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Research Article

Some New Volterra-Fredholm-Type Discrete Inequalities and Their Applications in the Theory of Difference Equations

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Some new Volterra-Fredholm-type discrete inequalities in two independent variables are established, which provide a handy tool in the study of qualitative and quantitative properties of solutions of certain difference equations. The established results extend some known results in the literature.

1. Introduction

In the research of solutions of certain differential and difference equations, if the solutions are unknown, then it is necessary to study their qualitative and quantitative properties such as boundedness, uniqueness, and continuous dependence on initial data. The Gronwall-Bellman inequality [1, 2] and its various generalizations which provide explicit bounds play a fundamental role in the research of this domain. Many such generalized inequalities (e.g., see [3–30] and the references therein) have been established in the literature including the known Ou-Liang's inequality [3]. In [8], Ma generalized the discrete version of Ou-Liang's inequality in two variables to Volterra-Fredholm form for the first time, which has proved to be very useful in the study of qualitative as well as quantitative properties of solutions of certain Volterra-Fredholm-type difference equations. But since then, few results on Volterra-Fredholm-type discrete inequalities have been established. Recent results in this direction include the work of Ma [9] to our knowledge. We notice, in the analysis of some certain Volterra-Fredholm-type difference equations with more complicated forms, that the bounds provided by the earlier inequalities are inadequate and it is necessary to seek some new Volterra-Fredholm-type discrete inequalities in order to obtain a diversity of desired results.

Our aim in this paper is to establish some new generalized Volterra-Fredholm-type discrete inequalities, which extend Ma's work in [9], and provide new bounds for unknown functions lying in these inequalities. We will illustrate the usefulness of the established results by applying them to study the boundedness, uniqueness, and continuous dependence on initial data of solutions of certain more complicated Volterra-Fredholm-type difference equations.

Throughout this paper, $\mathbb{R}$ denotes the set of real numbers $\mathbb{R}_+ = [0, \infty)$, and $\mathbb{Z}$ denotes the set of integers, while $\mathbb{N}_0$ denotes the set of nonnegative integers. Let $\Omega := ([m_0, M] \times [n_0, N]) \bigcap \mathbb{Z}^2$, where $m_0, n_0 \in \mathbb{Z}$ and $M, N \in \mathbb{Z} \bigcup \{\infty\}$ are two constants. $l_1, l_2 \in \mathbb{Z}$ are two constants, and $K_i > 0$, $i = 1, 2, 3, 4$, are all constants. If U is a lattice, then we denote the set of all $\mathbb{R}$-valued functions on U by $\wp(U)$ and denote the set of all $\mathbb{R}_+$-valued functions on U by $\wp_+(U)$. Finally, for a function $f \in \wp_+(U)$, we have $\sum_{s=m_0}^{m_1} f = 0$ provided $m_0 > m_1$.

2. Main Results

Lemma 2.1 (see [15]). *Assume that $a \geq 0$, $p \geq q \geq 0$, and $p \neq 0$ then for any $K > 0$*

$$a^{q/p} \leq \frac{q}{p} K^{(q-p)/p} a + \frac{p-q}{p} K^{q/p}. \tag{2.1}$$

Lemma 2.2. *Suppose that $u(m,n) \in \wp_+(\Omega)$, $b(s,t,m,n) \in \wp_+(\Omega^2)$, $\alpha \geq 0$ is a constant. If b is nondecreasing in the third variable, then, for $(m,n) \in \Omega$,*

$$u(m,n) \leq \alpha + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s,t,m,n) u(s,t) \tag{2.2}$$

implies that

$$u(m,n) \leq \alpha \exp\left\{ \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s,t,m,n) \right\}. \tag{2.3}$$

Lemma 2.3. *Suppose that $u(m,n), a(m,n), b(m,n) \in \wp_+(\Omega)$. If $a(m,n)$ is nondecreasing in the first variable, then, for $(m,n) \in \Omega$,*

$$u(m,n) \leq a(m,n) + \sum_{s=m_0}^{m-1} b(s,n) u(s,n) \tag{2.4}$$

implies that

$$u(m,n) \leq a(m,n) \prod_{s=m_0}^{m-1} [1 + b(s,n)]. \tag{2.5}$$

Remark 2.4. Lemma 2.3 is a direct variation of [19, Lemma 2.5(β_1)], and we note $a(m,n) \geq 0$ here.

Theorem 2.5. *Suppose that* $u(m,n), a(m,n) \in \wp_+(\Omega)$, $b_i(s,t,m,n), c_i(s,t,m,n) \in \wp_+(\Omega^2)$, $i = 1,2,\ldots,l_1$, $d_i(s,t,m,n), e_i(s,t,m,n) \in \wp_+(\Omega^2)$, $i = 1,2,\ldots,l_2$ *with* b_i, c_i, d_i, e_i *nondecreasing in the last two variables.* p, q_i, r_i *are nonnegative constants with* $p \geq q_i$, $p \geq r_i$, $i = 1,2,\ldots,l_1$, $p \neq 0$, *while* h_i, j_i *are nonnegative constants with* $p \geq h_i$, $p \geq j_i$, $i = 1,2,\ldots,l_2$. *If, for* $(m,n) \in \Omega$, $u(m,n)$ *satisfies*

$$u^p(m,n) \leq a(m,n) + \sum_{i=1}^{l_1} \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} \left[b_i(s,t,m,n) u^{q_i}(s,t) + \sum_{\xi=m_0}^{s} \sum_{\eta=n_0}^{t} c_i(\xi,\eta,m,n) u^{r_i}(\xi,\eta) \right] + \sum_{i=1}^{l_2} \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} \left[d_i(s,t,m,n) u^{h_i}(s,t) + \sum_{\xi=m_0}^{s} \sum_{\eta=n_0}^{t} e_i(\xi,\eta,m,n) u^{j_i}(\xi,\eta) \right], \tag{2.6}$$

then

$$u(m,n) \leq \left\{ a(m,n) + \frac{J(M,N)}{1-\mu(M,N)} C(m,n) \right\}^{1/p}, \tag{2.7}$$

