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Research Article

On the Fractional Difference Equations of Order $(2, q)$

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This paper presents a kind of new definition of fractional difference, fractional summation, and fractional difference equations and gives methods for explicitly solving fractional difference equations of order $(2, q)$.

1. Introduction

As is well known, there is a large quantity of research on what is usually called integer-order difference equations and integer-order differential equations. Since the study is started very early by many famous mathematicians, such as Leibniz, Bernoulli, Euler, and Lagrange, many systematic works were established, and much classical content was included in the textbooks [1–3]. Moreover, it is also well known that the theory of integer-order difference equations have many similar performances to the theory of integer-order differential equations. However, the study on the ordinary fractional differential equations is just a beginning of exploration in the recent two decades. For example, in their encyclopedic monograph [4] on the fractional integrals and derivatives, Samko et al. summed up the results of the fractional calculus and established the existence and uniqueness of the solution of ordinary fractional differential equations and so on. Miller and Ross [5] made a significant contribution to the solution of ordinary fractional differential equations by using the transcendental function and Laplace transform, as well as fractional Green function method; they researched linear fractional differential equations with constant coefficients skillfully and systematically and obtained a great deal of excellent results. These results have aroused a great interest for mathematicians [6–11]. After then, two new comprehensive monographs [12, 13] on fractional differential equations have been published one after another.

It is natural to ask whether the corresponding fractional difference theory and fractional summation theory can be established or what is the corresponding theory on fractional difference equations. These problems have been researched by many mathematicians, such as Samko et al., who gave the definition for the fractional difference with series type in Section 21 of their book [4]. This definition is useful for solving the numerical solution of the fractional differential equations. However, this definition has some limitations: for example, when the order ν is negative, this definition is unable to guarantee its convergence. Moreover, such a series type of definition, even the most simple fractional difference equations, cannot give their exact solution. Without doubt, they have not obtained the similar performance for the fractional differential equations.

The purpose of this paper is to give the new definitions of fractional difference, and fractional summation, as well as fractional difference equations. In particular, making use of our definitions, the fractional difference equations can be solved successfully, and its theory has a miraculous analogy with the theory on fractional differential equations. Limited to the length of the paper, we only give the explicit solution of the fractional differential equations of order $(2, q)$. Nevertheless, this method for solving fractional difference equations is not trivial. For another further systemic results, one can see our monograph [14].

2. Definitions of Fractional Difference and Fractional Summation

Let $x(n)$ be a real-valued sequence, $n \in Z$. Let us start from backward difference and give some basic definitions.

Definition 2.1. One calls

$$\nabla x(n) = x(n) - x(n-1) \tag{2.1}$$

one-order backward difference of $x(n)$ and calls

$$\nabla^k x(n) = \nabla \nabla^{(k-1)} x(n) \tag{2.2}$$

k-order backward difference of $x(n)$, where k is a positive integer.

Definition 2.2. One calls

$$\nabla^{-1} x(n) = \sum_{r=0}^{n} x(r) \tag{2.3}$$

one-order summation of $x(n)$ and calls

$$\nabla^{-k} x(n) = \nabla^{-1} \nabla^{-(k-1)} x(n) \tag{2.4}$$

k-order summation of $x(n)$, where k is a positive integer.

Definition 2.3. Set

$$(x)^{(n)} \triangleq x(x+1)\cdots(x+n-1), \tag{2.5}$$

where n is a positive integer, x is real, and $(x)^{(n)}$ is called rising factorial function. And define

$$\begin{bmatrix} x \\ n \end{bmatrix} \triangleq \frac{x(x+1)\cdots(x+n-1)}{n!}. \tag{2.6}$$

For backward difference of order m, where m is positive integer, we have

Lemma 2.4. *Assume that m is a positive integer, then*

$$\nabla^{-m}x(n) = \sum_{r=0}^{n}\begin{bmatrix} m \\ n-r \end{bmatrix}x(r) = \begin{bmatrix} m \\ n \end{bmatrix} * x(n), \tag{2.7}$$

where $\left[\begin{smallmatrix} m \\ n-r \end{smallmatrix}\right] = m(m+1)\cdots(m+n-r-1)/(n-r)!$, and $*$ is convolution symbol.

Proof. By Definition 2.2, we have $\nabla^{-1}x(n) = \sum_{r=0}^{n} x(r)$, and then

$$\begin{aligned}
\nabla^{-2}x(n) &= \nabla^{-1}\left(\nabla^{-1}x(n)\right) = \sum_{r=0}^{n}\nabla^{-1}x(r) \\
&= \sum_{r=0}^{n}\sum_{s=0}^{r}x(s) = \sum_{s=0}^{n}\sum_{r=s}^{n}x(s) = \sum_{s=0}^{n}(n-s+1)x(s), \\
\nabla^{-3}x(n) &= \nabla^{-1}\left(\nabla^{-2}x(n)\right) = \sum_{r=0}^{n}\nabla^{-2}x(r) = \sum_{r=0}^{n}\sum_{s=0}^{r}(r-s+1)x(s) \\
&= \sum_{s=0}^{n}\sum_{r=s}^{n}(r-s+1)x(s) = \frac{1}{2!}\sum_{s=0}^{n}(n-s+1)(n-s+2)x(s), \\
\nabla^{-4}x(n) &= \frac{1}{3!}\sum_{s=0}^{n}(n-s+1)(n-s+2)(n-s+3)x(s), \\
&\vdots
\end{aligned} \tag{2.8}$$

