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Research Article

Differences of Composition Operators on the Space of Bounded Analytic Functions in the Polydisc

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This paper gives some estimates of the essential norm for the difference of composition operators induced by φ and ψ acting on the space, $H^\infty(\mathbb{D}^n)$, of bounded analytic functions on the unit polydisc $\mathbb{D}^n$, where φ and ψ are holomorphic self-maps of $\mathbb{D}^n$. As a consequence, one obtains conditions in terms of the Carathéodory distance on $\mathbb{D}^n$ that characterizes those pairs of holomorphic self-maps of the polydisc for which the difference of two composition operators on $H^\infty(\mathbb{D}^n)$ is compact.

1. Introduction

Let $\mathbb{D}^n$ be the unit polydisc of $\mathbb{C}^n$ with boundary $\partial\mathbb{D}^n$. If $n = 1$, we will denote the unit disk $\mathbb{D}^1$ simply by $\mathbb{D}$. The class of allholomorphic functions on $\mathbb{D}^n$ will be denoted by $H(\mathbb{D}^n)$, while by $H^\infty(\mathbb{D}^n)$, we denote the space of all bounded analytic functions in the unit polydisc with the norm $\|f\|_\infty = \sup_{z\in\mathbb{D}^n}|f(z)|$.

Let $\varphi(z) = (\varphi_1(z), \dots, \varphi_n(z))$ and $\psi(z) = (\psi_1(z), \dots, \psi_n(z))$ be holomorphic self-maps of $\mathbb{D}^n$. The composition operator, C_φ, is defined by

$$C_\varphi(f)(z) = f(\varphi(z)) \tag{1.1}$$

for any $f \in H(\mathbb{D}^n)$ and $z \in \mathbb{D}^n$.

Let X be a Banach space. Recall that the essential norm of a continuous linear operator $T : X \to X$ is the distance from T to the compact operators, that is,

$$\|T\|_e = \inf\{\|T - K\|; K : X \longrightarrow X \text{ is compact}\}. \tag{1.2}$$

Notice that $\|T\|_e = 0$ if and only if T is compact, so that estimates on $\|T\|_e$ lead to conditions for T to be compact.

In the past few decades, boundedness, compactness, and essential norms of composition and closely related operators between various spaces of holomorphic functions have been studied by many authors (see, e.g., the following papers mostly in the settings of the unit ball and the unit polydisc [1–23] and the references therein). Recently, several papers focused on studying the mapping properties of the difference of two composition operators, that is, of an operator of the form

$$T = C_\varphi - C_\psi. \tag{1.3}$$

One of the first results of this type, in the setting of the Hardy space $H^2(\mathbb{D})$, belongs to Berkson [24]. There, it was shown that if φ is an analytic self-map of the unit disk $\mathbb{D}$ whose radial limit function φ^* satisfies $|\varphi^*(\zeta)| = 1$ for $\zeta \in E \subset \partial\mathbb{D}$, meas $(E) > 0$, then for any analytic self-map ψ of the disk, $\psi \neq \varphi$,

$$\|C_\varphi - C_\psi\| \geq \sqrt{\frac{\text{meas}\,(E)}{2}}, \tag{1.4}$$

where meas denotes the normalized Lebesgue measure on $\partial\mathbb{D}$, which means that C_φ is isolated in the operator norm topology. Some other conditions for isolation in the same setting are obtained in [15].

In [25], MacCluer et al., among other results, characterized the compactness of the difference of two composition operators on $H^\infty(\mathbb{D})$ in terms of the Poincaré distance. In [26], isolated points and essential components of composition operators on $H^\infty(\mathbb{D})$ are studied. In [27, 28], the authors have independently extended the result to $H^\infty(B_n)$ space, where B_n is the unit ball of $\mathbb{C}^n$. In [29], Moorhouse showed that if the pseudohyperbolic distance between the image values φ and ψ converges to zero as $z \to \zeta$ for every point ζ at which φ and ψ have finite angular derivative, then the difference $C_\varphi - C_\psi$ yields a compact operator. Differences of composition operators on the Bloch and the little Bloch space are studied in [30, 31]. Motivated by these results, we give some upper and lower estimates of the essential norm for the difference of composition operators induced by φ and ψ acting on the space $H^\infty(\mathbb{D}^n)$, where φ and ψ are analytic self-maps of $\mathbb{D}^n$. As a consequence, one obtains conditions in terms of the Carthéodory distance on $\mathbb{D}^n$ that characterize those pairs of holomorphic self-maps of the polydisc for which the difference of two composition operators on $H^\infty(\mathbb{D}^n)$ is compact.