provided that $\mu(M,N) < 1$, *where*

$$\begin{aligned} J(m,n) = {} & \sum_{i=1}^{l_1} \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} \Biggl\{ b_i(s,t,m,n) \left[\frac{q_i}{p} K_1^{(q_i-p)/p} a(s,t) + \frac{p-q_i}{p} K_1^{q_i/p} \right] \\ & + \sum_{\xi=m_0}^{s} \sum_{\eta=n_0}^{t} c_i(\xi,\eta,m,n) \left[\frac{r_i}{p} K_2^{(r_i-p)/p} a(\xi,\eta) + \frac{p-r_i}{p} K_2^{r_i/p} \right] \Biggr\} \\ & + \sum_{i=1}^{l_2} \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} \Biggl\{ d_i(s,t,m,n) \left[\frac{h_i}{p} K_3^{(h_i-p)/p} a(s,t) + \frac{p-h_i}{p} K_3^{h_i/p} \right] \\ & + \sum_{\xi=m_0}^{s} \sum_{\eta=n_0}^{t} e_i(\xi,\eta,m,n) \left[\frac{j_i}{p} K_4^{(j_i-p)/p} a(\xi,\eta) + \frac{p-j_i}{p} K_4^{j_i/p} \right] \Biggr\}, \end{aligned} \tag{2.8}$$

$$\mu(m,n) = \sum_{i=1}^{l_2} \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} \left\{ d_i(s,t,m,n) \frac{h_i}{p} K_3^{(h_i-p)/p} C(s,t) + \sum_{\xi=m_0}^{s} \sum_{\eta=n_0}^{t} e_i(\xi,\eta,m,n) \frac{j_i}{p} K_4^{(j_i-p)/p} C(\xi,\eta) \right\}, \tag{2.9}$$

$$C(m,n) = \exp\left\{ \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} B(s,t,m,n) \right\}, \tag{2.10}$$

$$B(s,t,m,n) = \sum_{i=1}^{l_1} \left[b_i(s,t,m,n) \frac{q_i}{p} K_1^{(q_i-p)/p} + \sum_{\xi=m_0}^{s} \sum_{\eta=n_0}^{t} c_i(\xi,\eta,m,n) \frac{r_i}{p} K_2^{(r_i-p)/p} \right]. \tag{2.11}$$

Proof. Denote

$$\begin{aligned} v(m,n) = {} & \sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left[b_i(s,t,m,n)u^{q_i}(s,t) + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} c_i(\xi,\eta,m,n)u^{r_i}(\xi,\eta)\right] \\ & + \sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left[d_i(s,t,m,n)u^{h_i}(s,t) + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} e_i(\xi,\eta,m,n)u^{j_i}(\xi,\eta)\right]. \end{aligned} \tag{2.12}$$

Then, we have

$$u(m,n) \le [a(m,n) + v(m,n)]^{1/p}, \tag{2.13}$$

and, furthermore, from Lemma 2.1 we have

$$\begin{aligned} u^{q_i}(m,n) \le [a(m,n)+v(m,n)]^{q_i/p} \le {} & \frac{q_i}{p}K_1^{(q_i-p)/p}[a(m,n)+v(m,n)] \\ & + \frac{p-q_i}{p}K_1^{q_i/p}, \quad i = 1,2,\dots,l_1, \\ u^{r_i}(m,n) \le [a(m,n)+v(m,n)]^{r_i/p} \le {} & \frac{r_i}{p}K_2^{(r_i-p)/p}[a(m,n)+v(m,n)] \\ & + \frac{p-r_i}{p}K_2^{r_i/p}, \quad i = 1,2,\dots,l_1, \\ u^{h_i}(m,n) \le [a(m,n)+v(m,n)]^{h_i/p} \le {} & \frac{h_i}{p}K_3^{(h_i-p)/p}[a(m,n)+v(m,n)] \\ & + \frac{p-h_i}{p}K_3^{h_i/p}, \quad i = 1,2,\dots,l_2, \\ u^{j_i}(m,n) \le [a(m,n)+v(m,n)]^{j_i/p} \le {} & \frac{j_i}{p}K_4^{(j_i-p)/p}[a(m,n)+v(m,n)] \\ & + \frac{p-j_i}{p}K_4^{j_i/p}, \quad i = 1,2,\dots,l_2. \end{aligned} \tag{2.14}$$

So

$$\begin{aligned}
v(m,n) \le \sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\Bigg\{& b_i(s,t,m,n)\left[\frac{q_i}{p}K_1^{(q_i-p)/p}(a(s,t)+v(s,t))+\frac{p-q_i}{p}K_1^{q_i/p}\right] \\
&+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}c_i(\xi,\eta,m,n)\left[\frac{r_i}{p}K_2^{(r_i-p)/p}(a(\xi,\eta)+v(\xi,\eta))+\frac{p-r_i}{p}K_2^{r_i/p}\right]\Bigg\} \\
+\sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\Bigg\{& d_i(s,t,m,n)\left[\frac{h_i}{p}K_3^{(h_i-p)/p}(a(s,t)+v(s,t))+\frac{p-h_i}{p}K_3^{h_i/p}\right] \\
&+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}e_i(\xi,\eta,m,n)\left[\frac{j_i}{p}K_4^{(j_i-p)/p}(a(\xi,\eta)+v(\xi,\eta))+\frac{p-j_i}{p}K_4^{j_i/p}\right]\Bigg\} \\
= H(m,n)+\sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\Bigg[& b_i(s,t,m,n)\frac{q_i}{p}K_1^{(q_i-p)/p}v(s,t) \\
&+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}c_i(\xi,\eta,m,n)\frac{r_i}{p}K_2^{(r_i-p)/p}v(\xi,\eta)\Bigg],
\end{aligned} \tag{2.15}$$

where $H(m,n) = J(m,n)+\sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\{d_i(s,t,m,n)(h_i/p)K_3^{(h_i-p)/p}v(s,t)+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}e_i(\xi,\eta,m,n)(j_i/p)K_4^{(j_i-p)/p}v(\xi,\eta)\}$ and $J(m,n)$ is defined in (2.8). Then, using that $H(m,n)$ is nondecreasing in every variable, we obtain

$$\begin{aligned}
v(m,n) &\le H(M,N)+\sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\Bigg[b_i(s,t,m,n)\frac{q_i}{p}K_1^{(q_i-p)/p}v(s,t) \\
&\qquad+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}c_i(\xi,\eta,m,n)\frac{r_i}{p}K_2^{(r_i-p)/p}v(\xi,\eta)\Bigg] \\
&\le H(M,N)+\sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\Bigg[b_i(s,t,m,n)\frac{q_i}{p}K_1^{(q_i-p)/p} \\
&\qquad+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}c_i(\xi,\eta,m,n)\frac{r_i}{p}K_2^{(r_i-p)/p}\Bigg]v(s,t) \\
&= H(M,N)+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}B(s,t,m,n)v(s,t),
\end{aligned} \tag{2.16}$$

where $B(s,t,m,n)$ is defined in (2.11).