By recursion, we have

$$\nabla^{-m}x(n) = \frac{1}{(m-1)!}\sum_{s=0}^{n}(n-s+1)(n-s+2)\cdots(n-s+m-1)x(s). \tag{2.9}$$

Since $\left[\begin{smallmatrix} m \\ n-s \end{smallmatrix}\right] = m(m+1)\cdots(m+n-s-1)/(n-s)! = (m+n-s-1)!/(n-s)!(m-1)! = (n-s+1)(n-s+2)\cdots(n-s+m-1)/(m-1)!$, by Definition 2.3 we can rewrite the above form as follows:

$$\begin{aligned} \nabla^{-m}x(n) &= \frac{1}{(m-1)!}\sum_{s=0}^{n}(n-s+1)^{(m-1)}x(s) \\ &= \sum_{s=0}^{n}\begin{bmatrix} m \\ n-s \end{bmatrix}x(s) = \begin{bmatrix} m \\ n \end{bmatrix} * x(n). \end{aligned} \tag{2.10}$$

□

Now we extend formula (2.7) to the general positive real number. It is obvious that the right side of (2.7) is also meaningful for any positive real number $\nu > 0$; based on this observation we give the definition of the fractional summation as follows.

Definition 2.5. Leting $\nu > 0$ be an arbitrary positive real number, one calls

$$\nabla^{-\nu}x(n) = \begin{bmatrix} \nu \\ n \end{bmatrix} * x(n) = \sum_{r=0}^{n}\begin{bmatrix} \nu \\ n-r \end{bmatrix}x(r) \tag{2.11}$$

ν-order summation of $x(n)$, where $*$ is the convolution symbol.

Next, we give the definition of the fractional difference as follows.

Definition 2.6. Let m be the smallest positive integer which is greater than $\mu > 0$. Then the fractional difference of $x(n)$ of order μ is defined by

$$\nabla^{\mu}x(n) = \nabla^{m}\nabla^{-(m-\mu)}x(n). \tag{2.12}$$

For example, we set again $x(n) = \left[\begin{smallmatrix} \nu \\ n \end{smallmatrix}\right] (\nu > 0, \mu > 0)$, then

$$\begin{gathered} \nabla^{-(m-\mu)}x(n) = \begin{bmatrix} m-\mu+\nu \\ n \end{bmatrix}, \\ \begin{aligned} \nabla\begin{bmatrix} m-\mu+\nu \\ n \end{bmatrix} &= \begin{bmatrix} m-\mu+\nu \\ n \end{bmatrix} - \begin{bmatrix} m-\mu+\nu \\ n-1 \end{bmatrix} \\ &= \frac{(m-\mu+\nu)(m-\mu+\nu+1)\cdots(m-\mu+\nu+n-1)}{n!} \\ &\quad - \frac{(m-\mu+\nu)(m-\mu+\nu+1)\cdots(m-\mu+\nu+n-1-1)}{(n-1)!} \\ &= \frac{(m-1-\mu+\nu)(m-\mu+\nu)\cdots(m-1-\mu+\nu+n-1)}{n!} \\ &= \begin{bmatrix} m-\mu+\nu-1 \\ n \end{bmatrix}. \end{aligned} \end{gathered} \tag{2.13}$$

By induction, it is not difficult to verify that

$$\nabla^{m}\begin{bmatrix} m-\mu+\nu \\ n \end{bmatrix} = \begin{bmatrix} \nu-\mu \\ n \end{bmatrix}. \tag{2.14}$$

Then by Definition 2.6, we have

$$\nabla^{\mu}\begin{bmatrix} \nu \\ n \end{bmatrix} = \begin{bmatrix} \nu-\mu \\ n \end{bmatrix}. \tag{2.15}$$

Definition 2.7. One calls

$$\sum_{n=0}^{\infty} x(n) z^{-n}, \quad (|z| > R) \tag{2.16}$$

a Z-transform of $x(n)$ and denotes it by $X(z)$ or $Z[x(n)]$, where R is the absolutely convergent radius of complex series.

Definition 2.8. Leting $f(n), g(n)$ be two sequences, one calls

$$\sum_{r=0}^{n} f(r) g(n-r) \tag{2.17}$$

an convolution of the $f(n)$ and $g(n)$ and denotes it by

$$f(n) * g(n) = \sum_{r=0}^{n} f(r) g(n-r). \tag{2.18}$$

Firstly, the following convolution theorem is well known.

Theorem 2.9. *Let* $Z[f(n)] = F(z), Z[g(n)] = G(z)$*, then*

$$Z\left[f(n) * g(n)\right] = Z\left[f(n)\right] Z\left[g(n)\right] = F(z) G(z). \tag{2.19}$$

Secondly, we have the Z-transform of function $\left[\begin{smallmatrix} k \\ n \end{smallmatrix}\right]$.

