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Research Article

Multiple Positive Solutions for Singular Periodic Boundary Value Problems of Impulsive Differential Equations in Banach Spaces

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By means of the fixed point theory of strict set contraction operators, we establish a new existence theorem on multiple positive solutions to a singular boundary value problem for second-order impulsive differential equations with periodic boundary conditions in a Banach space. Moreover, an application is given to illustrate the main result.

1. Introduction

The theory of impulsive differential equations describes processes that experience a sudden change of their state at certain moments. In recent years, a great deal of work has been done in the study of the existence of solutions for impulsive boundary value problems, by which a number of chemotherapy, population dynamics, optimal control, ecology, industrial robotics, and physics phenomena are described. For the general aspects of impulsive differential equations, we refer the reader to the classical monograph [1]. For some general and recent works on the theory of impulsive differential equations, we refer the reader to [2–14]. Meanwhile, the theory of ordinary differential equations in abstract spaces has become a new important branch (see [15–18]). So it is interesting and important to discuss the existence of positive solutions for impulsive boundary value problem in a Banach space.

Let $(E, \|\cdot\|)$ be a real Banach space, $J = [0, 2\pi]$, $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = 2\pi$, $J_0 = [0, t_1]$, and $J_i = (t_i, t_{i+1}]$, $i = 1, \ldots, m$. Note that $PC[J, E] = \{u : u$ is a map from J into E such that $u(t)$ is continuous at $t \neq t_k$ and left continuous at $t = t_k$ and $u(t_k^+)$ exist, $k = 1, 2, \dots, m\}$, and it is also a Banach space with norm

$$\|u\|_{PC} = \sup_{t\in J}\|u(t)\|. \tag{1.1}$$

Let the Banach space E be partially ordered by a cone P of E; that is, $x \leq y$ if and only if $y - x \in P$, and $PC[J, E]$ is partially ordered by $K = \{u \in PC[J, E] : u(t) \in P, t \in J\} : u \leq v$ if and only if $v - u \in K$; that is, $u(t) \leq v(t)$ for all $t \in J$.

In this paper, we consider the following singular periodic boundary value problem with impulsive effects in Banach E

$$\begin{gathered} -u''(t) + M^2u(t) = f(t, u(t)), \quad t \in J,\ t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k)), \\ \Delta u'\big|_{t=t_k} = -\overline{I}_k(u(t_k)), \quad k = 1, 2, \dots, m, \\ u(0) = u(2\pi), \qquad u'(0) = u'(2\pi), \end{gathered} \tag{1.2}$$

where $M > 0$ is constant, $f(t, u)$ may be singular at $t = 0$ and/or $t = 2\pi$, $f \in C[(0, 2\pi) \times P, P]$, $I_k, \overline{I}_k \in C[P, P]$, $\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-)$, $\Delta u'|_{t=t_k} = u'(t_k^+) - u'(t_k^-)$, $k = 1, 2, \dots, m$, and $u^i(t_k^+)$ (resp., $u^i(t_k^-)$) denote the right limit (resp., left limit) of $u^i(t)$ at $t = t_k$, $i = 0, 1$.

In the special case where $E = \mathbb{R}^+ = [0, +\infty)$, and $I_k = \overline{I}_k = 0$, $k = 1, 2, \dots, m$, problem (1.2) is reduced to the usual second-order periodic boundary value problem. For example, in [19], the periodic boundary value problem:

$$\begin{gathered} -u''(t) + Mu(t) = f(t, u(t)), \quad t \in (0, 2\pi), \\ u(0) = u(2\pi), \qquad u'(0) = u'(2\pi), \end{gathered} \tag{1.3}$$

was proved to have at least one positive solution, by Jiang [19] .

In [20], the authors studied the multiplicity of positive solutions for IBVP(1.2) in $E = \mathbb{R}^+$; the main tool is the theory of fixed point index.