Since $b_i(s,t,m,n)$, $c_i(s,t,m,n)$ are nondecreasing in the last two variables, then $B(s,t,m,n)$ is also nondecreasing in the last two variables, and by a suitable application of Lemma 2.2 we obtain

$$v(m,n) \leq H(M,N)\exp\left\{\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}B(s,t,m,n)\right\} = H(M,N)C(m,n), \tag{2.17}$$

where $C(m,n)$ is defined in (2.10). Furthermore, considering the definition of $H(m,n)$ and (2.17) we have

$$\begin{aligned}
H(M,N) &= J(M,N) + \sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left\{d_i(s,t,M,N)\frac{h_i}{p}K_3^{(h_i-p)/p}v(s,t)\right.\\
&\qquad\left. + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}e_i(\xi,\eta,M,N)\frac{j_i}{p}K_4^{(j_i-p)/p}v(\xi,\eta)\right\}\\
&\leq J(M,N) + \sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left\{d_i(s,t,M,N)\frac{h_i}{p}K_3^{(h_i-p)/p}H(M,N)C(s,t)\right.\\
&\qquad\left. + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}e_i(\xi,\eta,m_1,n_1)\frac{j_i}{p}K_4^{(j_i-p)/p}H(M,N)C(\xi,\eta)\right\}\\
&= J(M,N) + H(M,N)\sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left\{d_i(s,t,M,N)\frac{h_i}{p}K_3^{(h_i-p)/p}C(s,t)\right.\\
&\qquad\left. + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}e_i(\xi,\eta,M,N)\frac{j_i}{p}K_4^{(j_i-p)/p}C(\xi,\eta)\right\}\\
&= J(M,N) + H(M,N)\mu(M,N),
\end{aligned} \tag{2.18}$$

where $\mu(m,n)$ is defined in (2.9). Then,

$$H(M,N) \leq \frac{J(M,N)}{1-\mu(M,N)}. \tag{2.19}$$

Combining (2.17) and (2.19) we deduce

$$v(m,n) \leq \frac{J(M,N)}{1-\mu(M,N)}C(m,n). \tag{2.20}$$

Then, combining (2.13) and (2.20), we obtain the desired result. □

Corollary 2.6. *Suppose that* $g_{1i}(m,n), g_{2i}(m,n), b_{1i}(m,n), c_{1i}(m,n) \in \wp_+(\Omega)$, $i = 1,2,\ldots,l_1$ *with* g_{1i}, g_{2i} *nondecreasing in every variable.* $d_{1i}(m,n), e_{1i}(m,n) \in \wp_+(\Omega)$, $i = 1,2,\ldots,l_2$. $u(m,n), a(m,n), p, q_i, r_i, h_i, j_i$ *are defined as in Theorem 2.5. If, for* $(m,n) \in \Omega$, $u(m,n)$ *satisfies*

$$\begin{aligned} u^p(m,n) \le a(m,n) &+ \sum_{i=1}^{l_1} g_{1i}(m,n) \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} \left[b_{1i}(s,t) u^{q_i}(s,t) + \sum_{\xi=m_0}^{s} \sum_{\eta=n_0}^{t} c_{1i}(\xi,\eta) u^{r_i}(\xi,\eta) \right] \\ &+ \sum_{i=1}^{l_2} g_{2i}(m,n) \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} \left[d_{1i}(s,t) u^{h_i}(s,t) + \sum_{\xi=m_0}^{s} \sum_{\eta=n_0}^{t} e_{1i}(\xi,\eta) u^{j_i}(\xi,\eta) \right], \end{aligned} \tag{2.21}$$

then

$$u(m,n) \le \left\{ a(m,n) + \frac{J(M,N)}{1-\mu(M,N)} C(M,N) \right\}^{1/p}, \tag{2.22}$$

provided that $\mu(M,N) < 1$, *where*

$$\begin{aligned} J(m,n) = &\sum_{i=1}^{l_1} g_{1i}(m,n) \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} \left\{ b_{1i}(s,t) \left[\frac{q_i}{p} K_1^{(q_i-p)/p} a(s,t) + \frac{p-q_i}{p} K_1^{q_i/p} \right] \right. \\ &\left. + \sum_{\xi=m_0}^{s} \sum_{\eta=n_0}^{t} c_{1i}(\xi,\eta) \left[\frac{r_i}{p} K_2^{(r_i-p)/p} a(\xi,\eta) + \frac{p-r_i}{p} K_2^{r_i/p} \right] \right\} \\ &+ \sum_{i=1}^{l_2} g_{2i}(m,n) \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} \left\{ d_{1i}(s,t) \left[\frac{h_i}{p} K_3^{(h_i-p)/p} a(s,t) + \frac{p-h_i}{p} K_3^{h_i/p} \right] \right. \\ &\left. + \sum_{\xi=m_0}^{s} \sum_{\eta=n_0}^{t} e_{1i}(\xi,\eta) \left[\frac{j_i}{p} K_4^{(j_i-p)/p} a(\xi,\eta) + \frac{p-j_i}{p} K_4^{j_i/p} \right] \right\}, \\ \mu(m,n) = &\sum_{i=1}^{l_2} \left\{ g_{2i}(m,n) \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} \left[d_{1i}(s,t) \frac{h_i}{p} K_3^{(h_i-p)/p} C(s,t) \right. \right. \\ &\left. \left. + \sum_{\xi=m_0}^{s} \sum_{\eta=n_0}^{t} e_{1i}(\xi,\eta) \frac{j_i}{p} K_4^{(j_i-p)/p} C(\xi,\eta) \right] \right\}, \\ C(m,n) = &\exp\left\{ \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} B(s,t,m,n) \right\}, \\ B(s,t,m,n) = &\sum_{i=1}^{l_1} g_{1i}(m,n) \left[b_{1i}(s,t) \frac{q_i}{p} K_1^{(q_i-p)/p} + \sum_{\xi=m_0}^{s} \sum_{\eta=n_0}^{t} c_{1i}(\xi,\eta) \frac{r_i}{p} K_2^{(r_i-p)/p} \right]. \end{aligned} \tag{2.23}$$

The proof of Corollary 2.6 can be completed by setting $b_i(s,t,m,n) = g_{1i}(m,n)b_{1i}(s,t)$, $c_i(s,t,m,n) = g_{1i}(m,n)c_{1i}(s,t)$, $d_i(s,t,m,n) = g_{2i}(m,n)d_{1i}(s,t)$, $e_i(s,t,m,n) = g_{2i}(m,n)e_{1i}(s,t)$ in Theorem 2.5.

