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Research Article

Fixed Points of Geraghty-Type Mappings in Various Generalized Metric Spaces

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Fixed point theorems for mappings satisfying Geraghty-type contractive conditions are proved in the frame of partial metric spaces, ordered partial metric spaces, and metric-type spaces. Examples are given showing that these results are proper extensions of the existing ones.

1. Introduction

Let $\mathcal{S}$ denote the class of real functions $\beta : [0,+\infty) \to [0,1)$ satisfying the condition

$$\beta(t_n) \longrightarrow 1 \quad \text{implies } t_n \longrightarrow 0. \tag{1.1}$$

An example of a function in $\mathcal{S}$ may be given by $\beta(t) = e^{-2t}$ for $t > 0$ and $\beta(0) \in [0,1)$. In an attempt to generalize the Banach contraction principle, M. Geraghty proved in 1973 the following.

Theorem 1.1 (see [1]). *Let (X,d) be a complete metric space, and let $f : X \to X$ be a self-map. Suppose that there exists $\beta \in \mathcal{S}$ such that*

$$d(fx, fy) \le \beta(d(x,y))d(x,y) \tag{1.2}$$

holds for all $x, y \in X$. Then f has a unique fixed point $z \in X$ and for each $x \in X$ the Picard sequence $\{f^n x\}$ converges to z when $n \to \infty$.

Recently, A. Amini-Harandi and H. Emami extended this result to partially ordered metric spaces as follows.

Theorem 1.2 (see [2]). *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Let $f : X \to X$ be an increasing self-map such that there exists $x_0 \in X$ with $x_0 \preceq fx_0$. Suppose that there exists $\beta \in \mathcal{S}$ such that* (1.2) *holds for all $x, y \in X$ with $x \succeq y$. Assume that either f is continuous or X is such that*

$$\text{if an increasing sequence } \{x_n\} \text{ in } X \text{ converges to } x \in X, \text{ then } x_n \preceq x \quad \forall n. \tag{1.3}$$

Then, f has a fixed point in X. If, moreover,

$$\text{for each } x, y \in X \text{ there exists } z \in X \text{ comparable with } x, y, \tag{1.4}$$

then the fixed point of f is unique.

Similar results were also obtained in [3, 4].

In recent years several authors have worked on domain theory in order to equip semantics domain with a notion of distance. In particular, Matthews [5] introduced the notion of a *partial metric space* as a part of the study of denotational semantics of dataflow networks, and obtained, among other results, a nice relationship between partial metric spaces and so-called weightable quasimetric spaces. He showed that the Banach contraction principle can be generalized to the partial metric context for applications in program verification. Subsequently, several authors (see, e.g., [6, 7]) studied fixed point theorems in partial metric spaces, as well as ordered partial metric spaces (see, e.g., [8, 9]).

Huang and Zhang introduced *cone metric spaces* in [10], replacing the set of real numbers by an ordered Banach space as the codomain for a metric. Cone metric spaces over normal cones inspired another generalization of metric spaces that were called *metric-type spaces* by Khamsi [11] (see also [12]; note that, in fact, spaces of this kind were used earlier under the name of b-spaces by Czerwik [13]).

In the present paper, we extend Theorems 1.1 and 1.2 to the frame of partial metric spaces, ordered partial metric spaces, and metric type spaces. Examples are given to distinguish new results from the existing ones.

2. Notation and Preliminary Results

2.1. Partial Metric Spaces

The following definitions and details can be seen in [5–9, 14, 15].

Definition 2.1. A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that, for all $x, y, z \in X$

(p_1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,

(p_2) $p(x, x) \leq p(x, y)$,

(p_3) $p(x, y) = p(y, x)$,

(p_4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X.

It is clear that, if $p(x, y) = 0$, then from (p_1) and (p_2) $x = y$. But if $x = y$, $p(x, y)$ may not be 0.

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$. A sequence $\{x_n\}$ in (X, p) converges to a point $x \in X$, with respect to τ_p, if $\lim_{n\to\infty} p(x, x_n) = p(x, x)$. This will be denoted as $x_n \to x$, $n \to \infty$ or $\lim_{n\to\infty} x_n = x$.

