Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2010, Article ID 134392, 8 pages
doi:10.1155/2010/134392

Research Article

Asymptotic Behaviors of Intermediate Points in the Remainder of the Euler-Maclaurin Formula

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Received 23 June 2010; Revised 22 October 2010; Accepted 5 December 2010

Academic Editor: Dumitru Baleanu

The Euler-Maclaurin formula is a very useful tool in calculus and numerical analysis. This paper is devoted to asymptotic expansion of the intermediate points in the remainder of the generalized Euler-Maclaurin formula when the length of the integral interval tends to be zero. In the special case we also obtain asymptotic behavior of the intermediate point in the remainder of the composite trapezoidal rule.

1. Introduction

It is well known that the Euler-Maclaurin formula is a formula used in the numerical evaluation of integral, which states that the value of an integral is equal to the sum of the value given by the trapezoidal rule and a series of terms involving the odd-numbered derivatives of the function at the end points of the integral interval. Specifically, for the function $f \in C^{2m+2}[a,b]$ the Euler-Maclaurin formula can be expressed as follows

$$\int_a^b f(x)dx = \frac{1}{2}\left(f(a)+f(b)\right) - \sum_{i=1}^{m} \frac{\widehat{B}_{2i}(b-a)^{2i}}{(2i)!}\left(f^{(2i-1)}(b) - f^{(2i-1)}(a)\right) + R_{2m+2}, \tag{1.1}$$

where

$$R_{2m+2} = -\frac{\widehat{B}_{2m+2} f^{(2m+2)}(\xi)}{(2m+2)!}(b-a)^{2m+3}, \tag{1.2}$$

and ξ is some point between a and b. The constants $\widehat{B}_i$ are known as Bernoulli numbers, which are defined by the equation

$$\frac{x}{e^x-1}=\sum_{k=0}^{\infty}\frac{\widehat{B}_k}{k!}x^k. \tag{1.3}$$

The first few of the Bernoulli numbers are $\widehat{B}_0=1$, $\widehat{B}_1=-1/2$, $\widehat{B}_2=1/6$, and $\widehat{B}_{2i-1}=0$ for all $i\geq 2$.

The Euler-Maclaurin formula was discovered independently by Leonhard Euler and Colin Maclaurin, and it has wide applications in calculus and numerical analysis. For example, the Euler-Maclaurin formula is often used to evaluate finite sums and infinite series when a and b are integers. Conversely, it is also used to approximate integrals by finite sums. Therefore, the Euler-Maclaurin formula provides the correspondence between sums and integrals. Besides, the Euler-Maclaurin formula may be used to derive a wide range of quadrature formulas including the Newton-Cotes formulas, and used for detailed error analysis in numerical quadrature.

The Euler-Maclaurin formula has many generalizations and extensions [1–6]. A direct generalization of the Euler-Maclaurin formula in the interval $[a,a+h]$ can be described as

$$\begin{aligned}\int_a^{a+h}f(x)dx&=\frac{h}{2n}(f(a)+f(a+h))+\frac{h}{n}\sum_{k=1}^{n-1}f\left(a+\frac{k}{n}h\right)\\&\quad-\sum_{i=1}^{m}\frac{\widehat{B}_{2i}}{(2i)!}\left(\frac{h}{n}\right)^{2i}\left(f^{(2i-1)}(a+h)-f^{(2i-1)}(a)\right)-\frac{n\widehat{B}_{2m+2}f^{(2m+2)}(\xi)}{(2m+2)!}\left(\frac{h}{n}\right)^{2m+3},\end{aligned} \tag{1.4}$$

where $\xi\in(a,a+h)$, n is a positive integer and m is a nonnegative integer. Obviously, when $n=1$, (1.4) reduces to (1.1). We also conclude that this equation has algebraic accuracy of $2m+1$ which is the same as (1.1).

