equiv \mathrm{G}(x)+\mathrm{H}(x).\end{aligned} \tag{3.18}$$

Recall that $\{V_j\}_j$ are mutually disjoint. If we set $\Phi_l(t)=t\log^{l}(\mathrm{e}+t)$, our inductive hypothesis (3.7) via (3.11) now tells us that

$$\begin{aligned}&w\left(\left\{x\in\mathcal{X}\setminus\Omega:\mathrm{H}(x)>\frac{\lambda}{4}\right\}\right)\\&\quad\lesssim\sum_{l=0}^{k-1}\sum_j\int_{V_j}\Phi_l\left(\frac{\left|b(y)-m_{B_j}(b)\right|^{k-l}|f(y)|}{\lambda}\right)d\mu(y)\inf_{z\in C_4B_j}M_{L(\log L)^{\tilde{l}+\delta}}(w\chi_{\mathcal{X}\setminus\Omega})(z)\\&\qquad+\sum_{l=0}^{k-1}\sum_j\int_{V_j}\Phi_l\left(\frac{\left|b(y)-m_{B_j}(b)\right|^{k-l}|m_{V_j}(f)|}{\lambda}\right)d\mu(y)\inf_{z\in B_j}M_{L(\log L)^{\tilde{l}+\delta}}(\chi_{\mathcal{X}\setminus\Omega}w)(z).\end{aligned} \tag{3.19}$$

An application of Lemma 3.2 then gives that

$$\begin{aligned}&\int_{V_j}\Phi_l\left(\frac{\left|b(y)-m_{B_j}(b)\right|^{k-l}|f(y)|}{\lambda}\right)d\mu(y)\\&\quad\lesssim\int_{V_j}\Phi_l\left(\frac{\left|b(y)-m_{C_4B_j}(b)\right|^{k-l}|f(y)|}{\lambda}\right)d\mu(y)\\&\qquad+\Phi_l\left(\left|m_{C_4B_j}(b)-m_{B_j}(b)\right|^{k-l}\right)\int_{V_j}\Phi_l\left(\frac{|f(y)|}{\lambda}\right)d\mu(y)\\&\quad\lesssim\int_{V_j}\exp\left(\frac{\left|b(y)-m_{C_4B_j}(b)\right|}{C_2}\right)d\mu(x)+\int_{V_j}\Phi_k\left(\frac{|f(x)|}{\lambda}\right)d\mu(x)\\&\quad\lesssim\int_{V_j}\Phi_k\left(\frac{|f(x)|}{\lambda}\right)d\mu(x).\end{aligned} \tag{3.20}$$

For each fixed j, notice that by (3.8) and Lemma 3.2,

$$\int_{V_j}\Phi_l\left(\frac{|b(y)-m_{B_j}(b)|^{k-l}|m_{V_j}(f)|}{\lambda}\right)d\mu(y)\lesssim\mu(B_j). \tag{3.21}$$

It then follows that

$$\begin{aligned}w\left(\left\{x\in\mathcal{X}\setminus\Omega:\mathrm{H}(x)>\frac{\lambda}{4}\right\}\right)&\lesssim\sum_j\int_{V_j}\Phi_k\left(\frac{|f(x)|}{\lambda}\right)M_{L(\log L)^{k+\delta}}w(x)d\mu(x)\\&\quad+\sum_j\mu(B_j)\inf_{y\in B_j}M_{L(\log L)^{k+\delta}}w(y)\\&\lesssim\int_{\mathcal{X}}\Phi_k\left(\frac{|f(x)|}{\lambda}\right)M_{L(\log L)^{k+\delta}}w(x)d\mu(x).\end{aligned} \tag{3.22}$$

It remains to prove that

$$w\left(\left\{x\in\mathcal{X}\setminus\Omega:\mathrm{G}(x)>\frac{\lambda}{4}\right\}\right)\lesssim\lambda^{-1}\int_{\mathcal{X}}|f(x)|M_{L(\log L)^k}w(x)d\mu(x). \tag{3.23}$$