In [21], the author considers the following periodic boundary value problem of second-order integrodifferential equations of mixed type in Banach space:

$$\begin{gathered} -u'' = f(t, u, Tu, Su), \quad t \in (0, 2\pi), \\ \Delta u|_{t=t_k} = I_k(u(t_k)), \\ \Delta u'\big|_{t=t_k} = -\overline{I}_k(u(t_k)), \quad k = 1, 2, \dots, m, \\ u(0) = u(2\pi), \qquad u'(0) = u'(2\pi), \end{gathered} \tag{1.4}$$

where $f \in C[J \times E \times E \times E, E]$, $I_k, \overline{I}_k \in C[E, E]$, and the operators T, S are given by

$$Tu(t) = \int_0^t k(t,s)u(s)ds, \qquad Su(t) = \int_0^{2\pi} k_1(t,s)u(s)ds, \tag{1.5}$$

with $k \in C[D, \mathbb{R}]$, $D = \{(t,s) \in \mathbb{R}^2 : 0 \leq s \leq t \leq 2\pi\}$, $k_1 \in C[J \times J, \mathbb{R}]$. By applying the monotone iterative technique and cone theory based on a comparison result, the author obtained an existence theorem of minimal and maximal solutions for the IBVP(1.4).

Motivated by the above facts, our aim is to study the multiplicity of positive solutions for IBVP(1.2) in a Banach space. By means of the fixed point index theory of strict set contraction operators, we establish a new existence theorem on multiple positive solutions for IBVP(1.2). Moreover, an application is given to illustrate the main result.

The rest of this paper is organized as follows. In Section 2, we present some basic lemmas and preliminary facts which will be needed in the sequel. Our main result and its proof are arranged in Section 3. An example is given to show the application of the result in Section 4.

2. Preliminaries

Let $T_r = \{x \in E : \|x\| \leq r\}$, $B_r = \{u \in PC[J,E] : \|u\|_{PC} \leq r\}$ $(r > 0)$; for $D \subset PC[J,E]$, we denote $D(t) = \{u(t) : u \in D\} \subset E(t \in J)$.$\alpha$ denotes the Kuratowski measure of non-compactness.

Let $PC^1[J,E] = \{u \mid u$ be a map from J into E such that $u(t)$ is continuously differentiable at $t \neq t_k$ and left continuous at $t = t_k$ and $u(t_k^+)$, $u'(t_k^-)$, $u'(t_k^+)$ exist, $k = 1, 2, \ldots, m\}$. Evidently, $PC^1[J,E]$ is a Banach space with norm

$$\|u\|_{PC^1} = \max\{\|u\|_{PC}, \|u'\|_{PC}\}. \tag{2.1}$$

Let $J' = J \setminus \{t_1, t_2, \ldots, t_m\}$; a map $u \in PC^1[J,E] \cap C^2[J',E]$ is a solution of IBVP(1.2) if it satisfies (1.2).

Now, we first give the following lemmas in order to prove our main result.

Lemma 2.1 (see [17]). *Let K be a cone in real Banach space E, and let Ω be a nonempty bounded open convex subset of K. Suppose that $A : \overline{\Omega} \to K$ is a strict set contraction and $A(\overline{\Omega}) \subset K$. Then the fixed-point index $i(A, \Omega, K) = 1$.*

Lemma 2.2 (see [21]). *$u \in PC^1[J,E] \cap C^2[J',E]$ is a solution of IBVP* (1.2) *if and only if $u \in PC[J,E]$ is a solution of the impulsive integral equation:*

$$u(t) = \int_0^{2\pi} G(t,s)f(s,u(s))ds + \sum_{k=1}^{m} \left[G(t,t_k)\overline{I}_k(u(t_k)) + H(t,t_k)I_k(u(t_k))\right], \tag{2.2}$$

where

$$G(t,s) = \left(2M\left(e^{2\pi M}-1\right)\right)^{-1}\begin{cases} e^{M(2\pi - t + s)} + e^{M(t-s)}, & 0 \le s \le t \le 2\pi, \\ e^{M(2\pi + t - s)} + e^{M(s-t)}, & 0 \le t \le s \le 2\pi, \end{cases}$$
$$H(t,s) = \left(2\left(e^{2\pi M}-1\right)\right)^{-1}\begin{cases} e^{M(2\pi - t + s)} - e^{M(t-s)}, & 0 \le s \le t \le 2\pi, \\ e^{M(s-t)} - e^{M(2\pi + t - s)}, & 0 \le t < s \le 2\pi. \end{cases} \tag{2.3}$$