If p is a partial metric on X, then the function $p^s : X \times X \to \mathbb{R}^+$ given by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y) \tag{2.1}$$

is a metric on X. Furthermore, $\lim_{n\to\infty} p^s(x_n, x) = 0$ if and only if

$$p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_n, x_m). \tag{2.2}$$

Example 2.2. (1) A basic example of a partial metric space is the pair $(\mathbb{R}^+, p)$, where $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}^+$. The corresponding metric is

$$p^s(x, y) = 2\max\{x, y\} - x - y = |x - y|. \tag{2.3}$$

(2) If (X, d) is a metric space and $c \geq 0$ is arbitrary, then

$$p(x, y) = d(x, y) + c \tag{2.4}$$

defines a partial metric on X and the corresponding metric is $p^s(x, y) = 2d(x, y)$.

Other examples of partial metric spaces which are interesting from a computational point of view may be found in [5, 15].

Remark 2.3. Clearly, a limit of a sequence in a partial metric space need not be unique. Moreover, the function $p(\cdot, \cdot)$ need not be continuous in the sense that $x_n \to x$ and $y_n \to y$ implies $p(x_n, y_n) \to p(x, y)$. For example, if $X = [0, +\infty)$ and $p(x, y) = \max\{x, y\}$ for $x, y \in X$, then for $\{x_n\} = \{1\}$, $p(x_n, x) = x = p(x, x)$ for each $x \geq 1$ and so, for example, $x_n \to 2$ and $x_n \to 3$ when $n \to \infty$.

Definition 2.4 (see [8]). Let (X, p) be a partial metric space. Then one has the following

(1) A sequence $\{x_n\}$ in (X, p) is called a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists (and is finite).

(2) The space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

Lemma 2.5 (see [5, 6]). *Let (X, p) be a partial metric space.*

(a) *$\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p^s).*

(b) *The space (X, p) is complete if and only if the metric space (X, p^s) is complete.*

Definition 2.6. Let X be a nonempty set. Then $(X, p, \preceq)$ is called an ordered partial metric space if:

(i) (X, p) is a partial metric space and (ii) $(X, \preceq)$ is a partially ordered set.

The space $(X, p, \preceq)$ is called regular if the following holds: if $\{x_n\}$ is a nondecreasing sequence in X with respect to $\preceq$ such that $x_n \to x \in X$ as $n \to \infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.

2.2. Some Auxiliary Results

Assertions similar to the following lemma (see, e.g., [16]) were used (and proved) in the course of proofs of several fixed point results in various papers.

Lemma 2.7. *Let (X, d) be a metric space, and let $\{x_n\}$ be a sequence in X such that*

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{2.5}$$

If $\{x_{2n}\}$ is not a Cauchy sequence, then there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the following four sequences tend to ε when $k \to \infty$:

$$d(x_{2m_k}, x_{2n_k}), \qquad d(x_{2m_k}, x_{2n_k+1}), \qquad d(x_{2m_k-1}, x_{2n_k}), \qquad d(x_{2m_k-1}, x_{2n_k+1}). \tag{2.6}$$

As a corollary we obtain the following.

Lemma 2.8. *Let (X, p) be a partial metric space, and let $\{x_n\}$ be a sequence in X such that*

$$\lim_{n\to\infty} p(x_{n+1}, x_n) = 0. \tag{2.7}$$

If $\{x_{2n}\}$ is not a Cauchy sequence in (X, p), then there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the following four sequences tend to ε when $k \to \infty$:

$$p(x_{2m_k}, x_{2n_k}), \qquad p(x_{2m_k}, x_{2n_k+1}), \qquad p(x_{2m_k-1}, x_{2n_k}), \qquad p(x_{2m_k-1}, x_{2n_k+1}). \tag{2.8}$$

Proof. Suppose that $\{x_n\}$ is a sequence in (X, p) satisfying (2.7) such that $\{x_{2n}\}$ is not Cauchy. According to Lemma 2.5, it is not a Cauchy sequence in the metric space (X, p^s), either. Applying Lemma 2.7 we get the sequences

$$p^s(x_{2m_k}, x_{2n_k}), \qquad p^s(x_{2m_k}, x_{2n_k+1}), \qquad p^s(x_{2m_k-1}, x_{2n_k}), \qquad p^s(x_{2m_k-1}, x_{2n_k+1}) \tag{2.9}$$

tending to some $2\varepsilon > 0$ when $k \to \infty$. Using definition (2.1) of the associated metric and (2.7) (which by (p_2) implies that also $\lim_{n\to\infty} p(x_n, x_n) = 0$), we get that the sequences (2.8) tend to ε when $k \to \infty$. □