Lemma 2.10. *If* $k > 0$*, then*

$$Z\left(\begin{bmatrix} k \\ n \end{bmatrix}\right) = \left(1 - z^{-1}\right)^{-k}. \tag{2.20}$$

Proof. By the Taylor expansion, one has

$$\begin{aligned}\left(1-z^{-1}\right)^{-k} &= 1+\sum_{n=1}^{\infty}\frac{(-k)(-k-1)(-k-2)\cdots(-k-n+1)}{n!}\left(-\frac{1}{z}\right)^{-n}\\ &= 1+\sum_{n=1}^{\infty}\frac{k(k+1)(k+2)\cdots(k+n-1)}{n!}\left(\frac{1}{z}\right)^{-n}.\end{aligned} \tag{2.21}$$

Hence

$$Z\left(\begin{bmatrix}k\\ n\end{bmatrix}\right)=\sum_{n=0}^{\infty}\frac{k(k+1)(k+2)\cdots(k+n-1)}{n!}\left(\frac{1}{z}\right)^{-n}=\left(1-z^{-1}\right)^{-k}. \tag{2.22}$$

As an application, let $\nu>0,\ \mu>0$, set $x(n)=\left[\begin{smallmatrix}\nu\\ n\end{smallmatrix}\right]$, then we have

$$\nabla^{-\mu}x(n)=\begin{bmatrix}\mu\\ n\end{bmatrix}*\begin{bmatrix}\nu\\ n\end{bmatrix}=\begin{bmatrix}\nu+\mu\\ n\end{bmatrix}, \tag{2.23}$$

because of

$$Z\left(\begin{bmatrix}\mu\\ n\end{bmatrix}*\begin{bmatrix}\nu\\ n\end{bmatrix}\right)=Z\left(\begin{bmatrix}\mu\\ n\end{bmatrix}\right)Z\left(\begin{bmatrix}\nu\\ n\end{bmatrix}\right)=\left(1-z^{-1}\right)^{-\mu-\nu}=Z\left(\begin{bmatrix}\mu+\nu\\ n\end{bmatrix}\right). \tag{2.24}$$

□

In general, the law of exponents $\nabla^{\mu}\nabla^{\nu}x(n)=\nabla^{\mu+\nu}x(n)$ is not necessarily valid for arbitrary real numbers μ and ν. For example, let $x(n)=1$, then $\nabla x(n)=0$ and $\nabla^{-1}\nabla x(n)=0$, while $\nabla^{-1}x(n)=n$ and $\nabla\nabla^{-1}x(n)=1$. But with additional caveats, the law of exponents can also hold. For $\mu,\nu>0$ we have

Proposition 2.11. *One has* $\nabla^{-\mu}\nabla^{-\nu}x(n)=\nabla^{-(\nu+\mu)}x(n)$.

Proof. From the definition of fractional summation and convolution theorem we have

$$\begin{aligned}\nabla^{-\mu}\nabla^{-\nu}x(n) &= \begin{bmatrix}\mu\\ n\end{bmatrix}*\left(\begin{bmatrix}\nu\\ n\end{bmatrix}*x(n)\right)\\ &= \begin{bmatrix}\nu+\mu\\ n\end{bmatrix}*x(n)\\ &= \nabla^{-(\nu+\mu)}x(n).\end{aligned} \tag{2.25}$$

□

Proposition 2.12. *One has* $\nabla^{\mu}\nabla^{-\nu}x(n) = \nabla^{\mu-\nu}x(n)$.

Proof. By the definition of fractional difference, we get

$$\begin{aligned}
\nabla^{\mu}\nabla^{-\nu}x(n) &= \nabla^{m}\nabla^{-(m-\mu)}\nabla^{-\nu}x(n) \\
&= \nabla^{m}\nabla^{-(m-\mu+\nu)}x(n) \\
&= \nabla^{(\mu-\nu)}x(n).
\end{aligned} \tag{2.26}$$

□

Proposition 2.13. *One has* $\nabla\nabla^{-\nu}x(n) = \nabla^{-\nu}(\nabla x(n)) + x(-1)\left[\begin{smallmatrix}\nu\\ n\end{smallmatrix}\right]$.

Proof. We have

$$\begin{aligned}
\nabla\nabla^{-\nu}x(n) &= \nabla\left(\sum_{r=0}^{n}\begin{bmatrix}\nu\\ n-r\end{bmatrix}x(r)\right) \\
&= \sum_{r=0}^{n}\begin{bmatrix}\nu\\ n-r\end{bmatrix}x(r) - \sum_{r=0}^{n-1}\begin{bmatrix}\nu\\ n-1-r\end{bmatrix}x(r) \\
&= \sum_{r=0}^{n}\begin{bmatrix}\nu\\ n-r\end{bmatrix}x(r) - \sum_{r=1}^{n}\begin{bmatrix}\nu\\ n-r\end{bmatrix}x(r-1) \\
&= \sum_{r=0}^{n}\begin{bmatrix}\nu\\ n-r\end{bmatrix}(x(r) - x(r-1)) + x(-1)\begin{bmatrix}\nu\\ n\end{bmatrix} \\
&= \sum_{r=0}^{n}\begin{bmatrix}\nu\\ n-r\end{bmatrix}\nabla x(r) + x(-1)\begin{bmatrix}\nu\\ n\end{bmatrix} \\
&= \nabla^{-\nu}(\nabla x(n)) + x(-1)\begin{bmatrix}\nu\\ n\end{bmatrix}.
\end{aligned} \tag{2.27}$$

□

Proposition 2.14. *If* $0 < \mu + \nu < 1$, *then* $\nabla^{\mu}\nabla^{\nu}x(n) = \nabla^{\mu+\nu}x(n)$.