For each fixed j, let y_j and r_j be the center and radius of B_j, respectively. If $x\in\mathcal{X}\setminus\Omega$, then by the vanishing moment of h_j and the estimate (2.14), we obtain that

$$\begin{aligned}\widetilde{M}^*h_j(x)&\le\sup_{\epsilon>0}\int_{\mathcal{X}}|S_\epsilon(x,y)-S_\epsilon(x,y_j)||h_j(y)|d\mu(y)\\&\lesssim(r_j)^\gamma\frac{\{d(x,y_j)\}^{-\gamma}}{\mu(B(y_j,d(x,y_j)))}\int_{\mathcal{X}}|h_j(y)|d\mu(y).\end{aligned} \tag{3.24}$$

This in turn implies that

$$\begin{aligned}&w\left(\left\{x\in\mathcal{X}\setminus\Omega:\mathrm{G}(x)>\frac{\lambda}{4}\right\}\right)\\&\quad\lesssim\frac{1}{\lambda}\sum_j(r_j)^\gamma\int_{\mathcal{X}\setminus C_7B_j}\frac{|b(x)-m_{B_j}(b)|^k}{\mu(B(y_j,d(x,y_j)))}\frac{w(x)}{\{d(x,y_j)\}^\gamma}d\mu(x)\int_{\mathcal{X}}|h_j(y)|d\mu(y)\\&\quad\le\frac{1}{\lambda}\sum_j(r_j)^\gamma\sum_{l=1}^{\infty}\int_{(2^lC_7B_j)\setminus(2^{l-1}C_7B_j)}\frac{|b(x)-m_{B_j}(b)|^kw(x)}{\mu(B(y_j,d(x,y_j)))(d(x,y_j))^\gamma}d\mu(x)\int_{\mathcal{X}}|h_j(y)|d\mu(y)\\&\quad\lesssim\frac{1}{\lambda}\sum_j\int_{\mathcal{X}}|h_j(y)|d\mu(y)\inf_{y\in C_7B_j}M_{L(\log L)^k}w(y)\lesssim\frac{1}{\lambda}\int_{\mathcal{X}}|f(x)|M_{L(\log L)^k}w(x)d\mu(x),\end{aligned} \tag{3.25}$$

where in the second to the last inequality, we use the fact that for each fixed j, $y_j\in V_j$ and positive integer l, a standard argument involving the inequalities (1.22) and (2.18) yields

$$\int_{(2^l C_7 B_j)\setminus(2^{l-1} C_7 B_j)} \frac{|b(x)-m_{B_j}(b)|^k w(x)}{\mu(B(y_j,d(x,y_j)))(d(x,y_j))^{\gamma}} d\mu(x) \lesssim l^k (2^l r_j)^{-\gamma} \inf_{y\in C_7 B_j} M_{L(\log L)^k} w(y); \tag{3.26}$$

see also the proof of (2.25). We then complete the proof of Theorem 1.2. □

4. Proof of Theorem 1.3

This section is devoted to the proof of Theorem 1.3.

Lemma 4.1. *Let T be a Calderón-Zygmund operator. Then, there exists a positive constant C such that for all $\lambda > 0$, $f \in L_b^\infty(\mathcal{X})$, and weights w,*

$$\int_{\{x\in\mathcal{X}:\, T^* f(x)>\lambda\}} w(x) d\mu(x) \le C \int_{\mathcal{X}} \frac{|f(x)|}{\lambda} M_{L(logL)^{1+\delta}} w(x) d\mu(x). \tag{4.1}$$

Lemma 4.1 can be proved by a similar but more careful argument as that used in the proof of Theorem 1.2 in [8]. We omit the proof here for brevity.

Proof of Theorem 1.3. The argument here is similar to that used in the proof of Theorem 1.2, and we will only give an outline. Also, we proceed our proof by an inductive argument. By Lemma 4.1, it is obvious that (1.19) is true when $k = 0$. Now let k be a positive integer. For any fixed $\delta > 0$, and any nonnegative integer l with $0 \le l \le k-1$, we assume that for all $\lambda > 0$ and $f \in L_b^\infty(\mathcal{X})$,

$$w(\{x \in \mathcal{X} : T_{b,l}^* f(x) > \lambda\}) \lesssim \int_{\mathcal{X}} \frac{|f(x)|}{\lambda} \log^l\left(\mathrm{e} + \frac{|f(x)|}{\lambda}\right) M_{L(\log L)^{l+1+\delta}} w(x) d\mu(x). \tag{4.2}$$