2.3. Property (P)

Let X be a nonempty set and $f : X \to X$ a self-map. As usual, we denote by $F(f)$ the set of fixed points of f. Following Jeong and Rhoades [17], we say that the map f has property (P) if it satisfies $F(f) = F(f^n)$ for each $n \in \mathbb{N}$. The proof of the following lemma is the same as in the metric case [17, Theorem 1.1].

Lemma 2.9. *Let (X, p) be a partial metric space, and let $f : X \to X$ be a selfmap such that $F(f) \neq \emptyset$. Then f has property (P) if*

$$p\left(fx, f^2x\right) \leq \lambda p(x, fx) \tag{2.10}$$

holds for some $\lambda \in (0,1)$ and either (i) *for all $x \in X$ or* (ii) *for all $x \neq fx$.*

2.4. Metric Type Spaces

Definition 2.10 (see [11]). Let X be a nonempty set, $K \geq 1$ a real number, and let a function $D : X \times X \to \mathbb{R}$ satisfy the following properties:

(a) $D(x, y) = 0$ if and only if $x = y$;

(b) $D(x, y) = D(y, x)$ for all $x, y \in X$;

(c) $D(x, z) \leq K(D(x, y) + D(y, z))$ for all $x, y, z \in X$.

Then (X, D, K) is called a *metric type space*.

Obviously, for $K = 1$, metric type space is simply a metric space.

The notions such as *convergent sequence*, *Cauchy sequence*, and *complete space* are defined in an obvious way.

A metric type space may satisfy some of the following additional properties:

(d) $D(x, z) \leq K(D(x, y_1) + D(y_1, y_2) + \cdots + D(y_n, z))$ for arbitrary points $x, y_1, y_2, \ldots, y_n, z \in X$;

(e) function D is continuous in two variables, that is,

$$x_n \longrightarrow x \text{ and } y_n \longrightarrow y \text{ (in } (X, D, K)) \text{ implies } D(x_n, y_n) \longrightarrow D(x, y). \tag{2.11}$$

(The last condition is in the theory of symmetric spaces usually called "property (H_E)".)

Condition (d) was used instead of (c) in the original definition of a metric type space by Khamsi [11].

Note that weaker version of property (e):

(e′) $x_n \to x$ and $y_n \to x$ (in (X, D, K)) implies that $D(x_n, y_n) \to 0$

is satisfied in an arbitrary metric type space. It can also be proved easily that the limit of a sequence in a metric type space is unique. Indeed, if $x_n \to x$ and $x_n \to y$ (in (X, D, K)) and $D(x, y) = \varepsilon > 0$, then

$$0 \leq D(x, y) \leq K\left(D(x, x_n) + D(x_n, y)\right) < K\left(\frac{\varepsilon}{2K} + \frac{\varepsilon}{2K}\right) = \varepsilon \tag{2.12}$$

for sufficiently large n, which is impossible.

3. Results

3.1. Results in Partial Metric Spaces

Theorem 3.1. *Let (X, p) be a complete partial metric space, and let $f : X \to X$ be a self-map. Suppose that there exists $\beta \in \mathcal{S}$ such that*

$$p(fx, fy) \leq \beta(p(x, y))p(x, y) \tag{3.1}$$

holds for all $x, y \in X$. Then f has a unique fixed point $z \in X$ and for each $x \in X$ the Picard sequence $\{f^n x\}$ converges to z when $n \to \infty$.

Proof. Let $x_1 \in X$ be arbitrary, and let $x_{n+1} = fx_n$ for $n \in \mathbb{N}$. Consider the following two cases:

(1) $p(x_{n_0+1}, x_{n_0}) = 0$ for some $n_0 \in \mathbb{N}$;

(2) $p(x_{n+1}, x_n) > 0$ for each $n \in \mathbb{N}$.