Proof. We have

$$\begin{aligned}
\nabla^{\mu}\nabla^{\nu}x(n) &= \nabla^{\mu}\nabla\nabla^{-(1-\nu)}x(n) \\
&= \nabla^{\mu}\left(\nabla^{-(1-\nu)}x(n) + x(-1)\begin{bmatrix}1-\nu\\ n\end{bmatrix}\right) \\
&= \nabla^{-(1-(\mu+\nu))}\nabla x(n) + x(-1)\begin{bmatrix}1-\mu-\nu\\ n\end{bmatrix} \\
&= \nabla\nabla^{-(1-(\mu+\nu))}x(n) \\
&= \nabla^{\mu+\nu}x(n).
\end{aligned} \tag{2.28}$$

□

3. Fractional Difference Equations

An ordinary difference equation is an equation involving difference of a function, and the basic problem is to find a function that satisfies this equation. For example,

$$\left(\nabla^2 + a\nabla + b\nabla^0\right)x(n) = 0 \tag{3.1}$$

(where a and b are constants and $a + b + 1 \neq 0$) is a second-order ordinary linear difference equation with constant coefficients. The problem is to find nonidentically zero function $x(n)$ that satisfies (3.1). Therefore, it come as no surprise that we define a fractional difference equation as an equation involving fractional difference of a function. In particular, if k and q are positive integers and $\nu = 1/q$, then we call

$$L = \nabla^{k\nu} + a_1\nabla^{(k-1)\nu} + a_2\nabla^{(k-2)\nu} + \cdots + a_k\nabla^0 \tag{3.2}$$

a fractional difference operator of order (k, q), where a_i are constants, $i = 0, 1, \ldots, k$. Of course, there exist more complicated fractional difference operators, but (3.2) is more than sufficiently complex. Although we can solve the general equation $L(x(n)) = 0$, but limited to the length of this paper, we will only focus our attention on equations of order $(2, q)$, that is, on equations of the form

$$\left(\nabla^{2\nu} + a\nabla^{\nu} + b\nabla^0\right)x(n) = 0. \tag{3.3}$$

Our problem, of course, is to find a function $x(n)$ that satisfies (3.3). Let us briefly review some results in ordinary difference equations theory that may give us a hint as how to proceed.

In the difference equation (3.1), we already know that if $\alpha \neq 1$, $\beta \neq 1$ are the different roots of the indicial equation $p(t) = 0$, where

$$p(t) = t^2 + at + b, \tag{3.4}$$

then the solution of (3.1) is $x(n) = (1/(1-\alpha))^n$ or $x(n) = (1/(1-\beta))^n$, that is, to say, the solution is an exponential function. If $\alpha \neq \beta$, then we have two linearly independent solutions. If $\alpha = \beta$, then α is a double root of $p(t) = 0$, and $(1/(1-\alpha))^n$ and $n(1/(1-\alpha))^n$ are linearly independent solutions of (3.1).

Let us now attempt to use the above arguments in solving (3.3).

Define some special functions as follows:

$$\bigwedge\nolimits_n(-\mu, \lambda) = \nabla^{\mu}\lambda^n, \qquad \bigwedge\nolimits_n(0, \lambda) = \lambda^n. \tag{3.5}$$

It follows from Proposition 2.14 that

$$\begin{aligned}
\nabla^{\nu} \bigwedge_n (0, \lambda) &= \bigwedge_n (-\nu, \lambda), \\
\nabla^{\nu} \bigwedge_n (-\nu, \lambda) &= \bigwedge_n (-2\nu, \lambda), \\
&\vdots \\
\nabla^{\nu} \bigwedge_n (-(q-2)\nu, \lambda) &= \bigwedge_n (-(q-1)\nu, \lambda).
\end{aligned} \tag{3.6}$$

Making use of $qv = 1$ and Proposition 2.13, we have

$$\begin{aligned}
\nabla^{\nu} \bigwedge_n (-(q-1)\nu, \lambda) &= \bigwedge_n (-1, \lambda) + \begin{bmatrix} 0 \\ n \end{bmatrix} \\
&= \left(1 - \frac{1}{\lambda}\right) \bigwedge_n (0, \lambda) + \begin{bmatrix} 0 \\ n \end{bmatrix}.
\end{aligned} \tag{3.7}$$

From $\left[\begin{smallmatrix} 0 \\ n \end{smallmatrix}\right] = 0$ we clearly see that

$$\nabla^{\nu} \bigwedge_n (-(q-1)\nu, \lambda) = \left(1 - \frac{1}{\lambda}\right) \bigwedge_n (0, \lambda). \tag{3.8}$$

The significance of these applications is that if we apply the operator ∇^{ν} to

$$\bigwedge_n (0, \lambda), \bigwedge_n (-\nu, \lambda), \ldots, \bigwedge_n (-(q-1)\nu, \lambda), \tag{3.9}$$

then we get a cyclic permutation of the same functions. That is, no new functions are introduced.