2. Notation and background

The pseudohyperbolic distance on the unit disk is defined by

$$\beta(z, w) = \left|\frac{z - w}{1 - \bar{z}w}\right|, \quad z, w \in \mathbb{D}. \tag{2.1}$$

It is easy to see that $0 \leq \beta(z, w) \leq 1$.

Definition 2.1. The Poincaré distance ρ on $\mathbb{D}$ is

$$\rho(z,w) := \tanh^{-1}\beta(z,w) = \frac{1}{2}\ln\frac{1+\beta(z,w)}{1-\beta(z,w)} \tag{2.2}$$

for $z,w \in \mathbb{D}$.

Definition 2.2. The Carathéodory pseudodistance on a domain $G \subset \mathbb{C}^n$ is given by

$$c_G(z,w) := \sup\{\rho(f(z),f(w)) : f \in H(G,\mathbb{D})\} \tag{2.3}$$

for $z,w \in G$, where $H(G,\mathbb{D})$ denotes the class of holomorphic mappings from G to $\mathbb{D}$.

If we put

$$c_G^*(z,w) := \sup\{\beta(f(z),f(w)) : f \in H(G,\mathbb{D})\}, \quad z,w \in G, \tag{2.4}$$

then by the monotonicity of the function $h(x) = \ln((1+x)/(1-x))$ on $[0,1)$ and the inequality $h(x) \geq 2x$, $x \in [0,1)$, we have that

$$c_G = \tanh^{-1}(c_G^*) \geq c_G^*. \tag{2.5}$$

Next, we introduce the following pseudodistance on G:

$$d_G(z,w) := \sup\{|f(z)-f(w)| : f \in H(G,\mathbb{D})\}. \tag{2.6}$$

For the case $G = \mathbb{D}$, it is known that (see [32])

$$d_{\mathbb{D}}(z,w) = \frac{2-2\sqrt{1-\beta(z,w)^2}}{\beta(z,w)}. \tag{2.7}$$

Hence, the Poincaré metric on $\mathbb{D}$ is

$$\rho(z,w) = \tanh^{-1}\beta(z,w) = \ln\frac{2+d_{\mathbb{D}}(z,w)}{2-d_{\mathbb{D}}(z,w)}. \tag{2.8}$$

It is easy to see that for $z,w \in G$,

$$\begin{aligned} d_G(z,w) &= \sup\{|g(f(z))-g(f(w))| : g \in H(\mathbb{D},\mathbb{D}), f \in H(G,\mathbb{D})\} \\ &= \sup_{f\in H(G,\mathbb{D})} d_{\mathbb{D}}(f(z),f(w)). \end{aligned} \tag{2.9}$$

Since the map $t \to \ln((2+t)/(2-t))$ is strictly increasing on $[0,2)$, it follows that

$$\begin{aligned}\ln\frac{2+d_G}{2-d_G} &= \sup_{f\in H(G,\mathbb{D})}\ln\frac{2+d_{\mathbb{D}}(f(z),f(w))}{2-d_{\mathbb{D}}(f(z),f(w))}\\ &= \sup_{f\in H(G,\mathbb{D})}\rho(f(z),f(w))\\ &= c_G(z,w),\end{aligned} \tag{2.10}$$

or equivalently for any domain G and any $z,w\in G$,

$$\begin{aligned}d_G(z,w) &= \frac{2-2\sqrt{1-(\tanh c_G(z,w))^2}}{\tanh c_G(z,w)}\\ &= \frac{2-2\sqrt{1-(c_G^*(z,w))^2}}{c_G^*(z,w)}.\end{aligned} \tag{2.11}$$

It is well known that $c^*_{\mathbb{D}^n}(z,w)=\max_{1\le j\le n}\beta(z_j,w_j)$ (see [33, Corollary 2.2.4]). So we have

$$d_{\mathbb{D}^n}(z,w)=\frac{2-2\sqrt{1-(\max_{1\le j\le n}\beta(z_j,w_j))^2}}{\max_{1\le j\le n}\beta(z_j,w_j)}. \tag{2.12}$$

Before formulating and proving the main theorem, we give some notations. For any $\delta\in(0,1)$, define

$$E_\delta^j := \{z\in\mathbb{D}^n : |\varphi_j(z)| \vee |\psi_j(z)| > 1-\delta\}, \tag{2.13}$$

and we put $E_\delta=\cup_{j=1}^n E_\delta^j$, where $\vee$ means the maximum of two real numbers.