By simple calculations, we obtain that for $(t,s) \in J \times J$,

$$l_0 := \frac{e^{\pi M}}{M(e^{2\pi M}-1)} \le G(t,s) \le \frac{e^{2\pi M}+1}{2M(e^{2\pi M}-1)} := l_1, \tag{2.4}$$

$$|H(t,s)| \le \frac{1}{2}, \qquad MG(t,s) + H(t,s) > 0. \tag{2.5}$$

To establish the existence of multiple positive solutions in $PC^1[J,E] \cap C^2[J',E]$ of IBVP(1.2), let us list the following assumptions:

(A1) $\|f(t,x)\| \le g(t)\|h(x)\|$, $t \in (0,2\pi)$, $x \in P$, where $g : (0,2\pi) \to (0,\infty)$ is continuous and

$h : P \to P$ is bounded and continuous and satisfies $\int_0^{2\pi} g(s)ds < +\infty$.

(A2) $h(x)$ in (A1) satisfies

$$cl_1 \int_0^{2\pi} g(s)ds + l_1 \sum_{k=1}^{m} h_k + \frac{1}{2}\sum_{k=1}^{m} c_k < 1,$$
$$dl_1 \int_0^{2\pi} g(s)ds + l_1 \sum_{k=1}^{m} e_k + \frac{1}{2}\sum_{k=1}^{m} d_k < 1, \tag{2.6}$$

where

$$c = \overline{\lim_{\|x\|\to 0}} \frac{\|h(x)\|}{\|x\|}, \qquad d = \overline{\lim_{\|x\|\to +\infty}} \frac{\|h(x)\|}{\|x\|},$$
$$c_k = \overline{\lim_{\|x\|\to 0}} \frac{\|I_k\|}{\|x\|}, \qquad d_k = \overline{\lim_{\|x\|\to +\infty}} \frac{\|I_k\|}{\|x\|}, \tag{2.7}$$
$$h_k = \overline{\lim_{\|x\|\to 0}} \frac{\left\|\bar{I}_k\right\|}{\|x\|}, \qquad e_k = \overline{\lim_{\|x\|\to +\infty}} \frac{\left\|\bar{I}_k\right\|}{\|x\|}.$$

(A3) For any $r > 0$ and $[a, b] \subset (0, 2\pi)$, f is uniformly continuous on $[a, b] \times T_r$.

(A4) There exist $L, L_k, H_k \geq 0$ such that $\alpha(f(t, D)) \leq L\alpha(D)$, $\alpha(I_k(D)) \leq L_k\alpha(D)$, $\alpha(\overline{I}_k(D)) \leq H_k\alpha(D)$ $(k = 1, \ldots, m)$, and $4\pi L l_1 + l_1 \sum_{k=1}^{m} H_k + (1/2) \sum_{k=1}^{m} L_k < 1$, for $t \in (0, 2\pi)$, and $D \subset P$ is bounded.

(A5) For any $x \in P$, $\overline{I}_k(x) \geq M I_k(x)$;

(A6) P is a solid cone, and there exist $u_0 \in \overset{\circ}{P}$, $J_0' = [a', b'] \subset J$ such that $t \in J'_0$, $x \geq u_0$ imply $f(t, x) \geq \overline{h}(t)u_0$, $\overline{h} \in C(J_0', [0, +\infty))$, and $l := l_0 \int_{a'}^{b'} \overline{h}(s)ds > 1$.

Define an operator A as follows:

$$(Au)(t) = \int_0^{2\pi} G(t, s) f(s, u(s))ds + \sum_{k=1}^{m} \left[G(t, t_k)\overline{I}_k(u(t_k)) + H(t, t_k)I_k(u(t_k))\right], \quad t \in J. \tag{2.8}$$

Lemma 2.3. *Assuming (A1) and (A4) hold, then, for any $r > 0$, $A : PC[J, P] \cap B_r \to PC[J, P]$ is bounded and continuous.*

Proof. According to (A1) and (A4), we obtain that A is a bounded operator. In the following, we will show that A is continuous.