Therefore, we will choose a linear combination of these functions as a candidate for a solution of (3.3), say

$$x(n) = B_0 \bigwedge_n (0, \lambda) + B_1 \bigwedge_n (-\nu, \lambda) + \cdots + B_{(q-1)} \bigwedge_n (-(q-1)\nu, \lambda), \tag{3.10}$$

where B_i, $i = 1, 2, \ldots, q-1$, λ are arbitrary constants for the moment. From our preceding arguments, we have

$$\nabla^{\nu} x(n) = \left(1 - \frac{1}{\lambda}\right) B_{(q-1)} \bigwedge_n (0, \lambda) + B_0 \bigwedge_n (-\nu, \lambda) + \cdots + B_{(q-2)} \bigwedge_n (-(q-1)\nu, \lambda). \tag{3.11}$$

Now, if $\nabla^{2\nu} x(n)$ has the same cyclic property, then we may calculate $(\nabla^{2\nu} + a\nabla^{\nu} + b\nabla^{0}) x(n)$. It will be a linear combination of $\Lambda_n(0, \lambda), \Lambda_n(-v, \lambda), \ldots, \Lambda_n(-(q-1)v, \lambda)$ whose coefficients are functions of the $B's$ and λ. Then perhaps we can choose $B_0, B_1, \ldots, B_{q-1}, \lambda$ such that the coefficients of the $\Lambda_n(-kv, \lambda)$ functions vanish. If so, we will have a solution of (3.3).

Let us calculate $\nabla^{2\nu}x(n)$; it follows from Proposition 2.14 that

$$\begin{aligned}\nabla^{2\nu}x(n) &= B_0\bigwedge_n(-2\nu,\lambda) + B_1\bigwedge_n(-3\nu,\lambda) + \cdots + B_{(q-3)}\bigwedge_n(-(q-1)\nu,\lambda)\\ &\quad + B_{(q-2)}\nabla^{2\nu}\bigwedge_n(-(q-2)\nu,\lambda) + B_{(q-1)}\nabla^{2\nu}\bigwedge_n(-(q-1)\nu,\lambda).\end{aligned}\tag{3.12}$$

From the definition of fractional difference and Propositions 2.12 and 2.13 we clearly see that

$$\begin{aligned}\nabla^{2\nu}\bigwedge_n(-(q-2)\nu,\lambda) &= \bigwedge_n(-1,\lambda) + x(-1)\begin{bmatrix}0\\ n\end{bmatrix} = \left(1-\frac{1}{\lambda}\right)\bigwedge_n(0,\lambda),\\ \nabla^{2\nu}\bigwedge_n(-(q-1)\nu,\lambda) &= \left(1-\frac{1}{\lambda}\right)\bigwedge_n(-\nu,\lambda) + \lambda^{-1}\begin{bmatrix}-\nu\\ n\end{bmatrix}.\end{aligned}\tag{3.13}$$

Thus

$$\begin{aligned}\nabla^{2\nu}x(n) &= \left(1-\frac{1}{\lambda}\right)B_{(q-2)}\bigwedge_n(0,\lambda) + \left(1-\frac{1}{\lambda}\right)B_{(q-1)}\bigwedge_n(-\nu,\lambda) + B_0\bigwedge_n(-2\nu,\lambda)\\ &\quad + B_1\bigwedge_n(-3\nu,\lambda) + \cdots + B_{(q-3)}\bigwedge_n(-(q-1)\nu,\lambda) + B_{q-1}\lambda^{-1}\begin{bmatrix}-\nu\\ n\end{bmatrix}.\end{aligned}\tag{3.14}$$

We note that $\nabla^{2\nu}x(n)$ have cyclical property, that is, only the terms of the form $\Lambda_n(-kv,\lambda), k = 0,1,\ldots,q-1$ appeared; we also have the unwanted term $\lambda^{-1}\left[\begin{smallmatrix}-\nu\\ n\end{smallmatrix}\right]$. Now, we deal with the later term. From (3.10)–(3.14), we may compute $(\nabla^{2\nu} + a\nabla^{\nu} + b\nabla^0)x(n)$. From the coefficients of $\Lambda_n(-kv,\lambda)$ terms, we get

$$\begin{aligned}\left(\nabla^{2\nu} + a\nabla^{\nu} + b\nabla^0\right)x(n) &= \left[\left(1-\frac{1}{\lambda}\right)B_{(q-2)} + a\left(1-\frac{1}{\lambda}\right)B_{(q-1)} + bB_0\right]\bigwedge_n(0,\lambda)\\ &\quad + \left[\left(1-\frac{1}{\lambda}\right)B_{(q-1)} + aB_0 + bB_1\right]\bigwedge_n(-\nu,\lambda)\\ &\quad + \sum_{k=0}^{q-3}\left[B_k + aB_{k+1} + bB_{k+2}\right]\bigwedge_n(-(k+2)\nu,\lambda) + B_{(q-1)}\lambda^{-1}\begin{bmatrix}-\nu\\ n\end{bmatrix}.\end{aligned}\tag{3.15}$$