Lemma 2.3 (see [34]). *Let $\{z_n\}$ be a sequence in $\mathbb{D}$ with $|z_n|\to 1$ as $n\to\infty$. Then, there is a subsequence $\{z_{n_j}\}$ of $\{z_n\}$, a positive number M, and a sequence of functions $f_m\in H^\infty(\mathbb{D})$ such that*

(i) $f_m(z_{n_j})=\delta_m^j$,

(ii) $\sum_m|f_m(z)|\le M<\infty$ *for any* $z\in\mathbb{D}$,

(the symbol δ_m^j is equal to 1 if $m=j$ and 0, otherwise.)

Lemma 2.4. *Let Ω be a domain in $\mathbb{C}^n$, $f\in H(\Omega)$. If a compact set K and a neighborhood G of K satisfy $K\subset G\subset\subset\Omega$ (i.e., G is relative compact in Ω) and $\eta=\operatorname{dist}(K,\partial G)>0$, then*

$$\sup_{z\in K}\left|\frac{\partial f}{\partial z_j}(z)\right|\le\frac{\sqrt{n}}{\eta}\sup_{z\in G}|f(z)|, \tag{2.14}$$

for each $j\in\{1,\dots,n\}$.

Proof. Since $\eta = \operatorname{dist}(K, \partial G) > 0$ for any $a \in K$, the polydisc

$$P_a = \left\{ (z_1, \dots, z_n) \in \mathbb{C}^n : \left| z_j - a_j \right| < \frac{\eta}{\sqrt{n}},\ j = 1, \dots, n \right\} \tag{2.15}$$

is contained in G. Using Cauchy's inequality, we have

$$\left| \frac{\partial f}{\partial z_j}(a) \right| \le \frac{\sqrt{n}}{\eta} \sup_{z \in \partial_0 P_a} \left| f(z) \right| \le \frac{\sqrt{n}}{\eta} \sup_{z \in G} \left| f(z) \right|, \tag{2.16}$$

as desired (where $\partial_0 P_a$ is the distinguished boundary of P_a). □

Lemma 2.5. *For fixed* $0 < \delta < 1$, *let* $F_\delta = \{ z \in \mathbb{D}^n : \max_{1 \le j \le n} |z_j| > 1 - \delta \}$. *Then,*

$$\lim_{r \to 1} \sup_{\|f\|_\infty = 1} \sup_{z \in F_\delta^c} \left| f(z) - f(rz) \right| = 0 \tag{2.17}$$

for any f *in the unit ball of* $H^\infty(\mathbb{D}^n)$ *(where* F_δ^c *denotes the complement of* F_δ *relative to* $\mathbb{D}^n$*).*

Proof. We have

$$\begin{aligned} &\sup_{z \in F_\delta^c} \left| f(z) - f(rz) \right| \\ &\quad = \sup_{z \in F_\delta^c} \left| \sum_{j=1}^{n} \left(f(rz_1, rz_2, \dots, rz_{j-1}, z_j, \dots, z_n) - f(z_1, rz_2, \dots, rz_j, z_{j+1}, \dots, z_n) \right) \right| \\ &\quad \le \sup_{z \in F_\delta^c} \sum_{j=1}^{n} \int_r^1 |z_j| \left| \frac{\partial f}{\partial z_j}(rz_1, \dots, rz_{j-1}, tz_j, z_{j+1}, \dots, z_n) \right| dt \\ &\quad \le (1 - r) \sup_{z \in F_\delta^c} \sum_{j=1}^{n} \left| \frac{\partial f}{\partial z_j}(z) \right|. \end{aligned} \tag{2.18}$$

Consider $F_{\delta/2}^c$, then $F_\delta^c \subset F_{\delta/2}^c$ and $\operatorname{dist}(F_{\delta/2}^c, \partial \mathbb{D}^n) = \delta/2$.