Recently, some interests have been focused on the study of the mean value theorem for integrals and differentiations [7–14]. The aim of the present paper is to deal with asymptotic expansions of the intermediate points in the generalized Euler-Maclaurin formula when the length of the integral interval tends to be zero. The rest of this paper is organized as follows. In the second section, the Bell polynomials as a standard mathematical are introduced in detail. In the third section, we give asymptotic behavior of the intermediate points in the remainder of the generalized Euler-Maclaurin formula (1.4). As a special case, asymptotic behavior of the intermediate points in remainder of the composite trapezoidal formula is also presented.

2. Bell Polynomials

The Bell polynomials [15] extensively studied by Bell arise in combinatorial analysis, and they have been applied in many different frameworks. The exponential partial Bell polynomials are the polynomials

$$B_{n,k}=B_{n,k}(x_1,x_2,\ldots,x_{n-k+1}) \tag{2.1}$$

in an infinite number of variables $x_1, x_2, \ldots$, defined by the series expansion

$$\frac{1}{k!}\left(\sum_{m\geq 1} x_m \frac{t^m}{m!}\right)^k = \sum_{n\geq k} B_{n,k}\frac{t^n}{n!}, \quad k = 0, 1, 2, \ldots. \tag{2.2}$$

Their explicit expressions are given by the formula

$$B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) = \sum \frac{n!}{a_1!(1!)^{a_1} a_2!(2!)^{a_2}\cdots} {x_1}^{a_1} {x_2}^{a_2}\ldots, \tag{2.3}$$

where the summation takes place over all nonnegative integers $a_1, a_2, \ldots$, such that $a_1 + 2a_2 + \cdots = n$ and $a_1 + a_2 + \cdots = k$. For example, we have

$$\begin{gathered} B_{0,0} = 1, \qquad B_{1,1} = x_1, \qquad B_{2,1} = x_2, \qquad B_{2,2} = {x_1}^2, \qquad B_{3,1} = x_3, \\ B_{3,2} = 3x_1x_2, \qquad B_{3,3} = {x_1}^3, \ldots, \quad B_{n,1} = x_n, \qquad B_{n,n} = {x_1}^n. \end{gathered} \tag{2.4}$$

For more important properties the reader is referred to [15].

3. Asymptotic Expansions of Intermediate Points

In this section, we will consider asymptotic behavior of the point ξ in (1.4). Before the main result is given we first present an essential lemma.

Lemma 3.1 (see [15]). *The following identity:*

$$\sum_{k=1}^{n} k^r = \frac{1}{r+1}\sum_{k=0}^{r} \widehat{B}_k \binom{r+1}{k} (n+1)^{r+1-k}, \tag{3.1}$$

holds, where r is a positive integer.

This lemma gives relations between the sum of powers of the first n integers and the Bernoulli numbers. Now, we turn to the asymptotic behavior of the point ξ in (1.4). Namely, the following theorem is our main result.

Theorem 3.2. *Let p, q be integers and $p \geq 1$, $q \geq 0$. Assume that f is a function admitting in a neighborhood of the point a of $\mathbb{R}$ a derivative of order $2m+2+p+q$ such that $f^{(2m+2+p+q)}$ is continuous at a. If $f^{(2m+3)}(a) = \cdots = f^{(2m+1+p)}(a) = 0$, and $f^{(2m+2+p)}(a) \neq 0$, then*

$$\xi = a + \sum_{i=1}^{q+1} c_i h^i + o\left(h^{q+1}\right) \quad (h \longrightarrow 0). \tag{3.2}$$

The coefficients c_i are given by the recurrence formula

$$c_1 = \left(\frac{(2m+2)!p!}{(2m+3+p)!B_{2m+2}} \sum_{j=0}^{p} \binom{2m+3+p}{2m+2+j} \frac{\widehat{B}_{2m+2+j}}{n^j} \right)^{1/p}, \qquad c_{i+1} = R_i(c_1, \ldots, c_i) \quad (i = 1, \ldots, q) \tag{3.3}$$