Case 1. Under this assumption we get that

$$p(x_{n_0+2}, x_{n_0+1}) = p(fx_{n_0+1}, fx_{n_0}) \leq \beta(p(x_{n_0+1}, x_{n_0}))p(x_{n_0+1}, x_{n_0}) = \beta(0) \cdot 0 = 0, \tag{3.2}$$

and it follows that $p(x_{n_0+2}, x_{n_0+1}) = 0$. By induction, we obtain that $p(x_{n+1}, x_n) = 0$ for all $n \geq n_0$ and so $x_n = x_{n_0}$ for all $n \geq n_0$. Hence, $\{x_n\}$ is a Cauchy sequence, converging to x_{n_0} which is a fixed point of f.

Case 2. We will prove first that in this case the sequence $p(x_{n+1}, x_n)$ is decreasing and tends to 0 as $n \to \infty$.

For each $n \in \mathbb{N}$ we have that

$$0 < p(x_{n+2}, x_{n+1}) = p(fx_{n+1}, fx_n) \leq \beta(p(x_{n+1}, x_n))p(x_{n+1}, x_n) < p(x_{n+1}, x_n). \tag{3.3}$$

Hence, $p(x_{n+1}, x_n)$ is decreasing and bounded from below, thus converging to some $q \geq 0$. Suppose that $q > 0$. Then, it follows from (3.3) that

$$\frac{p(x_{n+2}, x_{n+1})}{p(x_{n+1}, x_n)} \leq \beta(p(x_{n+1}, x_n)) < 1, \tag{3.4}$$

where from, passing to the limit when $n \to \infty$, we get that $\lim_{n\to\infty}\beta(p(x_{n+1}, x_n)) = 1$. Using property (1.1) of the function β, we conclude that $\lim_{n\to\infty}p(x_{n+1}, x_n) = 0$, that is, $q = 0$, a contradiction. Hence, $\lim_{n\to\infty}p(x_{n+1}, x_n) = 0$ is proved.

In order to prove that $\{x_n\}$ is a Cauchy sequence in (X, p), suppose the contrary. As was already proved, $p(x_{n+1}, x_n) \to 0$ as $n \to \infty$, and so, using (p_2), $p(x_n, x_n) \to 0$ as $n \to \infty$. Hence, using (2.1), we get that $p^s(x_{n+1}, x_n) \to 0$ as $n \to \infty$. Using Lemma 2.8, we obtain that there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the following four sequences tend to ε when $k \to \infty$:

$$p(x_{2m_k}, x_{2n_k}), \qquad p(x_{2m_k}, x_{2n_k+1}), \qquad p(x_{2m_k-1}, x_{2n_k}), \qquad p(x_{2m_k-1}, x_{2n_k+1}). \tag{3.5}$$

Putting in the contractive condition $x = x_{2m_k-1}$ and $y = x_{2n_k}$, it follows that

$$p(x_{2m_k}, x_{2n_k+1}) \le \beta(p(x_{2m_k-1}, x_{2n_k}))p(x_{2m_k-1}, x_{2n_k}) < p(x_{2m_k-1}, x_{2n_k}). \tag{3.6}$$

Hence,

$$\frac{p(x_{2m_k}, x_{2n_k+1})}{p(x_{2m_k-1}, x_{2n_k})} \le \beta(p(x_{2m_k-1}, x_{2n_k})) < 1 \tag{3.7}$$

and $\lim_{k\to\infty}\beta(p(x_{2m_k-1}, x_{2n_k})) = 1$. Since $\beta \in \mathcal{S}$, it follows that $\lim_{k\to\infty}p(x_{2m_k-1}, x_{2n_k}) = 0$, which is in contradiction with $\varepsilon > 0$.