From Lemma 2.4, we have that for each $j \in \{1, \dots, n\}$

$$\sup_{z \in F_\delta^c} \left| \frac{\partial f}{\partial z_j}(z) \right| \le \frac{2\sqrt{n}}{\delta} \sup_{z \in F_{\delta/2}^c} \left| f(z) \right|. \tag{2.19}$$

From this and (2.18), it follows that

$$\sup_{z \in F_\delta^c} \left| f(z) - f(rz) \right| \le \frac{2(1 - r) n \sqrt{n}}{\delta} \|f\|_\infty. \tag{2.20}$$

Taking the supremum in (2.20) over the unit ball in $H^\infty(\mathbb{D}^n)$, then letting $r \to 1$ in (2.20), the lemma follows. □

3. Main theorem

In this section, we will state our main result and give its proof.

Theorem 3.1. *Let* $\varphi, \psi : \mathbb{D}^n \to \mathbb{D}^n$ *and* $C_\varphi - C_\psi : H^\infty(\mathbb{D}^n) \to H^\infty(\mathbb{D}^n)$. *Then,*

$$\frac{1}{M}\Psi \le \|C_\varphi - C_\psi\|_e \le \frac{4 - 4\sqrt{1-\Psi^2}}{\Psi}, \tag{3.1}$$

where $\Psi := \max_{1\le k\le n}\lim_{\delta\to 0}\sup_{z\in E_\delta^k}\max_{1\le j\le n}\beta(\varphi_j(z),\psi_j(z))$ *and* M *is a positive constant.*

Proof. First, we consider the upper estimate. For fixed $r \in (0,1)$, it is easy to check that both $C_{r\varphi}$ and $C_{r\psi}$ are compact operators. Therefore,

$$\|C_\varphi - C_\psi\|_e \le \|C_\varphi - C_\psi - C_{r\varphi} + C_{r\psi}\|. \tag{3.2}$$

Now, for any $0 < \delta < 1$,

$$\begin{aligned}\|C_\varphi - C_\psi - C_{r\varphi} + C_{r\psi}\| &= \sup_{\|f\|_\infty = 1}\|(C_\varphi - C_\psi - C_{r\varphi} + C_{r\psi})f\|_\infty \\ &= \sup_{\|f\|_\infty=1}\sup_{z\in\mathbb{D}^n}|f(\varphi(z)) - f(\psi(z)) - f(r\varphi(z)) + f(r\psi(z))| \\ &\le \sup_{\|f\|_\infty=1}\sup_{z\in E_\delta}|f(\varphi(z)) - f(\psi(z)) - f(r\varphi(z)) + f(r\psi(z))| \\ &\quad + \sup_{\|f\|_\infty=1}\sup_{z\in E_\delta^c}|f(\varphi(z)) - f(\psi(z)) - f(r\varphi(z)) + f(r\psi(z))| \\ &= I_1 + I_2.\end{aligned} \tag{3.3}$$

From Lemma 2.5, we can choose r sufficiently close to 1 such that I_2 is sufficiently small.

Applying the Schwarz-Pick lemma on the function $\phi(z) = rz$, $r \in (0,1)$, and by the monotony of the function $f(x) = (2 - 2\sqrt{1-x^2})/x$, we obtain

$$\begin{aligned}I_1 &\le \sup_{\|f\|_\infty=1}\sup_{z\in E_\delta}\big(|f(\varphi(z)) - f(\psi(z))| + |-f(r\varphi(z)) + f(r\psi(z))|\big) \\ &= \sup_{z\in E_\delta}\sup_{\|f\|_\infty=1}\big(|f(\varphi(z)) - f(\psi(z))| + |-f(r\varphi(z)) + f(r\psi(z))|\big) \\ &\le \sup_{z\in E_\delta}\big(d_{\mathbb{D}^n}(\varphi(z),\psi(z)) + d_{\mathbb{D}^n}(r\varphi(z), r\psi(z))\big) \\ &\le 2\sup_{z\in E_\delta}\frac{2 - 2\big(1 - \max_{1\le j\le n}(\beta(\varphi_j(z),\psi_j(z)))^2\big)^{1/2}}{\max_{1\le j\le n}\beta(\varphi_j(z),\psi_j(z))} \\ &= \frac{4 - 4\big(1 - \sup_{z\in E_\delta}\max_{1\le j\le n}(\beta(\varphi_j(z),\psi_j(z)))^2\big)^{1/2}}{\sup_{z\in E_\delta}\max_{1\le j\le n}\beta(\varphi_j(z),\psi_j(z))}\end{aligned} \tag{3.4}$$