Corollary 2.7. *Suppose that $u(m,n), a(m,n), b_i(s,t,m,n), c_i(s,t,m,n), d_i(s,t,m,n), e_i(s,t,m,n)$ are defined as in Theorem 2.5. If, for $(m,n) \in \Omega$, $u(m,n)$ satisfies*

$$\begin{aligned} u(m,n) \le a(m,n) &+ \sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left[b_i(s,t,m,n)u(s,t) + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} c_i(\xi,\eta,m,n)u(\xi,\eta)\right] \\ &+ \sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left[d_i(s,t,m,n)u(s,t) + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} e_i(\xi,\eta,m,n)u(\xi,\eta)\right], \end{aligned} \tag{2.24}$$

then

$$u(m,n) \le a(m,n) + \frac{J(M,N)}{1-\mu(M,N)}C(m,n), \tag{2.25}$$

provided that $\mu(M,N) < 1$, where

$$\begin{aligned} J(m,n) &= \sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left\{b_i(s,t,m,n)a(s,t) + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} c_i(\xi,\eta,m,n)a(\xi,\eta)\right\} \\ &\quad + \sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left\{d_i(s,t,m,n)a(s,t) + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} e_i(\xi,\eta,m,n)a(\xi,\eta)\right\}, \\ \mu(m,n) &= \sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left\{d_i(s,t,m,n)C(s,t) + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} e_i(\xi,\eta,m,n)C(\xi,\eta)\right\}, \\ C(m,n) &= \exp\left\{\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} B(s,t,m,n)\right\}, \\ B(s,t,m,n) &= \sum_{i=1}^{l_1}\left[b_i(s,t,m,n) + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} c_i(\xi,\eta,m,n)\right]. \end{aligned} \tag{2.26}$$

Theorem 2.8. *Suppose that $w(m,n) \in \wp_+(\Omega)$, $u, a, b_i, c_i, d_i, e_i, p, q_i, r_i, h_i, j_i$ are defined as in Theorem 2.5. Furthermore, assume that $a(m,n)$ is nondecreasing in the first variable. If, for $(m,n) \in \Omega$, $u(m,n)$ satisfies*

$$
\begin{aligned}
u^p(m,n) \le a(m,n) &+ \sum_{s=m_0}^{m-1} w(s,n)u^p(m,n) \\
&+ \sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left[b_i(s,t,m,n)u^{q_i}(s,t) + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} c_i(\xi,\eta,m,n)u^{r_i}(\xi,\eta)\right] \\
&+ \sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left[d_i(s,t,m,n)u^{h_i}(s,t) + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} e_i(\xi,\eta,m,n)u^{j_i}(\xi,\eta)\right],
\end{aligned}
\tag{2.27}
$$

then

$$
u(m,n) \le \left\{\left[a(m,n) + \frac{\overline{J}(M,N)}{1-\overline{\mu}(M,N)}\overline{C}(m,n)\right]\overline{w}(m,n)\right\}^{1/p},
\tag{2.28}
$$

provided that $\overline{\mu}(M,N) < 1$, *where*

$$
\begin{aligned}
\overline{J}(m,n) = &\sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\Bigg\{\overline{b}_i(s,t,m,n)\left[\frac{q_i}{p}K_1^{(q_i-p)/p}a(s,t) + \frac{p-q_i}{p}K_1^{q_i/p}\right] \\
&\qquad + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}\overline{c}_i(\xi,\eta,m,n)\left[\frac{r_i}{p}K_2^{(r_i-p)/p}a(\xi,\eta) + \frac{p-r_i}{p}K_2^{r_i/p}\right]\Bigg\} \\
&+ \sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\Bigg\{\overline{d}_i(s,t,m,n)\left[\frac{h_i}{p}K_3^{(h_i-p)/p}a(s,t) + \frac{p-h_i}{p}K_3^{h_i/p}\right] \\
&\qquad + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}\overline{e}_i(\xi,\eta,m,n)\left[\frac{j_i}{p}K_4^{(j_i-p)/p}a(\xi,\eta) + \frac{p-j_i}{p}K_4^{j_i/p}\right]\Bigg\},
\end{aligned}
\tag{2.29}
$$

$$
\begin{aligned}
&\overline{b}_i(s,t,m,n) = b_i(s,t,m,n)(\overline{w}(s,t))^{q_i/p}, \\
&\overline{c}_i(s,t,m,n) = c_i(s,t,m,n)(\overline{w}(s,t))^{r_i/p}, \quad i = 1,2,\ldots,l_1,
\end{aligned}
\tag{2.30}
$$

$$
\begin{aligned}
&\overline{d}_i(s,t,m,n) = d_i(s,t,m,n)(\overline{w}(s,t))^{h_i/p}, \\
&\overline{e}_i(s,t,m,n) = e_i(s,t,m,n)(\overline{w}(s,t))^{j_i/p}, \quad i = 1,2,\ldots,l_2,
\end{aligned}
\tag{2.31}
$$

$$\overline{w}(m,n)=\prod_{s=m_0}^{m-1}[1+w(s,n)],$$

$$\overline{\mu}(m,n)=\sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left\{\overline{d}_i(s,t,m,n)\frac{h_i}{p}K_3^{(h_i-p)/p}\overline{C}(s,t)\right.$$
$$\left.+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}\overline{e}_i(\xi,\eta,m,n)\frac{j_i}{p}K_4^{(j_i-p)/p}\overline{C}(\xi,\eta)\right\} \tag{2.32}$$

$$C(m,n)=\exp\left\{\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\overline{B}(s,t,m,n)\right\}, \tag{2.33}$$

$$\overline{B}(s,t,m,n)=\sum_{i=1}^{l_1}\left[\overline{b}_i(s,t,m,n)\frac{q_i}{p}K_1^{(q_i-p)/p}+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}\overline{c}_i(\xi,\eta,m,n)\frac{r_i}{p}K_2^{(r_i-p)/p}\right]. \tag{2.34}$$