Let $\{u_n\}, \{u\} \subset PC[J, P] \cap B_r$, and $\|u_n - u\|_{PC} \to 0$. Next we show that $\|Au_n - Au\|_{PC} \to 0$. By (A1), $\{(Au_n)(t)\}$ is equicontinuous on each J_i $(i = 0, \ldots, m)$. By the Lebesgue dominated convergence theorem and (2.4), we have

$$\begin{aligned}
&\|Au_n(t) - Au(t)\| \\
&\quad \leq \left\| \int_0^{2\pi} G(t, s)\big(f(s, u_n(s)) - f(s, u(s))\big)ds \right\| + \sum_{k=1}^{m} G(t, t_k) \left\| \overline{I}_k(u_n(t_k)) - \overline{I}_k(u(t_k)) \right\| \\
&\qquad + \sum_{k=1}^{m} |H(t, t_k)| \|I_k(u_n(t_k)) - I_k(u(t_k))\| \\
&\quad \leq l_1 \int_0^{2\pi} \|f(s, u_n(s)) - f(s, u(s))\| ds + l_1 \sum_{k=1}^{m} \left\| \overline{I}_k(u_n(t_k)) - \overline{I}_k(u(t_k)) \right\| \\
&\qquad + \frac{1}{2} \sum_{k=1}^{m} \|I_k(u_n(t_k)) - I_k(u(t_k))\| \longrightarrow 0 \quad (n \longrightarrow \infty).
\end{aligned} \tag{2.9}$$

In view of the Ascoli-Arzela theorem, $\{Au_n\}$ is a relatively compact set in $PC[J, E]$. In the following we will verify that $\|Au_n - Au\|_{PC} \to 0$ $(n \to \infty)$.

If this is not true, then there are $\varepsilon_0 > 0$ and $\{u_{ni}\} \subset \{u_n\}$ such that $\|Au_{ni} - Au\|_{PC} \geq \varepsilon_0$ $(i = 1, 2, \ldots)$. Since $\{Au_n\}$ is a relatively compact set, there exists a subsequence of $\{Au_{ni}\}$ which converges to $v \in PC[J, P]$, without loss of generality, and we assume that $\lim_{i \to \infty} Au_{ni} = v$, that is, $\lim_{i \to \infty} \|Au_{ni} - v\|_{PC} = 0$, so $v = Au$, which imply a contradiction. Therefore A is continuous. □

Lemma 2.4. *Assuming (A1), (A3), and (A4) hold, then, for any $R > 0$, $A : PC[J,P] \cap B_R \to PC[J,P]$ is a strict set contraction operator.*

Proof. For any $R > 0$, $S \subset PC[J,P] \cap B_R$, by (A1), AS is bounded and equicontinuous on each J_i, $i = 0,\dots,m$, and by [17],

$$\alpha_{PC}(AS) = \sup_{t\in J} \alpha((AS)(t)), \tag{2.10}$$

where $(AS)(t) = \{Au(t) : u \in S, t \in J\}$.

Let

$$\begin{aligned} D &= \left\{\int_0^{2\pi} G(t,s)f(s,u(s))ds : u \in S\right\}, \\ D_\delta &= \left\{\int_\delta^{2\pi-\delta} G(t,s)f(s,u(s))ds : u \in S\right\}, \quad 0 < \delta < \min\{\pi, t_1, 2\pi - t_m\}. \end{aligned} \tag{2.11}$$

By (A1) and (2.4), for any $u \in S$,

$$\begin{aligned} &\left\|\int_\delta^{2\pi-\delta} G(t,s)f(s,u(s))ds - \int_0^{2\pi} G(t,s)f(s,u(s))ds\right\| \\ &\quad \le l_1 \max_{x\in T_R}\|h(u)\| \int_0^\delta g(s)ds + l_1 \max_{u\in T_R}\|h(x)\| \int_{2\pi-\delta}^{2\pi} g(s)ds. \end{aligned} \tag{2.12}$$

In view of (2.12) and (A1), we have $d_H(D_\delta, D) \to 0$ $(\delta \to 0^+)$, where $d_H(D_\delta, D)$ denotes the Hausdorff distance of D and D_δ.