with

$$\begin{aligned} R_i(c_1, \ldots, c_i) &\\ = \frac{1}{t_0^{1/p}} &\left[\frac{s_0^{1/p}}{i!} \sum_{k=0}^{i} \binom{1/p}{k} k! B_{i,k}(\phi_1, \phi_2, \ldots, \phi_{i+1-k}) \right. \\ &\left. - \frac{t_0^{1/p}}{(i+1)!} \sum_{k=2}^{i+1} k B_{i+1,k}(1!c_1, 2!c_2, \ldots, (i+2-k)!c_{i+2-k}) \sum_{j=0}^{k-1} \binom{\frac{1}{p}}{j} j! B_{k-1,j}(\psi_1, \psi_2, \ldots, \psi_{k-j}) \right], \end{aligned} \tag{3.4}$$

where

$$s_k = \frac{f^{(2m+2+p+k)}(a)}{(2m+3+p+k)!} \sum_{j=0}^{p+k} \binom{2m+3+p+k}{2m+2+j} \frac{\widehat{B}_{2m+2+j}}{n^j}, \qquad t_k = \frac{\widehat{B}_{2m+2}}{(2m+2)!} \frac{f^{(2m+2+p+k)}(a)}{(p+k)!}, \qquad \phi_k = \frac{s_k}{s_0}, \quad \psi_k = \frac{t_k}{t_0}, \quad k = 0, 1, \ldots, q. \tag{3.5}$$

Proof. For convenience, we let

$$A = \int_a^{a+h} f(x)dx, \qquad C = \frac{n\widehat{B}_{2m+2} f^{(2m+2)}(\xi)}{(2m+2)!} \left(\frac{h}{n} \right)^{2m+3},$$
$$B = \frac{h}{2n}(f(a) + f(a+h)) + \frac{h}{n} \sum_{k=1}^{n-1} f\left(a + \frac{k}{n}h \right) - \sum_{i=1}^{m} \frac{\widehat{B}_{2i}}{(2i)!} \left(\frac{h}{n} \right)^{2i} \left(f^{(2i-1)}(a+h) - f^{(2i-1)}(a) \right). \tag{3.6}$$

According to (1.4), we have $A = B - C$. We first consider A. Using the Taylor expansion, we have

$$A = \sum_{k=0}^{2m+2+p+q} \frac{f^{(k)}(a)}{(k+1)!} h^{k+1} + o\left(h^{2m+3+p+q} \right), \quad h \longrightarrow 0. \tag{3.7}$$

When $h \to 0$, using the Taylor expansion again we have

$$B = hf(a) + \frac{1}{2n}\sum_{k=1}^{2m+2+p+q}\frac{f^{(k)}(a)}{k!}h^{k+1} + \frac{h}{n}\sum_{k=1}^{n-1}\sum_{j=1}^{2m+2+p+q}\frac{f^{(j)}(a)}{j!}\left(\frac{k}{n}h\right)^{j}$$
$$-\sum_{k=1}^{m}\frac{\widehat{B}_{2k}}{(2k)!}\left(\frac{h}{n}\right)^{2k}\sum_{j=2k}^{2m+2+p+q}\frac{f^{(j)}(a)}{(j+1-2k)!}h^{j+1} + o\left(h^{2m+3+p+q}\right)$$
$$= hf(a) + \frac{f'(a)}{2!}h^2 + \frac{1}{2n}\sum_{k=2}^{2m+2+p+q}\frac{f^{(k)}(a)}{k!}h^{k+1} + \sum_{k=2}^{2m+2+p+q}\frac{f^{(k)}(a)}{k!}\left(\frac{h}{n}\right)^{k+1}\sum_{j=1}^{n-1}j^{k}$$
$$-\sum_{k=2}^{2m+1}\frac{f^{(k)}(a)}{(k+1)!}h^{k+1}\sum_{j=1}^{\lfloor k/2\rfloor}\binom{k+1}{2j}\frac{\widehat{B}_{2j}}{n^{2j}} - \sum_{k=2m+2}^{2m+2+p+q}\frac{f^{(k)}(a)}{(k+1)!}h^{k+1}\sum_{j=1}^{m}\binom{k+1}{2j}\frac{\widehat{B}_{2j}}{n^{2j}} + o\left(h^{2m+3+p+q}\right), \tag{3.8}$$