Proof. Denote

$$z(m,n)=a(m,n)+\sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left[b_i(s,t,m,n)u^{q_i}(s,t)+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}c_i(\xi,\eta,m,n)u^{r_i}(\xi,\eta)\right]$$
$$+\sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left[d_i(s,t,m,n)u^{h_i}(s,t)+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}e_i(\xi,\eta,m,n)u^{j_i}(\xi,\eta)\right]. \tag{2.35}$$

Then, we have

$$u^p(m,n)\le z(m,n)+\sum_{s=m_0}^{m-1}w(s,n)u^p(m,n). \tag{2.36}$$

Obviously $z(m,n)$ is nondecreasing in the first variable. So by Lemma 2.3 we obtain

$$u^p(m,n)\le z(m,n)\prod_{s=m_0}^{m-1}[1+w(s,n)]=z(m,n)\overline{w}(m,n), \tag{2.37}$$

where $\overline{w}(m,n)=\prod_{s=m_0}^{m-1}[1+w(s,n)]$. Define

$$v(m,n)=\sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left[b_i(s,t,m,n)u^{q_i}(s,t)+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}c_i(\xi,\eta,m,n)u^{r_i}(\xi,\eta)\right]$$
$$+\sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left[d_i(s,t,m,n)u^{h_i}(s,t)+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}e_i(\xi,\eta,m,n)u^{j_i}(\xi,\eta)\right]. \tag{2.38}$$

Then,

$$u(m,n) \le \left[(a(m,n)+v(m,n))\overline{w}(m,n)\right]^{1/p}, \tag{2.39}$$

and, furthermore, by (2.39) and Lemma 2.1 we have

$$\begin{aligned}
v(m,n) \le{}& \sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\Bigg\{ b_i(s,t,m,n)\left[(a(s,t)+v(s,t))\overline{w}(s,t)\right]^{q_i/p} \\
&\qquad + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} c_i(\xi,\eta,m,n)\left[(a(\xi,\eta)+v(\xi,\eta))\overline{w}(\xi,\eta)\right]^{r_i/p}\Bigg\} \\
&+ \sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\Bigg\{ d_i(s,t,m,n)\left[(a(s,t)+v(s,t))\overline{w}(s,t)\right]^{h_i/p} \\
&\qquad + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} e_i(\xi,\eta,m,n)\left[(a(\xi,\eta)+v(\xi,\eta))\overline{w}(\xi,\eta)\right]^{j_i/p}\Bigg\} \\
\le{}& \sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\Bigg\{ b_i(s,t,m,n)(\overline{w}(s,t))^{q_i/p}\left[\frac{q_i}{p}K_1^{(q_i-p)/p}(a(s,t)+v(s,t)) + \frac{p-q_i}{p}K_1^{q_i/p}\right] \\
&\qquad + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} c_i(\xi,\eta,m,n)(\overline{w}(\xi,\eta))^{r_i/p} \\
&\qquad \times\left[\frac{r_i}{p}K_2^{(r_i-p)/p}(a(\xi,\eta)+v(\xi,\eta)) + \frac{p-r_i}{p}K_2^{r_i/p}\right]\Bigg\} \\
&+ \sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\Bigg\{ d_i(s,t,m,n)(\overline{w}(s,t))^{h_i/p}\left[\frac{h_i}{p}K_3^{(q_i-p)/p}(a(s,t)+v(s,t)) + \frac{p-h_i}{p}K_3^{h_i/p}\right] \\
&\qquad + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} e_i(\xi,\eta,m,n)(\overline{w}(\xi,\eta))^{j_i/p} \\
&\qquad \times\left[\frac{j_i}{p}K_4^{j_i-p/p}(a(\xi,\eta)+v(\xi,\eta)) + \frac{p-j_i}{p}K_4^{j_i/p}\right]\Bigg\} \\
={}& \sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\Bigg\{ \overline{b}_i(s,t,m,n)\left[\frac{q_i}{p}K_1^{(q_i-p)/p}(a(s,t)+v(s,t)) + \frac{p-q_i}{p}K_1^{q_i/p}\right] \\
&\qquad + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} \overline{c}_i(\xi,\eta,m,n)\left[\frac{r_i}{p}K_2^{(r_i-p)/p}(a(\xi,\eta)+v(\xi,\eta)) + \frac{p-r_i}{p}K_2^{r_i/p}\right]\Bigg\}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left\{\overline{d}_i(s,t,m,n)\left[\frac{h_i}{p}K_3^{(q_i-p)/p}(a(s,t)+v(s,t))+\frac{p-h_i}{p}K_3^{h_i/p}\right]\right.\\
&\qquad\left.+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}\overline{e}_i(\xi,\eta,m,n)\left[\frac{j_i}{p}K_4^{(j_i-p)/p}(a(\xi,\eta)+v(\xi,\eta))+\frac{p-j_i}{p}K_4^{j_i/p}\right]\right\}\\
&=\overline{H}(m,n)+\sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left[\overline{b}_i(s,t,m,n)\frac{q_i}{p}K_1^{(q_i-p)/p}v(s,t)\right.\\
&\qquad\left.+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}\overline{c}_i(\xi,\eta,m,n)\frac{r_i}{p}K_2^{(r_i-p)/p}v(\xi,\eta)\right],
\end{aligned}
\tag{2.40}
$$

where $\overline{H}(m,n)=\overline{J}(m,n)+\sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\{\overline{d}_i(s,t,m,n)(h_i/p)K_3^{(h_i-p)/p}v(s,t)+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}\overline{e}_i(\xi,\eta,m,n)j_i/pK_4^{(j_i-p)/p}v(\xi,\eta)\}$, and $\overline{J}(m,n)$, $\overline{b}_i$, $\overline{c}_i$, $\overline{d}_i$, $\overline{e}_i$ are defined in (2.29)–(2.31) respectively.