$$\le \frac{4 - 4\big(1 - \max_{1\le k\le n}\sup_{z\in E_\delta^k}\max_{1\le j\le n}(\beta(\varphi_j(z),\psi_j(z)))^2\big)^{1/2}}{\max_{1\le k\le n}\sup_{z\in E_\delta^k}\max_{1\le j\le n}\beta(\varphi_j(z),\psi_j(z))}. \tag{3.5}$$

By direct calculation, it is easy to check that

$$\lim_{\delta\to 0}\max_{1\le k\le n}\sup_{z\in E_\delta^k}\max_{1\le j\le n}\beta(\varphi_j(z),\psi_j(z)) = \max_{1\le k\le n}\lim_{\delta\to 0}\sup_{z\in E_\delta^k}\max_{1\le j\le n}\beta(\varphi_j(z),\psi_j(z)). \tag{3.6}$$

From which, and letting $\delta \to 0$ in (3.5), the upper estimate in (3.1) follows.

Now, we turn to the lower estimate.

Let

$$a_k = \lim_{\delta\to 0}\sup_{z\in E_\delta^k}\max_{1\le j\le n}\beta\big(\varphi_j(z),\psi_j(z)\big), \quad k = 1,\dots,n. \tag{3.7}$$

If we set $\delta_m = 1/m$, then $\delta_m \to 0$ as $m \to \infty$, and there exists $z_m \in E_{\delta_m}^j$ and some j such that

$$\lim_{m\to\infty}\beta\big(\varphi_j(z_m),\psi_j(z_m)\big) = a_k. \tag{3.8}$$

Since $z_m \in E_{\delta_m}^j$ and $\delta_m \to 0$, we have $|\varphi_j(z_m)| \to 1$ or $|\phi_j(z_m)| \to 1$. Without loss of generality, we can assume $|\varphi_j(z_m)| \to 1$. Let $w_m = \varphi_j(z_m)$, by Lemma 2.3, we have that there is a subsequence of w_m (we may denote it again by w_m), a positive number M, and a sequence of functions $f_m \in H^\infty(\mathbb{D})$ such that

(i) $f_m(w_k) = \delta_m^k$,

(ii) $\sum_m |f_m(z)| \le M < \infty$ for any $z \in \mathbb{D}$.

Now, for any $z \in \mathbb{D}^n$, we define $\widetilde{f}_m(z) := f_m(z^j)$, where z^j is the jth component of z, then $\sum_m |\widetilde{f}_m(z)| \le M < \infty$.

Next we claim that $\widetilde{f}_m$ converge weakly to 0. Let $\lambda \in H^\infty(\mathbb{D}^n)^*$. For any natural N, there exist some unimodular sequences α_m such that

$$\begin{aligned}\sum_{m=0}^N\big|\lambda(\widetilde{f}_m)\big| &= \sum_{m=0}^N\alpha_m\lambda(\widetilde{f}_m)\\ &= \lambda\left(\sum_{m=0}^N\alpha_m\widetilde{f}_m\right)\\ &\le \|\lambda\|\left\|\sum_{m=0}^N\alpha_m\widetilde{f}_m\right\|_\infty\\ &\le \|\lambda\|M.\end{aligned} \tag{3.9}$$

Thus, $\lambda(\widetilde{f}_m) \to 0$ as $m \to \infty$, that is, $\widetilde{f}_m$ converge weakly to 0.

Set

$$g_m(z) = \frac{\widetilde{f}_m(z)}{M}\frac{z^j - \psi_j(z_m)}{1 - \overline{\psi_j(z_m)}z^j}, \quad m \in \mathbb{N}. \tag{3.10}$$

Then, $\|g_m\|_\infty \leq 1$ and similarly to $\widetilde{f}_m$, it is easy to see that g_m converge weakly to 0. Thus, for any compact operator K, we have $\|Kg_m\|_\infty \to 0$ as $m \to \infty$.