Therefore,

$$\lim_{\delta\to 0^+} \alpha(D_\delta) = \alpha(D). \tag{2.13}$$

Next we will estimate $\alpha(D_\delta)$. Since

$$\int_\delta^{2\pi-\delta} G(t,s)f(s,u(s))ds \in (2\pi - 2\delta)\overline{\text{co}}(\{G(t,s)f(s,u(s)) : s \in [\delta, 2\pi - \delta]\}), \tag{2.14}$$

thus

$$\begin{aligned} \alpha(D_\delta) &= \alpha\left(\left\{\int_\delta^{2\pi-\delta} G(t,s)f(s,u(s))ds : u \in S\right\}\right) \\ &\le 2(\pi - \delta)\alpha(\overline{\text{co}}\{G(t,s)f(s,u(s)) : s \in [\delta, 2\pi - \delta], u \in S\}) \\ &\le 2\pi\alpha(\{G(t,s)f(s,u(s)) : s \in [\delta, 2\pi - \delta], u \in S\}) \\ &\le 2\pi l_1 \alpha(f(I_\delta \times S(I_\delta))), \end{aligned} \tag{2.15}$$

where $I_\delta = [\delta, 2\pi - \delta]$, $S(I_\delta) = \{u(t) : t \in I_\delta, u \in S\}$.

By (A3) and (A4), it is not difficult to prove that

$$\alpha\big(f(I_\delta \times S(I_\delta))\big) = \max_{t\in I_\delta} \alpha\big(f(t, S(I_\delta))\big) \le L\alpha(S(I_\delta)) \le L\alpha(S(J)). \tag{2.16}$$

By [17], we have

$$L\alpha(S(J)) \le 2L\alpha_{PC}(S). \tag{2.17}$$

Let $\delta \to 0^+$, and making use of the fact that $\lim_{\delta\to 0^+}\alpha(D_\delta) = \alpha(D)$, we obtain

$$\alpha(D) \le 2\pi l_1 2L\alpha_{PC}(S) = 4\pi L l_1 \alpha_{PC}(S). \tag{2.18}$$

It is clear that

$$\alpha\left(\left\{\sum_{k=1}^{m} G(t,t_k)\overline{I}_k(u(t_k)) : u \in S\right\}\right) \le \left(l_1 \sum_{k=1}^{m} H_k\right)\alpha_{PC}(S), \tag{2.19}$$

$$\alpha\left(\left\{\sum_{k=1}^{m} H(t,t_k)I_k(u(t_k)) : u \in S\right\}\right) \le \left(\frac{1}{2}\sum_{k=1}^{m} L_k\right)\alpha_{PC}(S). \tag{2.20}$$

Hence, according to (2.18)–(2.20), we have

$$\alpha_{PC}(AS) \le \left(4\pi L l_1 + l_1\sum_{k=1}^{m} H_k + \frac{1}{2}\sum_{k=1}^{m} L_k\right)\alpha_{PC}(S). \tag{2.21}$$

By (A4) and Lemma 2.3, A is a strict set contraction operator from $PC[J,P]$ into $PC[J,P]$. □

3. Main Result

Theorem 3.1. *Assuming that (A1)–(A6) hold, then the IBVP* (1.2) *has at least two positive solutions u_1 and u_2 satisfying*

$$u_1(t) \ge l u_0(t), \quad \text{for } t \in J_0' = [a', b'],\ l > 1, \tag{3.1}$$

where l was specified in (A6).

Proof. First we verify that there exists $\delta > 0$ such that $\|v\| \ge \delta$ for $v \ge u_0$. If this is not true, then there exists $\{v_n\} \subset E$which satisfies $v_n \ge u_0$ and $\|v_n\| < (1/n)$ $(n = 1, 2, \ldots)$, so we have $u_0 \le \theta$, which is a contradiction with $u_0 \in \overset{\circ}{P}$.