Similar to the process of (2.15)–(2.20) we deduce

$$
v(m,n)\le\frac{\overline{J}(M,N)}{1-\overline{\mu}(M,N)}\overline{C}(m,n),
\tag{2.41}
$$

where $\overline{\mu}(m,n)$, $\overline{C}(m,n)$ are defined in (2.32) and (2.33).

Combining (2.39) and (2.41), we get the desired result. □

Remark 2.9. If we set $b_i(s,t,m,n)=g_{1i}(m,n)b_{1i}(s,t)$, $c_i(s,t,m,n)=g_{1i}(m,n)c_{1i}(s,t)$, $d_i(s,t,m,n)=g_{2i}(m,n)d_{1i}(s,t)$, $e_i(s,t,m,n)=g_{2i}(m,n)e_{1i}(s,t)$ or set $p=q_i=r_i=h_i=j_i=1$ in Theorem 2.8, then immediately we get two corollaries which are similar to Corollaries 2.6 and 2.7, and we omit the details for them.

Theorem 2.10. *Suppose that u, a, b_i, c_i, d_i, e_i, p, q_i, r_i, h_i, j_i are defined as in Theorem 2.5. $L_i,T_i:\Omega\times\mathbb{R}_+\to\mathbb{R}_+$, $i=1,2,\ldots,l_2$, satisfies $0\le L_i(m,n,u)-L_i(m,n,v)\le T_i(m,n,v)(u-v)$ for $u\ge v\ge 0$. If, for $(m,n)\in\Omega$, $u(m,n)$ satisfies*

$$
\begin{aligned}
u^p(m,n)\le a(m,n)&+\sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left[b_i(s,t,m,n)u^{q_i}(s,t)+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}c_i(\xi,\eta,m,n)u^{r_i}(\xi,\eta)\right]\\
&+\sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left[d_i(s,t,m,n)L_i\left(s,t,u^{h_i}(s,t)\right)+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}e_i(\xi,\eta,m,n)u^{j_i}(\xi,\eta)\right],
\end{aligned}
\tag{2.42}
$$

then

$$u(m,n) \le \left\{ a(m,n) + \frac{\widehat{J}(M,N)}{1-\widehat{\mu}(M,N)} \widehat{C}(m,n) \right\}^{1/p}, \tag{2.43}$$

provided that $\widehat{\mu}(M,N) < 1$, *where*

$$\begin{aligned}
\widehat{J}(m,n) = {} & \sum_{i=1}^{l_1} \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} \left\{ b_i(s,t,m,n) \left[\frac{q_i}{p} K_1^{(q_i-p)/p} a(s,t) + \frac{p-q_i}{p} K_1^{q_i/p} \right] \right. \\
& \left. + \sum_{\xi=m_0}^{s} \sum_{\eta=n_0}^{t} c_i(\xi,\eta,m,n) \left[\frac{r_i}{p} K_2^{(r_i-p)/p} a(\xi,\eta) + \frac{p-r_i}{p} K_2^{r_i/p} \right] \right\} \\
& + \sum_{i=1}^{l_2} \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} \left\{ d_i(s,t,m,n) L_i \left[s,t, \frac{h_i}{p} K_3^{(h_i-p)/p} a(s,t) + \frac{p-h_i}{p} K_3^{h_i/p} \right] \right. \\
& \left. + \sum_{\xi=m_0}^{s} \sum_{\eta=n_0}^{t} e_i(\xi,\eta,m,n) \left[\frac{j_i}{p} K_4^{(j_i-p)/p} a(\xi,\eta) + \frac{p-j_i}{p} K_4^{j_i/p} \right] \right\},
\end{aligned} \tag{2.44}$$

$$\begin{aligned}
\widehat{\mu}(m,n) = {} & \sum_{i=1}^{l_2} \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} \left\{ \widehat{d}_i(s,t,m,n) \frac{h_i}{p} K_3^{(h_i-p)/p} C(s,t) \right. \\
& \left. + \sum_{\xi=m_0}^{s} \sum_{\eta=n_0}^{t} e_i(\xi,\eta,m,n) \frac{j_i}{p} K_4^{(j_i-p)/p} C(\xi,\eta) \right\},
\end{aligned} \tag{2.45}$$

$$\widehat{d}_i(s,t,m,n) = d_i(s,t,m,n) T_i \left[s,t, \frac{h_i}{p} K_3^{(h_i-p)/p} a(s,t) + \frac{p-h_i}{p} K_3^{h_i/p} \right], \quad i = 1,2,\ldots,l_2, \tag{2.46}$$

$$\widehat{C}(m,n) = \exp\left\{ \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} \widehat{B}(s,t,m,n) \right\}, \tag{2.47}$$

$$\widehat{B}(s,t,m,n) = \sum_{i=1}^{l_1} \left[b_i(s,t,m,n) \frac{q_i}{p} K_1^{(q_i-p)/p} + \sum_{\xi=m_0}^{s} \sum_{\eta=n_0}^{t} c_i(\xi,\eta,m,n) \frac{r_i}{p} K_2^{(r_i-p)/p} \right]. \tag{2.48}$$

Proof. Denote

$$\begin{aligned}
v(m,n) &= \sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left[b_i(s,t,m,n)u^{q_i}(s,t) + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} c_i(\xi,\eta,m,n)u^{r_i}(\xi,\eta)\right] \\
&\quad + \sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left[d_i(s,t,m,n)L_i\Big(s,t,u^{h_i}(s,t)\Big) + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} e_i(\xi,\eta,m,n)u^{j_i}(\xi,\eta)\right].
\end{aligned} \tag{2.49}$$