Now, we have

$$\begin{aligned}
J &= \left\|C_\varphi - C_\psi - K\right\| \\
&\geq \limsup_{m\to\infty}\left\|(C_\varphi - C_\psi - K)g_m\right\|_\infty \\
&\geq \limsup_{m\to\infty}\left(\left\|(C_\varphi - C_\psi)g_m\right\|_\infty - \left\|Kg_m\right\|_\infty\right) \\
&= \limsup_{m\to\infty}\sup_{z\in\mathbb{D}^n}\left|g_m(\varphi(z)) - g_m(\psi(z))\right| \\
&\geq \frac{1}{M}\limsup_{m\to\infty}\sup_{z\in\mathbb{D}^n}\left|\frac{\varphi_j(z)-\psi_j(z_m)}{1-\overline{\psi_j(z_m)}\varphi_j(z)}\widetilde{f}_m(\varphi(z)) - \frac{\psi_j(z)-\psi_j(z_m)}{1-\overline{\psi_j(z_m)}\psi_j(z)}\widetilde{f}_m(\psi(z))\right| \\
&\geq \frac{1}{M}\limsup_{m\to\infty}\frac{\left|\varphi_j(z_m)-\psi_j(z_m)\right|}{\left|1-\overline{\psi_j(z_m)}\varphi_j(z_m)\right|} \\
&= \frac{1}{M}\lim_{m\to\infty}\beta(\varphi_j(z_m),\psi_j(z_m)) \\
&= \frac{1}{M}a_k.
\end{aligned} \tag{3.11}$$

Then, we have

$$\left\|C_\varphi - C_\psi\right\|_e \geq \frac{1}{M}\max_{1\leq k\leq n}\lim_{\delta\to 0}\sup_{z\in E_\delta^k}\max_{1\leq j\leq n}\beta\left(\varphi_j(z),\psi_j(z)\right), \tag{3.12}$$

finishing the proof of the theorem. □

Corollary 3.2. *The operator $C_\varphi - C_\psi$ is compact if and only if*

$$\max_{1\leq k\leq n}\lim_{\delta\to 0}\sup_{z\in E_\delta^k}\max_{1\leq j\leq n}\beta\left(\varphi_j(z),\psi_j(z)\right) = 0. \tag{3.13}$$

Proof. By using the inequality $(1-\sqrt{1-x^2})/x \leq x$ $(0 < x \leq 1)$ and the fact that T is compact if and only if $\|T\|_e = 0$, the corollary follows by Theorem 3.1. □

Example 3.3. Let $n = 2$, $\varphi(z) = (z_1, (1/2)z_2)$, and $\psi(z) = (z_1, (1/3)z_2)$. Then, $\beta(\varphi_1(z), \psi_1(z)) = 0$ and $\beta(\varphi_2(z), \psi_2(z)) = (1/6)(|z_2|/(1-(1/6)|z_2|^2))$. A direct calculation shows that

$$\max_{1\leq k\leq 2}\lim_{\delta\to 0}\sup_{z\in E_\delta^k}\max_{1\leq j\leq 2}\beta\left(\varphi_j(z),\psi_j(z)\right) = \frac{1}{5} > 0; \tag{3.14}$$

so by Corollary 3.2, $C_\varphi - C_\psi$ is not compact.

Example 3.4. Let $n = 2$, $p > 1$, $0 < c \leq 1$, $\varphi(z) = ((z_1 + 1)/2, (1/2)z_2)$, and

$$\psi(z) = \left(\frac{z_1 + 1}{2} + c\left(\frac{1 - z_1}{2}\right)^p, \frac{1}{2}z_2\right), \tag{3.15}$$

where we choose the usual branch of the logarithm of w_1, $\operatorname{Re} w_1 > 0$, in order to define $((1 - z_1)/2)^p$. By [35], ψ is a self-map of $\mathbb{D}^2$, whenever c is small. Moreover, $\max_{1\leq j\leq 2}\beta(\varphi_j(z), \psi_j(z)) = \beta(\varphi_1(z), \psi_1(z))$. By Corollary 3.2 and the proof of Example 1 of [25], we have, for these c,

(1) if $1 < p \leq 2$, then $C_\varphi - C_\psi$ is noncompact;

(2) if $2 < p < \infty$, then $C_\varphi - C_\psi$ is compact.

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