By (A2), there exist $c' > c$, $c_k' > c_k$, $d' > d$, $d_k' > d_k$, $h_k' > h_k$, and $e_k' > e_k$, and

$$0 < r_1 < \delta, \qquad r_2 > \max\{\delta, l\|u_0\|\} \tag{3.2}$$

satisfy

$$c'l_1\int_0^{2\pi} g(s)ds + l_1\sum_{k=1}^m h'_k + \frac{1}{2}\sum_{k=1}^m c'_k < 1, \tag{3.3}$$

$$b := d'l_1\int_0^{2\pi} g(s)ds + l_1\sum_{k=1}^m e'_k + \frac{1}{2}\sum_{k=1}^m d'_k < 1. \tag{3.4}$$

For $x \in T_{r_1} \cap P$,

$$\|h(x)\| \le c'\|x\|, \qquad \|I_k(x)\| \le c'_k\|x\|, \qquad \left\|\overline{I}_k(x)\right\| \le h'_k\|x\|. \tag{3.5}$$

For $\|x\| \ge r_2$ and $x \in P$,

$$\|h(x)\| \le d'\|x\|, \qquad \|I_k(x)\| \le d'_k\|x\|, \qquad \left\|\overline{I}_k(x)\right\| \le e'_k\|x\|. \tag{3.6}$$

Therefore, for any $x \in P$, we have

$$\|h(x)\| \le d'\|x\| + M', \qquad \|I_k(x)\| \le d'_k\|x\| + M', \qquad \left\|\overline{I}_k(x)\right\| \le e'_k\|x\| + M', \tag{3.7}$$

where

$$\begin{gathered} M' = \max\{M_0, M_1, \dots, M_m, K_1, \dots, K_m\}, \qquad M_0 = \sup\{\|h(x)\| : x \in T_{r_2} \cap P\}, \\ M_k = \sup\{\|I_k(x)\| : x \in T_{r_2} \cap P\}, \quad K_k = \sup\left\{\left\|\overline{I}_k(x)\right\| : x \in T_{r_2} \cap P\right\} \quad (k = 1, 2, \dots, m). \end{gathered} \tag{3.8}$$

Let $r_3 = r_2 + (1-b)^{-1}G$, $G = M'[l_1\int_0^{2\pi} g(s)ds + ml_1 + m/2]$, $U_1 = \{u \in PC[J, P] : \|u\|_{PC} < r_1\}$, $U_2 = \{u \in PC[J, P] : \|u\|_{PC} < r_3\}$, $U_3 = \{u \in PC[J, P] : \|u\|_{PC} < r_3,\ u(t) \ge lu_0$ for $t \in J'_0$ and $l > 1\}$. It is clear that U_1, U_2, U_3 are nonempty, bounded, and convex open sets in $PC[J, P]$, and $\overline{U}_1 = PC[J, P] \cap B_{r_1}$, $\overline{U}_2 = PC[J, P] \cap B_{r_3}$, and $\overline{U}_3 = \{u \in \overline{U}_2 : u(t) \ge lu_0, t \in J'_0\}$.

From (3.2), we obtain

$$U_1 \subset U_2, \qquad U_3 \subset U_2, \qquad U_1 \cap U_3 = \emptyset. \tag{3.9}$$

According to Lemma 2.4, $A : \overline{U}_2 \to PC[J, P]$ is a strict set contraction operator, and for $u \in \overline{U}_2$, by (2.4) and (3.7), we obtain

$$\begin{aligned} \|(Au)(t)\| &= \left\|\int_0^{2\pi} G(t,s)f(s,u(s))ds + \sum_{k=1}^m\left[G(t,t_k)\overline{I}_k(u(t_k)) + H(t,t_k)I_k(u(t_k))\right]\right\| \\ &\le l_1\int_0^{2\pi} g(s)ds\|h(u)\| + \sum_{k=1}^m\left(l_1\left\|\overline{I}_k\right\| + \frac{1}{2}\|I_k\|\right) \end{aligned}$$

$$\begin{aligned}
&\leq l_1 \int_0^{2\pi} g(s)ds\left(d'\|u\| + M'\right) + l_1 \sum_{k=1}^{m}\left(e'_k\|u\| + M'\right) + \frac{1}{2}\sum_{k=1}^{m}\left(d'_k\|u\| + M'\right) \\
&= \left[d' l_1 \int_0^{2\pi} g(s)ds + l_1 \sum_{k=1}^{m} e'_k + \frac{1}{2}\sum_{k=1}^{m} d'_k\right]\|u\| + M'\left[l_1 \int_0^{2\pi} g(s)ds + m l_1 + \frac{m}{2}\right] \\
&= b\|u\| + G \\
&\leq b r_3 + G < r_3.
\end{aligned} \tag{3.10}$$