Since α is a root of the indicial equation $p(t) = 0$, hence

$$\alpha^2 + a\alpha + b = 0.\tag{3.16}$$

Comparing (3.16) with the terms under the summation sign in (3.15), we see that if B_k represent decreasing powers of α, then all these terms will vanish. Let

$$B_k = A\alpha^{-k}, \tag{3.17}$$

where A is an arbitrary nonzero factor independent of k. Then

$$\begin{aligned} B_k + aB_{k+1} + bB_{k+2} &= A\left(\alpha^{-k} + a\alpha^{-k-1} + b\alpha^{-k-2}\right) \\ &= A\alpha^{-k-2}\left(\alpha^2 + a\alpha + b\right) = 0. \end{aligned} \tag{3.18}$$

Therefore, (3.15) reduces to

$$\begin{aligned} \left(\nabla^{2\nu} + a\nabla^{\nu} + b\nabla^{0}\right)x(n) = A&\left[\left(1-\frac{1}{\lambda}\right)\alpha^{-q+2} + a\left(1-\frac{1}{\lambda}\right)\alpha^{-q+1} + b\right]\bigwedge\nolimits_n(0,\lambda) \\ &+ A\left[\left(1-\frac{1}{\lambda}\right)\alpha^{-q+1} + a + b\alpha^{-1}\right]\bigwedge\nolimits_n(\nu,\lambda) + A\alpha^{-q+1}\lambda^{-1}\begin{bmatrix}-\nu \\ n\end{bmatrix}. \end{aligned} \tag{3.19}$$

But the constant λ is still at our disposal. If we take $1 - 1/\lambda = \alpha^q$, then $\lambda = 1/(1-\alpha^q)$, and the above expression reduces to

$$\left(\nabla^{2\nu} + a\nabla^{\nu} + b\nabla^{0}\right)x(n) = A\alpha^{-q+1}(1-\alpha^q)\begin{bmatrix}-\nu \\ n\end{bmatrix}. \tag{3.20}$$

Since A is arbitrary, we choose $A = \alpha^{q-1}/(1-\alpha^q)$ such that the term on the right-hand side of (3.20) is independent of α.

From the choices of $B_0, B_1, \ldots, B_{(q-1)}, \lambda$ and A we clearly see that $x(n)$ and (3.20) can be rewritten as

$$\lambda_\alpha(n) = \frac{1}{1-\alpha^q}\sum_{k=0}^{q-1}\alpha^{q-k-1}\bigwedge\left(-k\nu, \frac{1}{1-\alpha^q}\right), \tag{3.21}$$

$$\left(\nabla^{2\nu} + a\nabla^{\nu} + b\nabla^{0}\right)\lambda_\alpha(n) = \begin{bmatrix}-\nu \\ n\end{bmatrix}, \tag{3.22}$$

respectively, where α is a zero of $p(t)$. If $\alpha = 0$, then we choose

$$\lambda_0(n) = \begin{bmatrix} \nu \\ n \end{bmatrix}. \tag{3.23}$$

Of course, $\lambda_\alpha(n)$ is not a solution of (3.3), since we still have the term $\left[\begin{smallmatrix} -\nu \\ n \end{smallmatrix}\right]$ on the right-hand side of (3.22), but we are getting close. We recall that $p(t) = 0$ have two zeros; let β be another zero. Set

$$\lambda_\beta(n) = \frac{1}{1-\beta^q} \sum_{k=0}^{q-1} \beta^{q-k-1} \bigwedge\left(-k\nu, \frac{1}{1-\beta^q}\right), \tag{3.24}$$

then similar arguments show that

$$\left(\nabla^{2\nu} + a\nabla^{\nu} + b\nabla^{0}\right)\lambda_\beta(n) = \begin{bmatrix} -\nu \\ n \end{bmatrix}. \tag{3.25}$$

Thus

$$\Psi(n) = \lambda_\alpha(n) - \lambda_\beta(n) \tag{3.26}$$

is the solution of (3.3).

Therefore, we have the following theorem.

Theorem 3.1. *If $\alpha \neq \beta$, then*

$$\Psi(n) = \lambda_\alpha(n) - \lambda_\beta(n) \tag{3.27}$$

is the solution of (3.3).