We need only consider the case that $\lambda > \|f\|_{L^1(\mathcal{X})}[\mu(\mathcal{X})]^{-1}$. For each fixed bounded function f with bounded support and $\lambda > \|f\|_{L^1(\mathcal{X})}[\mu(\mathcal{X})]^{-1}$, applying Lemma 3.1 to $|f|$ at level λ, with the same notation $\{B_j\}_j$, $\{V_j\}_j$, Ω as in the proof of Theorem 1.2, we decompose $f \equiv g + h$, where $g \equiv f\chi_{\mathcal{X}\setminus\cup_j V_j} + \sum_j m_{V_j}(f)\chi_{V_j}$ and $h \equiv \sum_j h_j$ with $h_j \equiv (f - m_{V_j}(f))\chi_{V_j}$. Applying the estimate (1.17), and a similar argument to that used to deal with the term $\widetilde{M}_{b,k} g$ gives us that

$$\begin{aligned} w\left(\left\{x \in (\mathcal{X}\setminus\Omega) : T_{b,k}^* g(x) > \frac{\lambda}{2}\right\}\right) &\lesssim \lambda^{-p_0} \int_{\mathcal{X}} |g(x)|^{p_0} M_{L(\log L)^{(k+1)p_0+\delta_1}} (w\chi_{\mathcal{X}\setminus\Omega})(x) d\mu(x) \\ &\lesssim \lambda^{-1} \int_{\mathcal{X}} |f(x)| M_{L(\log L)^{k+1+\delta}} w(x) d\mu(x), \end{aligned} \tag{4.3}$$

where $p_0 \in (1,\infty)$ and $\delta_1 > 0$ such that $(k+1)p_0 + \delta_1 < k+1+\delta$.

We now turn to the term $T_{b,k}^* h$. For any $x \in \mathcal{X}\setminus\Omega$ and $\epsilon > 0$, set

$$\begin{aligned} \mathrm{I}_1(x,\epsilon) &= \{j : \ \forall\, y \in C_4 B_j, d(x,y) \le \epsilon\}, \\ \mathrm{I}_2(x,\epsilon) &= \{j : \ \forall\, y \in C_4 B_j, d(x,y) > \epsilon\}, \\ \mathrm{I}_3(x,\epsilon) &= \{j : (C_4 B_j) \cap \{y \in \mathcal{X} : d(x,y) > \epsilon\} \ne \emptyset, (C_4 B_j) \cap \{y \in \mathcal{X} : d(x,y) \le \epsilon\} \ne \emptyset\}. \end{aligned} \tag{4.4}$$

It then follows that

$$\begin{aligned}\left|T_{\epsilon;b,k}h(x)\right| &\leq \left|T_{\epsilon;b,k}\left(\sum_{j\in \mathrm{I}_2(x,\epsilon)} h_j\right)(x)\right| + \left|T_{\epsilon;b,k}\left(\sum_{j\in \mathrm{I}_3(x,\epsilon)} h_j\right)(x)\right| \\ &\lesssim \left|\sum_{j\in \mathrm{I}_2(x,\epsilon)} \left(b(x)-m_{B_j}(b)\right)^k T_\epsilon h_j(x)\right| + \sum_{l=0}^{k-1}\left|T_{\epsilon;b,l}\left(\sum_j \left(b(\cdot)-m_{B_j}(b)\right)^{k-l}h_j\right)(x)\right| \\ &\quad + \sum_{l=0}^{k-1}\left|T_{\epsilon;b,l}\left(\sum_{j\in \mathrm{I}_3(x,\epsilon)} \left(b(\cdot)-m_{B_j}(b)\right)^{k-l}h_j\right)(x)\right| \\ &\quad + \left|T_{\epsilon;b,k}\left(\sum_{j\in \mathrm{I}_3(x,\epsilon)} h_j\right)(x)\right| \equiv \mathrm{U}_\epsilon(x)+\mathrm{V}_\epsilon(x)+\mathrm{W}_\epsilon(x)+\mathrm{X}_\epsilon(x).\end{aligned} \tag{4.5}$$