Then

$$u(m,n) \le [a(m,n) + v(m,n)]^{1/p}, \tag{2.50}$$

and, furthermore, from Lemma 2.1 we have

$$\begin{aligned}
v(m,n) \le& \sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\Bigg\{b_i(s,t,m,n)\left[\frac{q_i}{p}K_1^{(q_i-p)/p}(a(s,t)+v(s,t)) + \frac{p-q_i}{p}K_1^{q_i/p}\right] \\
&\qquad + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} c_i(\xi,\eta,m,n)\left[\frac{r_i}{p}K_2^{(r_i-p)/p}(a(\xi,\eta)+v(\xi,\eta)) + \frac{p-r_i}{p}K_2^{r_i/p}\right]\Bigg\} \\
&+ \sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\Bigg\{d_i(s,t,m,n)L_i\left[s,t,\frac{h_i}{p}K_3^{(h_i-p)/p}(a(s,t)+v(s,t)) + \frac{p-h_i}{p}K_3^{h_i/p}\right] \\
&\qquad + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} e_i(\xi,\eta,m,n)\left[\frac{j_i}{p}K_4^{(j_i-p)/p}(a(\xi,\eta)+v(\xi,\eta)) + \frac{p-j_i}{p}K_4^{j_i/p}\right]\Bigg\} \\
\le& \sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\Bigg\{b_i(s,t,m,n)\left[\frac{q_i}{p}K_1^{(q_i-p)/p}(a(s,t)+v(s,t)) + \frac{p-q_i}{p}K_1^{q_i/p}\right] \\
&\qquad + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} c_i(\xi,\eta,m,n)\left[\frac{r_i}{p}K_2^{(r_i-p)/p}(a(\xi,\eta)+v(\xi,\eta)) + \frac{p-r_i}{p}K_2^{r_i/p}\right]\Bigg\} \\
&+ \sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\Bigg\{d_i(s,t,m,n)\Bigg\{L_i\left[s,t,\frac{h_i}{p}K_3^{(h_i-p)/p}(a(s,t)+v(s,t)) + \frac{p-h_i}{p}K_3^{h_i/p}\right]
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad - L_i\left[s,t,\frac{h_i}{p}K_3^{(h_i-p)/p}a(s,t)+\frac{p-h_i}{p}K_3^{h_i/p}\right]\\
&\qquad\qquad + L_i\left[s,t,\frac{h_i}{p}K_3^{(h_i-p)/p}a(s,t)+\frac{p-h_i}{p}K_3^{h_i/p}\right]\Bigg\}\\
&\qquad + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}e_i(\xi,\eta,m,n)\left[\frac{j_i}{p}K_4^{(j_i-p)/p}(a(\xi,\eta)+v(\xi,\eta))+\frac{p-j_i}{p}K_4^{j_i/p}\right]\Bigg\}\\
&\le \sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\Bigg\{b_i(s,t,m,n)\left[\frac{q_i}{p}K_1^{(q_i-p)/p}(a(s,t)+v(s,t))+\frac{p-q_i}{p}K_1^{q_i/p}\right]\\
&\qquad + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}c_i(\xi,\eta,m,n)\left[\frac{r_i}{p}K_2^{(r_i-p)/p}(a(\xi,\eta)+v(\xi,\eta))+\frac{p-r_i}{p}K_2^{r_i/p}\right]\Bigg\}\\
&\quad + \sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\Bigg\{d_i(s,t,m,n)\bigg\{T_i\left[s,t,\frac{h_i}{p}K_3^{(h_i-p)/p}a(s,t)+\frac{p-h_i}{p}K_3^{h_i/p}\right]\frac{h_i}{p}K_3^{(h_i-p)/p}\\
&\qquad\qquad \times v(s,t)+L_i\left[s,t,\frac{h_i}{p}K_3^{(h_i-p)/p}a(s,t)+\frac{p-h_i}{p}K_3^{h_i/p}\right]\bigg\}\\
&\qquad + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}e_i(\xi,\eta,m,n)\left[\frac{j_i}{p}K_4^{(j_i-p)/p}(a(\xi,\eta)+v(\xi,\eta))+\frac{p-j_i}{p}K_4^{j_i/p}\right]\Bigg\}\\
&= \widehat{H}(m,n)+\sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\Bigg[b_i(s,t,m,n)\frac{q_i}{p}K_1^{(q_i-p)/p}v(s,t)\\
&\qquad + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}c_i(\xi,\eta,m,n)\frac{r_i}{p}K_2^{(r_i-p)/p}v(\xi,\eta)\Bigg],
\end{aligned}
\tag{2.51}
$$

where $\widehat{H}(m,n)=\widehat{J}(m,n)+\sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\{\widehat{d}_i(s,t,m,n)(h_i/p)K_3^{(h_i-p)/p}v(s,t)+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}e_i(\xi,\eta,m,n)(j_i/p)K_4^{(j_i-p)/p}v(\xi,\eta)\}$ and $\widehat{J}(m,n),\widehat{d}_i(s,t,m,n)$ are defined in (2.44) and (2.46) respectively.

Similar to the process of (2.15)–(2.20) we deduce

$$
v(m,n)\le\frac{\widehat{J}(M,N)}{1-\widehat{\mu}(M,N)}\widehat{C}(M,N),
\tag{2.52}
$$

where $\widehat{\mu}(m,n)$, $\widehat{C}(m,n)$ are defined in (2.45) and (2.47) respectively.

Combining (2.50) and (2.52), we get the desired result. □

Theorem 2.11. *Suppose that $w(m,n) \in \wp_+(\Omega)$, u, a, b_i, c_i, d_i, e_i, p, q_i, r_i, h_i, j_i are defined as in Theorem 2.5. Furthermore, assume $a(m,n)$ is nondecreasing in the first variable. L_i, T_i, $i = 1,2,\ldots,l_2$, are defined as in Theorem 2.10. If, for $(m,n) \in \Omega$, $u(m,n)$ satisfies*

$$\begin{aligned}
u^p(m,n) \le{}& a(m,n) + \sum_{s=m_0}^{m-1} w(s,n) u^p(m,n) \\
&+ \sum_{i=1}^{l_1} \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} \left[b_i(s,t,m,n) u^{q_i}(s,t) + \sum_{\xi=m_0}^{s} \sum_{\eta=n_0}^{t} c_i(\xi,\eta,m,n) u^{r_i}(\xi,\eta) \right] \\
&+ \sum_{i=1}^{l_2} \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} \left[d_i(s,t,m,n) L_i\Big(s,t,u^{h_i}(s,t)\Big) + \sum_{\xi=m_0}^{s} \sum_{\eta=n_0}^{t} e_i(\xi,\eta,m,n) u^{j_i}(\xi,\eta) \right],
\end{aligned} \tag{2.53}$$