If $\alpha \neq \beta$, then (3.27) represents a nonidentically zero solution of (3.3). Of course, if $\alpha = \beta$, then we have the trivial solution $\Psi(n) = 0$. However, we recall the same phenomenon in ordinary difference equation theory. If $\alpha = \beta$, then $n\lambda^n$ is a solution of (3.1). For (3.3), using a similar but more sophisticated argument, one has

$$x(n) = \sum_{k=-(q-1)}^{q-1} A_k \bigwedge\nolimits_n\left(-k\nu, \frac{1}{1-\alpha^q}\right) + \sum_{k=-(q-1)}^{q-1} B_k \nabla^{k\nu}(n+1)\left(\frac{1}{1-\alpha^q}\right)^n. \tag{3.28}$$

Noting that $(n+1)(1/(1-\alpha^q))^n$ is zero at $n=-1$, Proposition 2.14 is also valid for function $(n+1)(1/(1-\alpha^q))^n$ even if $\mu+\nu=1$. Setting $A_{q-1}=0$ and $B_{q-1}=0$, calculating $\nabla^\nu x(n)$ and $\nabla^{2\nu}x(n)$ as before, we know that no new functions are introduced. We obtain

$$\begin{aligned}
&\left(\nabla^{2\nu}-2\alpha\nabla^{\nu}+\alpha^2\nabla^0\right)x(n)\\
&\quad=\alpha^2A_{-(q-1)}\bigwedge\nolimits_n\left((q-1)\nu,\frac{1}{1-\alpha^q}\right)+\left(-2\alpha A_{-(q-1)}+\alpha^2A_{-(q-2)}\right)\bigwedge\nolimits_n\left((q-2)\nu,\frac{1}{1-\alpha^q}\right)\\
&\qquad+\sum_{k=3}^{q-1}\left(A_{-(q-k+2)}-2\alpha A_{-(q-k+1)}+\alpha^2A_{-(q-k)}\right)\bigwedge\nolimits_n\left((q-k)\nu,\frac{1}{1-\alpha^q}\right)\\
&\qquad+\left(A_{-2}-2\alpha A_{-1}+\alpha^2A_0+\alpha^qA_{q-2}+(1-\alpha^q)B_{q-2}\right)\bigwedge\nolimits_n\left(0,\frac{1}{1-\alpha^q}\right)\\
&\qquad+\sum_{k=1}^{q-1}\left(A_{k-2}-2\alpha A_{k-1}+\alpha^2A_k\right)\bigwedge\nolimits_n\left(-k\nu,\frac{1}{1-\alpha^q}\right)\\
&\qquad+\alpha^2B_{-(q-1)}\nabla^{-(q-1)\nu}(n+1)\left(\frac{1}{1-\alpha^q}\right)^n\\
&\qquad+\left(-2\alpha B_{-(q-1)}+\alpha^2B_{-(q-2)}\right)\nabla^{-(q-2)\nu}(n+1)\left(\frac{1}{1-\alpha^q}\right)^n\\
&\qquad+\sum_{k=3}^{q-1}\left(B_{-(q-k+2)}-2\alpha B_{-(q-k+1)}+\alpha^2B_{-(q-k)}\right)\nabla^{-(q-k)\nu}(n+1)\left(\frac{1}{1-\alpha^q}\right)^n\\
&\qquad+\left(B_{-2}-2\alpha B_{-1}+\alpha^2B_0+\alpha^qB_{q-2}\right)\nabla^{0}(n+1)\left(\frac{1}{1-\alpha^q}\right)^n\\
&\qquad+\sum_{k=1}^{q-1}\left(B_{k-2}-2\alpha B_{k-1}+\alpha^2B_k\right)\nabla^{k\nu}(n+1)\left(\frac{1}{1-\alpha^q}\right)^n.
\end{aligned}\tag{3.29}$$

Let $A_{-(q-k)}=(1-\alpha^q)(k+1)\alpha^{q-(k+1)}, 1\leqslant k\leqslant q-1$, then

$$A_{-(q-k+2)}-2\alpha A_{-(q-k+1)}+\alpha^2A_{-(q-k)}=0.\tag{3.30}$$

Let $A_k=(1-\alpha^q)(q-(k+1))\alpha^{-(k+1)}, 0\leqslant k\leqslant q-1$, then

$$A_{k-2}-2\alpha A_{k-1}+\alpha^2A_k=0.\tag{3.31}$$

Let $B_{-(q-k)}=(k+1)\alpha^{2q-(k+1)}, 1\leqslant k\leqslant q-1$, then

$$B_{-(q-k+2)}-2\alpha B_{-(q-k+1)}+\alpha^2B_{-(q-k)}=0.\tag{3.32}$$

Let $B_k = (2q-(k+1))\alpha^{-(k+1)}, 1 \leqslant k \leqslant q-1$, then when $k \geq 3$ we have

$$B_{k-2} - 2\alpha B_{k-1} + \alpha^2 B_k = 0,$$
$$A_{-2} - 2\alpha A_{-1} + \alpha^2 A_0 + \alpha^q A_{q-2} + (1-\alpha^q) B_{q-2} = 0. \tag{3.33}$$