Thus $\{x_n\}$ is a Cauchy sequence, both in (X, p) and in (X, p^s). Since these spaces are complete, it follows that sequence $\{x_n\}$ converges in the metric space (X, p^s), say $\lim_{n\to\infty}p^s(x_n, z) = 0$. Again from Lemma 2.5, we have

$$p(z, z) = \lim_{n\to\infty} p(x_n, z) = \lim_{n,m\to\infty} p(x_n, x_m). \tag{3.8}$$

Moreover since $\{x_n\}$ is a Cauchy sequence in the metric space (X, p^s), we have $\lim_{n,m\to\infty}p^s(x_n, x_m) = 0$ and so, by the definition of p^s, we have $\lim_{n,m\to\infty}p(x_n, x_m) = 0$. Then (3.8) implies that $p(z, z) = 0$ and

$$\lim_{n\to\infty} p(x_n, z) = p(z, z) = 0. \tag{3.9}$$

We will prove that z is a fixed point of f.

By (p_4), and using the contractive condition, we get that

$$\begin{aligned} p(z, fz) &\le p(z, x_{n+1}) + p(x_{n+1}, fz) - p(x_{n+1}, x_{n+1}) \\ &\le p(z, x_{n+1}) + p(fx_n, fz) \\ &\le p(z, x_{n+1}) + \beta(p(x_n, z))p(x_n, z) \\ &\le p(z, x_{n+1}) + p(x_n, z) \longrightarrow 0 + 0 = 0. \end{aligned} \tag{3.10}$$

Thus, $p(z, fz) = 0$ and $fz = z$.

Assume that $u \ne v$ are two fixed points of f. Then

$$0 < p(u, v) = p(fu, fv) \le \beta(p(u, v))p(u, v) < p(u, v), \tag{3.11}$$

a contradiction. Hence the fixed point of f is unique. The theorem is proved. □

Remark 3.2. It follows from Lemma 1, (viii) ⇔ (x) of the paper [18] of Jachymski, that under conditions of Theorem 3.1 there exists a continuous and nondecreasing function $\varphi : [0, +\infty) \to [0, +\infty)$ such that $\varphi(t) < t$ for all $t > 0$ and $p(fx, fy) \le \varphi(p(x, y))$ for all $x, y \in X$.

On the other hand, Romaguera [19] recently obtained a partial metric extension of the celebrated Boyd and Wong fixed point theorem, from which it follows that if (X, p) is a complete partial metric space and $f : X \to X$ is a map satisfying $p(fx, fy) \leq \varphi(p(x, y))$ for all $x, y \in X$, with a function φ with the aforementioned properties, then f has a unique fixed point. Hence, combining Jachymski's and Romaguera's results, an alternative proof of Theorem 3.1 is obtained.

Theorem 3.3. *If $f : X \to X$ satisfies conditions of Theorem 3.1, then it has property (P).*

Proof. By Theorem 3.1, the set of fixed points of f is a singleton, $F(f) = \{z\}$. Then also $z \in F(f^n)$ for all $n \in \mathbb{N}$. Let $v \in F(f^n)$ for some $n > 1$, and suppose that $z \neq v$, that is, $p(z, v) > 0$. Then

$$\begin{aligned} 0 &< p(z,v) \\ &= p\left(ff^{n-1}z, ff^{n-1}v\right) \\ &\leq \beta\left(p\left(f^{n-1}z, f^{n-1}v\right)\right)p\left(f^{n-1}z, f^{n-1}v\right) \\ &< p\left(f^{n-1}z, f^{n-1}v\right). \end{aligned} \tag{3.12}$$

We have that $f^{n-1}z \neq f^{n-1}v$ (otherwise $z = f^n z = f^n v = v$, which is excluded). It follows that

$$\begin{aligned} 0 &< p(z,v) \\ &< p\left(ff^{n-2}z, ff^{n-2}v\right) \\ &\leq \beta\left(p\left(f^{n-2}z, f^{n-2}v\right)\right)p\left(f^{n-2}z, f^{n-2}v\right) \\ &< p\left(f^{n-2}z, f^{n-2}v\right). \end{aligned} \tag{3.13}$$

Continuing, we obtain that

$$0 < p(z,v) < p\left(f^{n-1}x, f^{n-1}v\right) < \cdots < p(z,v), \tag{3.14}$$

a contradiction. Hence, $p(z, v) = 0$ and $z = v$, that is, $F(f) = F(f^n)$ for each $n \in \mathbb{N}$. □