where $\lfloor k/2\rfloor$ denotes the largest integer that is not greater than $k/2$. Since $\widehat{B}_0 = 1$, $\widehat{B}_1 = -1/2$, and $\widehat{B}_{2i-1} = 0$ for $i \geq 2$, by Lemma 3.1 there holds

$$B = hf(a) + \frac{f'(a)}{2!}h^2 + \frac{1}{2n}\sum_{k=2}^{2m+2+p+q}\frac{f^{(k)}(a)}{k!}h^{k+1} + \sum_{k=2}^{2m+2+p+q}\frac{f^{(k)}(a)}{(k+1)!}h^{k+1}\sum_{j=0}^{k}\binom{k+1}{j}\frac{\widehat{B}_j}{n^j}$$
$$-\sum_{k=2}^{2m+1}\frac{f^{(k)}(a)}{(k+1)!}h^{k+1}\sum_{j=1}^{\lfloor k/2\rfloor}\binom{k+1}{2j}\frac{\widehat{B}_{2j}}{n^{2j}} - \sum_{k=2m+2}^{2m+2+p+q}\frac{f^{(k)}(a)}{(k+1)!}h^{k+1}\sum_{j=1}^{m}\binom{k+1}{2j}\frac{\widehat{B}_{2j}}{n^{2j}} + o\left(h^{2m+3+p+q}\right)$$
$$= \sum_{k=0}^{2m+1}\frac{f^{(k)}(a)}{(k+1)!}h^{k+1} + \sum_{k=2m+2}^{2m+2+p+q}\frac{f^{(k)}(a)}{(k+1)!}h^{k+1}\left(1 + \sum_{j=2m+2}^{k}\binom{k+1}{j}\frac{\widehat{B}_j}{n^j}\right) + o\left(h^{2m+3+p+q}\right), \tag{3.9}$$

as $h \to 0$. Because $f^{(2m+3)}(a) = \cdots = f^{(2m+1+p)}(a) = 0$, we obtain

$$B - A = \frac{\widehat{B}_{2m+2}f^{(2m+2)}(a)}{(2m+2)!n^{2m+2}}h^{2m+3} + \sum_{k=2m+2+p}^{2m+2+p+q}\frac{f^{(k)}(a)}{(k+1)!}h^{k+1}\sum_{j=2m+2}^{k}\binom{k+1}{j}\frac{\widehat{B}_j}{n^j} + o\left(h^{2m+3+p+q}\right), \tag{3.10}$$

as $h \to 0$. By the Taylor expansion, it follows that

$$C = \frac{\widehat{B}_{2m+2}h^{2m+3}}{(2m+2)!n^{2m+2}}\left[f^{(2m+2)}(a) + \sum_{k=p}^{p+q}\frac{f^{(2m+2+k)}(a)}{k!}(\xi - a)^k + o(h^{p+q})\right], \quad h \longrightarrow 0. \tag{3.11}$$

Since $B - A = C$ and $0 < \xi - a < h$, we have

$$\left(\frac{\xi - a}{h}\right)^p \frac{\widehat{B}_{2m+2}}{(2m+2)!} \sum_{k=0}^{q} \frac{f^{(2m+2+p+k)}(a)}{(p+k)!} (\xi - a)^k$$
$$= \sum_{k=0}^{q} \frac{f^{(2m+2+p+k)}(a)}{(2m+3+p+k)!} h^k \sum_{j=0}^{p+k} \binom{2m+3+p+k}{2m+2+j} \frac{\widehat{B}_{2m+2+j}}{n^j} + o(h^q), \quad h \longrightarrow 0. \tag{3.12}$$