Notice that for $x \in (\mathcal{X}\setminus\Omega)$ and $j \in \mathrm{I}_2(x,\epsilon)$, we have that $T_\epsilon h_j(x) = Th_j(x)$. By the vanishing moment of h_j and the regularity condition (1.13), we have

$$\sup_{\epsilon>0}\mathrm{U}_\epsilon(x) \leq \sum_j \left|b(x)-m_{B_j}(b)\right|^k \int_{\mathcal{X}} \frac{\left(d(y,y_j)\right)^\tau}{\mu(B(y,d(x,y)))\left(d(x,y)\right)^\tau}\left|h_j(y)\right|d\mu(y) \tag{4.6}$$

and so

$$\begin{aligned}& w\left(\left\{x\in(\mathcal{X}\setminus\Omega) : \sup_{\epsilon>0}\mathrm{U}_\epsilon(x) > \frac{\lambda}{6}\right\}\right) \\ &\quad \lesssim \lambda^{-1}\sum_j \int_{\mathcal{X}} \left|h_j(y)\right|\left(d(y,y_j)\right)^\tau \int_{\mathcal{X}\setminus C_7B_j} \frac{\left|b(x)-m_{B_j}(b)\right|^k w(x)}{\mu(B(y,d(x,y)))\left(d(x,y)\right)^\tau}d\mu(x)d\mu(y) \\ &\quad \lesssim \lambda^{-1}\int_{\mathcal{X}} \left|f(x)\right| M_{L(\log L)^k} w(x)d\mu(x).\end{aligned} \tag{4.7}$$

Our inductive hypothesis (4.2), via the argument for the term H in the proof of Theorem 1.2, leads to

$$w\left(\left\{x\in(\mathcal{X}\setminus\Omega) : \sup_{\epsilon>0}\mathrm{V}_\epsilon(x) > \frac{\lambda}{4}\right\}\right) \lesssim \int_{\mathcal{X}} \frac{|f(x)|}{\lambda}\log^k\left(\mathrm{e}+\frac{|f(x)|}{\lambda}\right)M_{L(\log L)^{k+1+\delta}}w(x)d\mu(x). \tag{4.8}$$

Notice that for $x \in \mathcal{X}\setminus\Omega$ and $j \in \mathrm{I}_3(x,\epsilon)$, we have that $C_4B_j \subset \{B(x,C_8\epsilon)\setminus B(x,C_9\epsilon)\}$, where C_8 and C_9 with $C_8 > C_9$ are two positive constants. Therefore, for all $x \in \mathcal{X}\setminus\Omega$,

$$\sup_{\epsilon>0}\left(\mathrm{W}_\epsilon(x)+\mathrm{X}_\epsilon(x)\right) \lesssim \sum_{l=0}^{k} M_{b,l}\left(\sum_j \left|b-m_{B_j}(b)\right|^{k-l}\left|h_j\right|\right)(x). \tag{4.9}$$

This, along with Theorem 1.2 and an argument for the term H in the proof of Theorem 1.2, leads to

$$\begin{aligned}& w\left(\left\{x\in(\mathcal{X}\setminus\Omega) : \sup_{\epsilon>0}\left(\mathrm{W}_\epsilon(x)+\mathrm{X}_\epsilon(x)\right) > \frac{\lambda}{6}\right\}\right) \\ &\quad \lesssim \sum_{l=0}^{k}\sum_j \int_{V_j} \Phi_l\left(\frac{\left|b(y)-m_{B_j}(b)\right|^{k-l}\left|h_j(y)\right|}{\lambda}\right)M_{L(\log L)^{\tilde{l}+\delta}}\left(w\chi_{\mathcal{X}\setminus\Omega}\right)(y)d\mu(y) \\ &\quad \lesssim \int_{\mathcal{X}} \frac{|f(y)|}{\lambda}\log^k\left(\mathrm{e}+\frac{|f(y)|}{\lambda}\right)M_{L(\log L)^{\tilde{k}+\delta}}w(y)d\mu(y),\end{aligned} \tag{4.10}$$

where $\widetilde{l} = l$ when l is even and $\widetilde{l} = l + 1$ when l is odd. Combining the estimates for the terms $\sup_{\epsilon>0} \mathrm{U}_\epsilon$, $\sup_{\epsilon>0} \mathrm{V}_\epsilon$, and $\sup_{\epsilon>0}(\mathrm{W}_\epsilon + \mathrm{X}_\epsilon)$ gives us that

$$w\left(\left\{x \in (\mathcal{X} \setminus \Omega) : T^*_{b,k}h(x) > \frac{\lambda}{2}\right\}\right) \lesssim \int_{\mathcal{X}} \frac{|f(y)|}{\lambda} \log^k\left(\mathrm{e} + \frac{|f(y)|}{\lambda}\right) M_{L(\log L)^{k+1+\delta}} w(y) d\mu(y), \tag{4.11}$$

which completes the proof of Theorem 1.3. □

Acknowledgments

The authors would like to thank the referees for their many helpful suggestions and corrections, which improve the presentation of this paper. The first author is supported by National Natural Science Foundation of China (Grant no. 10671210), and the third author is supported by National Science Foundation for Distinguished Young Scholars (Grant no. 10425106) and NCET (Grant no. 04-0142) of Ministry of Education of China.