Let us set $B_{-2} - 2\alpha B_{-1} + \alpha^2 B_0 + \alpha^q B_{q-2} = 0$, then $B_0 = q\alpha^{q-1}$. Thus we can rewrite

$$\begin{aligned}
&\left(\nabla^{2\nu} - 2\alpha\nabla^{\nu} + \alpha^2\nabla^0\right)x(n) \\
&\quad = 2(1-\alpha^q)\alpha^q \bigwedge\nolimits_n\left((q-1)\nu, \frac{1}{1-\alpha^q}\right) - (1-\alpha^q)\alpha^{q-1}\bigwedge\nolimits_n\left((q-2)\nu, \frac{1}{1-\alpha^q}\right) \\
&\qquad + 2\alpha^{2q}\nabla^{-(q-1)\nu}(n+1)\left(\frac{1}{1-\alpha^q}\right)^n - \alpha^{2q-1}\nabla^{-(q-2)\nu}(n+1)\left(\frac{1}{1-\alpha^q}\right)^n \\
&\qquad - 2\alpha^q\nabla^{\nu}(n+1)\left(\frac{1}{1-\alpha^q}\right)^n + \alpha^{q-1}\nabla^{2\nu}(n+1)\left(\frac{1}{1-\alpha^q}\right)^n.
\end{aligned} \tag{3.34}$$

But

$$\begin{aligned}
\nabla^{\mu}(n+1)\left(\frac{1}{1-\alpha^q}\right)^n &= \nabla\nabla^{-(1-\mu)}(n+1)\left(\frac{1}{1-\alpha^q}\right)^n \\
&= \nabla^{-(1-\mu)}\nabla(n+1)\left(\frac{1}{1-\alpha^q}\right)^n \\
&= (1-\alpha^q)\bigwedge\nolimits_n\left(1-\mu, \frac{1}{1-\alpha^q}\right) + \alpha^q\nabla^{-(1-\mu)}(n+1)\left(\frac{1}{1-\alpha^q}\right)^n.
\end{aligned} \tag{3.35}$$

Thus

$$\left(\nabla^{2\nu} - 2\alpha\nabla^{\nu} + \alpha^2\nabla^0\right)x(n) = 0. \tag{3.36}$$

Therefore, we get a nontrivial solution of $(\nabla^{2\nu} - 2\alpha\nabla^{\nu} + \alpha^2\nabla^0)x(n) = 0$. Substituting A_k and B_k into $x(n)$, we get

$$\begin{aligned}
x(n) &= \sum_{k=0}^{q-2}\alpha^k(q-k)\nabla^{-(k+1)\nu}\left(\alpha^q(n+1)\left(\frac{1}{1-\alpha^q}\right)^n + (1-\alpha^q)\left(\frac{1}{1-\alpha^q}\right)^n\right) \\
&\quad + \alpha^{-1}(q-1)\left[\alpha^q(n+1)\left(\frac{1}{1-\alpha^q}\right)^n + (1-\alpha^q)\left(\frac{1}{1-\alpha^q}\right)^n\right] + \alpha^{q-1}(n+1)\left(\frac{1}{1-\alpha^q}\right)^n \\
&\quad + \sum_{k=2}^{q-1}\alpha^{-k}(q-k)\nabla^{(k-1)\nu}\left(\alpha^q(n+1)\left(\frac{1}{1-\alpha^q}\right)^n + (1-\alpha^q)\left(\frac{1}{1-\alpha^q}\right)^n\right) \\
&= \sum_{k=0}^{q-2}\alpha^k(q-k)\nabla^{-(k+1)\nu}\nabla(n+1)\left(\frac{1}{1-\alpha^q}\right)^n \\
&\quad + \alpha^{-1}(q-1)\nabla(n+1)\left(\frac{1}{1-\alpha^q}\right)^n + \alpha^{q-1}\nabla^0(n+1)\left(\frac{1}{1-\alpha^q}\right)^n \\
&\quad + \sum_{k=2}^{q-1}\alpha^{-k}(q-k)\nabla^{(k-1)\nu}\nabla(n+1)\left(\frac{1}{1-\alpha^q}\right)^n \\
&= \sum_{k=-(q-1)}^{q-1}\alpha^k(q-|k|)\nabla^{1-(k+1)\nu}\left((n+1)\left(\frac{1}{1-\alpha^q}\right)^n\right).
\end{aligned} \tag{3.37}$$

Thus

$$\Psi(n) = \sum_{k=-(q-1)}^{q-1}\alpha^k(q-|k|)\nabla^{1-(k+1)\nu}\left((n+1)\left(\frac{1}{1-\alpha^q}\right)^n\right) \tag{3.38}$$

is a nontrivial solution of (3.3).

Theorem 3.2. *If $\alpha = \beta \neq 0$, then*

$$\Psi(n) = \sum_{k=-(q-1)}^{q-1}\alpha^k(q-|k|)\nabla^{1-(k+1)\nu}\left((n+1)\left(\frac{1}{1-\alpha^q}\right)^n\right) \tag{3.39}$$

is the solution of (3.3).

Besides, if $\alpha = \beta = 0$, then (3.3) *becomes*

$$\nabla^{2\nu}x(n) = 0, \tag{3.40}$$

and its solution is

$$x(n) = \begin{bmatrix} 2\nu \\ n \end{bmatrix}. \tag{3.41}$$

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