Hence

$$A\left(\overline{U}_2\right) \subset U_2. \tag{3.11}$$

Similarly, $A : \overline{U}_1 \to PC[J, P]$ is a strict set contraction operator, and for $u \in \overline{U}_1$, by (3.3) and (3.5), we obtain

$$\begin{aligned}
\|(Au)(t)\| &= \left\| \int_0^{2\pi} G(t,s) f(s,u(s))ds + \sum_{k=1}^{m}\left[G(t,t_k)\overline{I}_k(u(t_k)) + H(t,t_k) I_k(u(t_k))\right]\right\| \\
&\leq c' l_1 \int_0^{2\pi} g(s)ds\|u\| + \sum_{k=1}^{m}\left(l_1 h'_k\|u\| + \frac{1}{2}c'_k\|u\|\right) \\
&= \left[c' l_1 \int_0^{2\pi} g(s)ds + l_1 \sum_{k=1}^{m} h'_k + \frac{1}{2}\sum_{k=1}^{m} c'_k\right]\|u\| \\
&< \|u\| \leq r_1,
\end{aligned} \tag{3.12}$$

so

$$A\left(\overline{U}_1\right) \subset U_1. \tag{3.13}$$

Let $u \in \overline{U}_3$, by (3.11), we have $\|Au\|_{PC} < r_3$.
By (2.5), (A5), and (A6), for $t \in J'_0$,

$$\begin{aligned}
(Au)(t) &= \int_0^{2\pi} G(t,s) f(s,u(s))ds + \sum_{k=1}^{m}\left[G(t,t_k)\overline{I}_k(u(t_k)) + H(t,t_k) I_k(u(t_k))\right] \\
&\geq \int_0^{2\pi} G(t,s) f(s,u(s))ds + \sum_{k=1}^{m}[M G(t,t_k) + H(t,t_k)] I_k(u(t_k)) \\
&\geq \int_0^{2\pi} G(t,s) f(s,u(s))ds
\end{aligned}$$

$$\begin{aligned}&\geq l_0 \int_{a'}^{b'} f(s,u(s))ds\\&\geq l_0 \int_{a'}^{b'} \overline{h}(s)ds\, u_0\\&= lu_0.\end{aligned} \tag{3.14}$$

So $Au \in U_3$, and

$$A\left(\overline{U}_3\right) \subset U_3. \tag{3.15}$$

According to (3.11)–(3.15) and Lemma 2.1, we have

$$i(A, U_j, PC[J, P]) = 1 \quad (j = 1, 2, 3). \tag{3.16}$$

Hence

$$\begin{aligned}&i\left(A, U_2 \setminus \left(\overline{U}_1 \cup \overline{U}_3\right), PC[J, P]\right)\\&\quad = i(A, U_2, PC[J, P]) - i(A, U_1, PC[J, P]) - i(A, U_3, PC[J, P])\\&\quad = -1.\end{aligned} \tag{3.17}$$

Thus, A has two fixed points u_1 and u_2 in U_3 and $U_2 \setminus (\overline{U}_1 \cup \overline{U}_3)$, respectively, which means $u_1(t)$ and $u_2(t)$ are positive solution of the IBVP (1.2), where $u_1(t) \geq lu_0$, for $t \in J_0'$ and $l > 1$. □

4. Example

To illustrate how our main result can be used in practice, we present an example.

Example 4.1. Consider the following problem:

$$\begin{gathered}-x_n''(t) + 4x_n = \frac{1}{\sqrt{t}}\left[3\left(2 + \frac{999}{\pi}t\right)\ln\left(1 + x_{n+1}^2\right) + \frac{x_n}{3}\right], \quad t \in J,\\ \Delta x_n|_{t=1/3} = \frac{1}{8}x_n\left(\frac{1}{3}\right),\\ \Delta x_n'|_{t=1/3} = -\frac{1}{4}x_n\left(\frac{1}{3}\right),\\ x_n(0) = x_n(2\pi), \qquad x_n'(0) = x_n'(2\pi),\end{gathered} \tag{4.1}$$

where $x_{m+n} = x_n$ $(n = 1, 2, \ldots, m)$.