References

[1] R. R. Coifman and G. Weiss, *Analyse Harmonique Non-Commutative sur Certains Espaces Homogènes*, vol. 242 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1971.

[2] R. A. Macías and C. Segovia, "Lipschitz functions on spaces of homogeneous type," *Advances in Mathematics*, vol. 33, no. 3, pp. 257–270, 1979.

[3] J. García-Cuerva, E. Harboure, C. Segovia, and J. L. Torrea, "Weighted norm inequalities for commutators of strongly singular integrals," *Indiana University Mathematics Journal*, vol. 40, no. 4, pp. 1397–1420, 1991.

[4] A. M. Alphonse, "An end point estimate for maximal commutators," *Journal of Fourier Analysis and Applications*, vol. 6, no. 4, pp. 449–456, 2000.

[5] D. Li, G. Hu, and X. Shi, "Weighted norm inequalities for the maximal commutators of singular integral operators," *Journal of Mathematical Analysis and Applications*, vol. 319, no. 2, pp. 509–521, 2006.

[6] G. Hu and W. Wang, "A weighted L^p estimate for the maximal commutator on spaces of homogeneous type," submitted.

[7] M. M. Rao and Z. D. Ren, *Theory of Orlicz Spaces*, vol. 146 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 1991.

[8] G. Hu, X. Shi, and Q. Zhang, "Weighted norm inequalities for the maximal singular integral operators on spaces of homogeneous type," *Journal of Mathematical Analysis and Applications*, vol. 336, no. 1, pp. 1–17, 2007.

[9] D. Cruz-Uribe, J. M. Martell, and C. Pérez, "Extrapolation from A_∞ weights and applications," *Journal of Functional Analysis*, vol. 213, no. 2, pp. 412–439, 2004.

[10] C. Pérez, "On sufficient conditions for the boundedness of the Hardy-Littlewood maximal operator between weighted L^p-spaces with different weights," *Proceedings of the London Mathematical Society*, vol. 71, no. 1, pp. 135–157, 1995.

[11] C. Pérez, "Sharp estimates for commutators of singular integrals via iterations of the Hardy-Littlewood maximal function," *Journal of Fourier Analysis and Applications*, vol. 3, no. 6, pp. 743–756, 1997.

[12] G. Pradolini and O. Salinas, "Commutators of singular integrals on spaces of homogeneous type," *Czechoslovak Mathematical Journal*, vol. 57, no. 1, pp. 75–93, 2007.

[13] G. Pradolini and O. Salinas, "Maximal operators on spaces of homogeneous type," *Proceedings of the American Mathematical Society*, vol. 132, no. 2, pp. 435–441, 2004.

[14] Y. Han, D. Müller, and D. Yang, "A theory of Besov and Triebel-Lizorkin spaces on metric measure spaces modeled on Carnot-Carathéodory spaces," submitted.

[15] J. Duoandikoetxea, *Fourier Analysis*, American Mathematical Society, Providence, RI, USA, 2001.

[16] M. Carozza and A. Passarelli di Napoli, "Composition of maximal operators," *Publicacions Matemàtiques*, vol. 40, no. 2, pp. 397–409, 1996.
[17] H. Aimar, "Singular integrals and approximate identities on spaces of homogeneous type," *Transactions of the American Mathematical Society*, vol. 292, no. 1, pp. 135–153, 1985.
[18] J. García-Cuerva and J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland, Amsterdam, The Netherlands, 1985.
[19] R. O'Neil, "Integral transforms and tensor products on Orlicz spaces and $L(p,q)$ spaces," *Journal d'Analyse Mathématique*, vol. 21, pp. 1–276, 1968.
[20] C. Pérez and G. Pradolini, "Sharp weighted endpoint estimates for commutators of singular integrals," *Michigan Mathematical Journal*, vol. 49, no. 1, pp. 23–37, 2001.