Example 3.4. Let $X = [0,1]$, $d(x,y) = 2|x-y|$, $p(x,y) = \max\{x,y\}$, $\beta(t) = e^{-t}/(t+1)$ for $t > 0$ and $\beta(0) \in [0,1)$. The mapping $f : (X, d) \to (X, d)$ defined by $fx = (1/6)x$ does not satisfy conditions of Theorem 1.1. Indeed, take $x = 1$, $y = 0$ and obtain that

$$\begin{aligned} d(f1, f0) &= 2d\left(\frac{1}{6}, 0\right) = 2\left|\frac{1}{6} - 0\right| = \frac{1}{3}, \\ \beta(d(1,0))d(1,0) &= \beta(2) \cdot 2 = \frac{e^{-2}}{2+1} \cdot 2 = \frac{2e^{-2}}{3} < \frac{1}{3}. \end{aligned} \tag{3.15}$$

On the other hand, take $x, y \in X$ with, for example, $x \geq y$. Then

$$p(fx, fy) = p\left(\frac{1}{6}x, \frac{1}{6}y\right) = \frac{1}{6}x,$$
$$\beta(p(x,y))p(x,y) = \beta(x) \cdot x = \frac{e^{-x}}{x+1} \cdot x \geq \frac{1}{6}x, \tag{3.16}$$

since $e^{-x}/(x+1) \geq 1/2e > 1/6$ for $x \in [0,1]$. Hence, f satisfies conditions of Theorem 3.1 and thus has a unique fixed point ($z = 0$).

3.2. Results in Ordered Partial Metric Spaces

Theorem 3.5. *Let $(X, p, \preceq)$ be a complete ordered partial metric space. Let $f : X \to X$ be an increasing self-map (with respect to. $\preceq$) such that there exists $x_0 \in X$ with $x_0 \preceq fx_0$. Suppose that there exists $\beta \in \mathcal{S}$ such that* (3.1) *holds for all comparable $x, y \in X$. Assume that either f is continuous or X is regular. Then, f has a fixed point in X. The set $F(f)$ of fixed points of f is a singleton if and only if it is well ordered.*

Proof. Take $x_0 \in X$ with $x_0 \preceq fx_0$ and, using monotonicity of f, form the sequence $x_n = fx_{n-1}$ with

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq \cdots. \tag{3.17}$$

Since x_{n-1} and x_n are comparable we can apply contractive condition to obtain

$$p(x_{n+1}, x_n) = p(fx_n, fx_{n-1}) \leq \beta(p(x_{n-1}, x_n))p(x_{n-1}, x_n) \leq p(x_{n-1}, x_n). \tag{3.18}$$

Proceeding as in the proof of Theorem 3.1 we obtain that $\lim_{n\to\infty} p(x_{n+1}, x_n) = 0$, that $\{x_n\}$ is a Cauchy sequence in (X, p) (and in (X, p^s)). Thus, it converges (in p and in p^s) to a point $z \in X$ such that

$$p(z,z) = \lim_{n\to\infty} p(x_n, z) = \lim_{n,m\to\infty} p(x_n, x_m). \tag{3.19}$$

Also, it follows as in the proof of Theorem 3.1 that

$$\lim_{n\to\infty} p(x_n, z) = p(z,z) = 0. \tag{3.20}$$

We will prove that z is a fixed point of f.

(i) Suppose that $f : (X, p) \to (X, p)$ is continuous. We have, by (p_4),

$$p(z, fz) \leq p(z, x_{n+1}) + p(fx_n, fz). \tag{3.21}$$

Passing to the limit when $n \to \infty$ and using continuity of f we get that

$$p(z, fz) \leq p(z,z) + p(fz, fz) = p(fz, fz) \leq p(z, fz) \quad (\text{by } (p_2)). \tag{3.22}$$

It follows that $p(z, fz) = p(fz, fz)$. Since $z \preceq z$, using contractive condition, we get that

$$p(fz, fz) \leq \beta(p(z, z))p(z, z) = 0 \tag{3.23}$$

and so $p(z, fz) = 0$ and $fz = z$.