Conclusion

IBVP (4.1) has at least two positive solutions $\{x_{1n}(t)\}$ and $\{x_{2n}(t)\}$ such that $x_{1n}(t) > 1$ for $t \in [\pi, 2\pi]$, $n = 1, 2, \ldots, m$.

Proof. Let $J = [0, 2\pi]$, $E = \mathbb{R}^m = \{x = (x_1, x_2, \ldots, x_m) : x_n \in \mathbb{R}, n = 1, 2, \ldots, m\}$; then, E is a Banach space with norm $\|x\| = \max_{1\le n\le m}|x_n|$. Let $P = \{(x_1, x_2, \ldots, x_m) : x_n \ge 0,\ n = 1, 2, \ldots, m\}$; then, P is a solid cone in E. Compared to IBVP (1.2), $f(t,x) = (f_1, f_2, \ldots, f_m)$, $f_n(t,x) = (1/\sqrt{t})[3(2 + (999/\pi)t)\ln(1 + x_{n+1}^2) + x_n/3)]$ is singular at $t = 0$. $I(x) = (I_1(x), I_2(x), \ldots, I_m(x))$, and $I_n(x) = (1/8)x_n(1/3)$. $\overline{I}(x) = (\overline{I}_1(x), \ldots, \overline{I}_m(x))$, and $\overline{I}_n(x) = (1/4)x_n(1/3)$, $n = 1, 2, \ldots, m$.

Next we will verify that the conditions in Theorem 3.1 are satisfied.

Let $g(t) = 1/\sqrt{t}$, $h(x) = (h_1(x), h_2(x), \ldots, h_m(x))$, and $h_n(x) = 6000\ln(1 + x_{n+1}^2) + x_n/3$. It is clear that $\|f(t,x)\| \le g(t)\|h(x)\|$, for $t \in (0, 2\pi)$ and $x \in E$, so (A1) is satisfied.

By simple calculations, we have $M = 2$, $c = d = 1/3$, $c_1 = d_1 = 1/8$, $h_1 = e_1 = 1/4$, $\int_0^{2\pi} g(s)ds = 5.01326$, $l_0 = 0.00093$, and $l_1 = 0.25$. Hence, $l_1\int_0^{2\pi} g(s)dsc + l_1 h_1 + (1/2)c_1 < 1$; that is, (A2) is satisfied.

Since E is a finite-dimensional space, it is obvious that (A3) and (A4) are satisfied.

It is clear that $\overline{I}_n(x) = (1/4)x_n(1/3)$ and $MI_n(x) = 2 \times (1/8)x_n(1/3) = (1/4)x_n(1/3)$, so $\overline{I}_n(x) = MI_n(x)$; that is, (A5) is satisfied.

Let $u_0 = (1, 1, \ldots, 1) \in \mathring{P}$ and $J_0' = [\pi, 2\pi] \subset [0, 2\pi]$; for $t \in J_0'$ and $x \ge u_0$, we have

$$f_n(t,x) = \frac{1}{\sqrt{t}}\left[\left(3\left(2 + \frac{999}{\pi}t\right)\ln\left(1 + x_{n+1}^2\right) + \frac{x_n}{3}\right)\right] > \frac{3000\ln 2}{\sqrt{t}}. \tag{4.2}$$

Let $\overline{h}(t) = 3000\ln 2/\sqrt{t}$; then, for $t \in J_0'$ and $x \ge u_0$, we obtain that $f(t,x) \ge \overline{h}(t)u_0$ and $l_0\int_\pi^{2\pi}\overline{h}(s)ds > 1$. Therefore (A6) is satisfied.

By Theorem 3.1, IBVP (4.1) has at least two positive solutions $\{x_{1n}(t)\}$ and $\{x_{2n}(t)\}$ and satisfies $x_{1n}(t) > 1$, $n = 1, 2, \ldots, m$. □

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