Let

$$s_k = \frac{f^{(2m+2+p+k)}(a)}{(2m+3+p+k)!} \sum_{j=0}^{p+k} \binom{2m+3+p+k}{2m+2+j} \frac{\widehat{B}_{2m+2+j}}{n^j}, \qquad t_k = \frac{\widehat{B}_{2m+2}}{(2m+2)!} \frac{f^{(2m+2+p+k)}(a)}{(p+k)!},$$
$$\phi_k = \frac{s_k}{s_0}, \qquad \psi_k = \frac{t_k}{t_0}, \quad k = 0, 1, \ldots, q. \tag{3.13}$$

Then (3.12) can be rewritten as

$$\left(\frac{\xi - a}{h}\right)^p t_0 \sum_{k=0}^{q} \psi_k (\xi - a)^k = s_0 \sum_{k=0}^{q} \phi_k h^k + o(h^q), \quad h \longrightarrow 0. \tag{3.14}$$

The rest proof is similar to that in [7, 8], and we omit it. □

Putting $m = 0$ in (1.4) we derive the composite trapezoidal rule as follows

$$\int_a^{a+h} f(x)dx = \frac{h}{2n}\bigl(f(a) + f(a+h)\bigr) + \frac{h}{n}\sum_{k=1}^{n-1} f\left(a + \frac{k}{n}h\right) - \frac{f''(\xi)}{12n^2}h^3, \tag{3.15}$$

where $\xi \in (a, a+h)$. When $h \to 0$, in view of Theorem 3.2 we obtain asymptotic behavior of the intermediate point ξ in above composite trapezoidal rule.

Corollary 3.3. *Let p, q be integers and $p \geq 1$, $q \geq 0$. Assume that f is a function admitting in a neighborhood of the point a of $\mathbb{R}$ a derivative of order $p + q + 2$ and that $f^{(p+q+2)}$ is continuous at a. If $f'''(a) = \cdots = f^{(p+1)}(a) = 0$, and $f^{(p+2)}(a) \neq 0$, then*

$$\xi = a + \sum_{i=1}^{q+1} c_i h^i + o\left(h^{q+1}\right) \quad (h \longrightarrow 0). \tag{3.16}$$

The coefficients c_i are given by the recurrence formula

$$c_1 = \left(\frac{1}{3} \frac{p!}{(p+3)!} \sum_{j=0}^{p} \binom{p+3}{j+2} \frac{\widehat{B}_{j+2}}{n^j} \right)^{1/p},$$
$$c_{i+1} = R_i(c_1, \ldots, c_i) \qquad (i = 1, \ldots, q) \tag{3.17}$$

with

$$\begin{aligned} &R_i(c_1, \ldots, c_i) \\ &= \frac{1}{t_0{}^{1/p}} \left[\frac{s_0{}^{1/p}}{i!} \sum_{k=0}^{i} \binom{1/p}{k} k! B_{i,k}(\phi_1, \phi_2, \ldots, \phi_{i+1-k}) - \frac{t_0{}^{1/p}}{(i+1)!} \right. \\ &\qquad \left. \times \sum_{k=2}^{i+1} k B_{i+1,k}(1!c_1, 2!c_2, \ldots, (i+2-k)!c_{i+2-k}) \sum_{j=0}^{k-1} \binom{1/p}{j} j! B_{k-1,j}(\psi_1, \psi_2, \ldots, \psi_{k-j}) \right], \end{aligned} \tag{3.18}$$

where

$$s_k = \frac{f^{(p+k+2)}(a)}{(p+k+3)!} \sum_{j=0}^{p+k} \binom{p+k+3}{j+2} \frac{\widehat{B}_{j+2}}{n^j}, \qquad t_k = \frac{1}{12} \frac{f^{(p+k+2)}(a)}{(p+k)!},$$
$$\phi_k = \frac{s_k}{s_0}, \qquad \psi_k = \frac{t_k}{t_0}, \quad k = 0, 1, \ldots, q. \tag{3.19}$$

Acknowledgment

This work is supported by the Zhejiang Province Natural Science Foundation (Grant no. Y6100021), and the Ningbo Natural Science Foundation (Grant nos. 2010A610099 and 2009A610082).

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