(ii) If (X, p) is regular, since $\{x_n\}$ is an increasing sequence tending to z, we have that $x_n \preceq z$ for each $n \in \mathbb{N}$. So we can apply (p_4) and contractive condition to obtain

$$\begin{aligned} p(z, fz) &\leq p(z, x_{n+1}) + p(fx_n, fz) \\ &\leq p(z, x_{n+1}) + \beta(p(x_n, z))p(x_n, z) \\ &\leq p(z, x_{n+1}) + p(x_n, z). \end{aligned} \tag{3.24}$$

Letting $n \to \infty$ we get

$$p(z, fz) \leq p(z, z) + p(z, z) = 0. \tag{3.25}$$

Hence, we again obtain that $fz = z$.

Let the set $F(f)$ of fixed points of f be well ordered, and suppose that there exist two distinct points $u, v \in F(f)$. Then these points are comparable, and we can apply the contractive condition to obtain

$$0 < p(u, v) = p(fu, fv) \leq \beta(p(u, v))p(u, v) < p(u, v), \tag{3.26}$$

a contradiction. Hence, the set $F(f)$ is a singleton. The converse is trivial. □

Example 3.6. Let $X = \{1, 2, 3\}$, and define the partial order $\preceq$ on X by

$$\preceq = \{(1,1), (2,2), (3,3), (1,3), (3,2), (1,2)\}. \tag{3.27}$$

Consider the function $f : X \to X$ given as $f = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \end{pmatrix}$, which is increasing with respect to $\preceq$.

Define first the metric d on X by $d(1,2) = d(1,3) = 1$, $d(2,3) = 1/2$, $d(x,x) = 0$ for $x \in X$ and $d(y,x) = d(x,y)$ for $x, y \in X$. Then (X, d) is a complete partially ordered metric space. The function $\beta : [0, +\infty) \to [0,1)$, defined by $\beta(t) = e^{-t}$, $t > 0$, and $\beta(0) \in [0,1)$, belongs to the class $\mathcal{S}$.

Take $x = 1$ and $y = 3$. Then

$$d(f1, f3) = d(3,2) = \frac{1}{2} > \frac{1}{e} = \beta(1) \cdot 1 = \beta(d(1,3))d(1,3). \tag{3.28}$$

Hence, conditions of Theorem 1.2 are not fulfilled and this theorem cannot be used to prove the existence of a fixed point of f.

Define now the partial metric p on X by $p(1,2) = p(1,3) = 1$, $p(2,3) = 1/9$, $p(x,y) = p(y,x)$ for $x, y \in X$; further $p(1,1) = 1$, $p(2,2) = 0$, and $p(3,3) = 1/10$ (it is easy to check conditions (p_1)–(p_4)). Let us check contractive condition (3.1) of Theorem 3.5:

$$\begin{gathered}
p(f1, f1) = p(3,3) = \frac{1}{10} < \frac{1}{e} = \beta(1) \cdot 1 = \beta(p(1,1))p(1,1), \\
p(f1, f2) = p(3,2) = \frac{1}{9} < \frac{1}{e} = \beta(1) \cdot 1 = \beta(p(1,2))p(1,2), \\
p(f1, f3) = p(3,2) = \frac{1}{9} < \frac{1}{e} = \beta(1) \cdot 1 = \beta(p(1,3))p(1,3), \\
p(f2, f2) = p(2,2) = 0 = 0 = \beta(0) \cdot 0 = \beta(p(2,2))p(2,2), \\
p(f2, f3) = p(2,2) = 0 < \frac{1}{9}e^{-1/9} = \beta\left(\frac{1}{9}\right) \cdot \frac{1}{9} = \beta(p(2,3))p(2,3), \\
p(f3, f3) = p(2,2) = 0 < \frac{1}{10}e^{-1/10} = \beta\left(\frac{1}{10}\right) \cdot \frac{1}{10} = \beta(p(3,3))p(3,3).
\end{gathered} \tag{3.29}$$

Hence, we can apply Theorem 3.5 to conclude that there is a unique fixed point of f (which is $z = 2$).

A variant of Theorem 1.2 which uses an altering function was obtained in [3, Theorems 2.2, 2.3]. Recall that $\psi : [0,+\infty) \to [0,+\infty)$ is called an *altering function* if it is continuous, increasing and $\psi^{-1}(0) = \{0\}$. We state a partial metric version of this result. The proof is omitted since it is similar to the previous one.

Theorem 3.7. *Let $(X, p, \preceq)$ be a complete ordered partial metric space. Let $f : X \to X$ be an increasing self-map (w.r.t. $\preceq$) such that there exists $x_0 \in X$ with $x_0 \preceq fx_0$. Suppose that there exist $\beta \in \mathcal{S}$ and an altering function ψ such that*

$$\psi(p(fx, fy)) \le \beta(p(x,y))\psi(p(x,y)) \tag{3.30}$$

holds for all comparable $x, y \in X$. Assume that either f is continuous or X is regular. Then, f has a fixed point in X. The set $F(f)$ of fixed points of f is a singleton if and only if it is well ordered.

3.3. Results in Metric Type Spaces

For the use in metric type spaces (with the given $K > 1$) we will consider the class of functions $\mathcal{S}_K$, where $\beta \in \mathcal{S}_K$ if $\beta : [0,+\infty) \to [0, 1/K)$ and has the property

$$\beta(t_n) \longrightarrow \frac{1}{K} \text{ implies } t_n \longrightarrow 0. \tag{3.31}$$

An example of a function in $\mathcal{S}_K$ is given by $\beta(t) = (1/K)e^{-t}$ for $t > 0$ and $\beta(0) \in [0, 1/K)$.

Theorem 3.8. *Let $K > 1$, and let (X, D, K) be a complete metric type space. Suppose that a mapping $f : X \to X$ satisfies the condition*

$$D(fx, fy) \leq \beta(D(x, y))D(x, y) \tag{3.32}$$

for all $x, y \in X$ and some $\beta \in \mathcal{S}_K$. Then f has a unique fixed point $z \in X$, and for each $x \in X$ the Picard sequence $\{f^n x\}$ converges to z in (X, D, K).

Proof. Using condition (3.32) it is easy to show that the fixed point of f in (X, D, K) is unique (if it exists) and that f is D-continuous in the sense that $x_n \to x$ implies that $fx_n \to fx$ in (X, D, K) (for details see [12]).

Let $x_0 \in X$ be arbitrary and $x_n = fx_{n-1}$ for $n \in \mathbb{N}$. If $x_{n_0+1} = x_{n_0}$ for some n_0, then it is easy to show that $x_n = x_{n_0}$ for $n \geq n_0$, and the proof is complete. Suppose that $x_n \neq x_{n+1}$ for all $n \geq 0$. Then, using (3.32), we get that

$$D(x_{n+1}, x_n) = D(fx_n, fx_{n-1}) \leq \beta(D(x_n, x_{n-1}))D(x_n, x_{n-1}) < \frac{1}{K}D(x_n, x_{n-1}). \tag{3.33}$$

By [12, Lemma 3.1], $\{x_n\}$ is a Cauchy sequence in (X, D, K). As this space is complete, $\{x_n\}$ converges to some $z \in X$ as $n \to \infty$. Obviously, also $fx_{n-1} = x_n \to z$ and continuity of f implies that $fx_{n-1} \to fz$. Since the limit of a sequence in a metric type space is unique, it follows that $fz = z$. □

Example 3.9. Let $X = \{0, 1, 3\}$ be equipped with the metric type function D given by $D(x, y) = (x - y)^2$ with $K = 2$. Consider the mapping $f : X \to X$ defined by $f(0) = 1$, $f1 = 1$, $f3 = 0$, and the function $\beta \in \mathcal{S}_K$ given by $\beta(t) = (1/2)e^{-t/9}$, $t > 0$, and $\beta(0) \in [0, 1/2)$. Then

$$\begin{aligned} D(f0, f1) &= D(1, 1) = 0 < \frac{1}{2}e^{-1/9} = \beta(1) \cdot 1 = \beta(D(0, 1))D(0, 1), \\ D(f0, f3) &= D(1, 0) = 1 < \frac{9}{2e} = \beta(9) \cdot 9 = \beta(D(0, 3))D(0, 3), \\ D(f1, f3) &= D(1, 0) = 1 < 2e^{-4/9} = \beta(4) \cdot 4 = \beta(D(1, 3))D(1, 3). \end{aligned} \tag{3.34}$$

Hence, f satisfies all the assumptions of Theorem 3.8 and thus it has a unique fixed point (which is $z = 1$).

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