then

$$u(m,n) \le \left\{ \left[a(m,n) + \frac{\widetilde{J}(M,N)}{1-\widetilde{\mu}(M,N)} \widetilde{C}(m,n) \right] \widetilde{w}(m,n) \right\}^{1/p}, \tag{2.54}$$

provided that $\widetilde{\mu}(M,N) < 1$, where

$$\begin{aligned}
\widetilde{J}(m,n) ={}& \sum_{i=1}^{l_1} \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} \left\{ \widetilde{b}_i(s,t,m,n) \left[\frac{q_i}{p} K_1^{(q_i-p)/p} a(s,t) + \frac{p-q_i}{p} K_1^{q_i/p} \right] \right. \\
&\left. + \sum_{\xi=m_0}^{s} \sum_{\eta=n_0}^{t} \widetilde{c}_i(\xi,\eta,m,n) \left[\frac{r_i}{p} K_2^{(r_i-p)/p} a(\xi,\eta) + \frac{p-r_i}{p} K_2^{r_i/p} \right] \right\} \\
&+ \sum_{i=1}^{l_2} \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} \left\{ d_i(s,t,m,n) L_i \left[s,t,(\widetilde{w}(s,t))^{h_i/p} \left(\frac{h_i}{p} K_3^{(h_i-p)/p} a(s,t) + \frac{p-h_i}{p} K_3^{h_i/p} \right) \right] \right. \\
&\left. + \sum_{\xi=m_0}^{s} \sum_{\eta=n_0}^{t} \widetilde{e}_i(\xi,\eta,m,n) \left[\frac{j_i}{p} K_4^{(j_i-p)/p} a(\xi,\eta) + \frac{p-j_i}{p} K_4^{j_i/p} \right] \right\},
\end{aligned}$$

$$\begin{aligned}
\widetilde{b}_i(s,t,m,n) &= b_i(s,t,m,n)(\widetilde{w}(s,t))^{q_i/p}, \\
\widetilde{c}_i(s,t,m,n) &= c_i(s,t,m,n)(\widetilde{w}(s,t))^{r_i/p}, \quad i = 1,2,\ldots,l_1, \\
\widetilde{d}_i(s,t,m,n) &= d_i(s,t,m,n)(\widetilde{w}(s,t))^{h_i/p} T_i \left[s,t,(\widetilde{w}(s,t))^{h_i/p} \left(\frac{h_i}{p} K_3^{(h_i-p)/p} a(s,t) + \frac{p-h_i}{p} K_3^{h_i/p} \right) \right], \\
\widetilde{e}_i(s,t,m,n) &= e_i(s,t,m,n)(\widetilde{w}(s,t))^{j_i/p}, \quad i = 1,2,\ldots,l_2,
\end{aligned}$$

$$\widetilde{w}(m,n)=\prod_{s=m_0}^{m-1}[1+w(s,n)],$$
$$\widetilde{\mu}(m,n)=\sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left\{\widetilde{d}_i(s,t,m,n)\frac{h_i}{p}K_3^{(h_i-p)/p}\widetilde{C}(s,t)\right.$$
$$\left.+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}\widetilde{e}_i(\xi,\eta,m,n)\frac{j_i}{p}K_4^{(j_i-p)/p}\widetilde{C}(\xi,\eta)\right\},$$
$$\widetilde{C}(m,n)=\exp\left\{\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\widetilde{B}(s,t,m,n)\right\},$$
$$\widetilde{B}(s,t,m,n)=\sum_{i=1}^{l_1}\left[\overline{b}_i(s,t,m,n)\frac{q_i}{p}K_1^{(q_i-p)/p}+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}\overline{c}_i(\xi,\eta,m,n)\frac{r_i}{p}K_2^{(r_i-p)/p}\right]. \tag{2.55}$$

The proof for Theorem 2.11 is similar to the combination of Theorems 2.8 and 2.10, and we omit the details here.

Remark 2.12. If we take $g_{1i}(m,n)\equiv 1$, $c_{1i}(m,n)\equiv 0$, $i=1,2,\ldots,l_1$, and $g_{2i}(m,n)\equiv 1$, $e_{1i}(m,n)\equiv 0$, $i=1,2,\ldots,l_2$ in Corollary 2.6, then Corollary 2.6 reduces to [9, Theorem 2.5]. If furthermore $l_1=l_2=1$, then Corollary 2.6 reduces to [9, Theorem 2.1]. If we take $b_i(s,t,m,n)=b_{1i}(s,t)$, $c_i(s,t,m,n)\equiv 0$, $i=1,2,\ldots,l_1$ and $d_i(s,t,m,n)=d_{1i}(s,t)$, $e_i(s,t,m,n)\equiv 0$, $h_i=1$, $i=1,2,\ldots,l_2$ in Theorem 2.10, then Theorem 2.10 reduces to [9, Theorem 2.7]. If furthermore $l_1=l_2=1$, then Theorem 2.10 reduces to [9, Theorem 2.6].

3. Applications

In this section, we will present some applications for the established results above and show that they are useful in the study of boundedness, uniqueness, and continuous dependence of solutions of certain difference equations.

Example 3.1. Consider the following Volterra-Fredholm sum-difference equation:

$$u^p(m,n)=a(m,n)+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left[F_1(s,t,m,n,u(s,t))+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}F_2(\xi,\eta,m,n,u(\xi,\eta))\right]$$
$$+\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{M-1}\left[G_1(s,t,m,n,u(s,t))+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}G_2(\xi,\eta,m,n,u(\xi,\eta))\right], \tag{3.1}$$

where $u(m,n), a(m,n) \in \wp(\Omega)$, $p \geq 1$ is an odd number, $F_i, G_i : \Omega^2 \times \mathbb{R} \to \mathbb{R}$, $i = 1,2$, M, N are two integers defined the same as in Theorem 2.5.

Theorem 3.2. *Suppose that $u(m,n)$ is a solution of* (3.1), *and* $|F_1(s,t,m,n,u)| \leq f_1(s,t,m,n)|u|^q$, $|F_2(s,t,m,n,u)| \leq f_2(s,t,m,n)|u|^r$, $|G_1(s,t,m,n,u)| \leq g_1(s,t,m,n)|u|^h$, $|G_2(s,t,m,n,u)| \leq g_2(s,t,m,n)|u|^j$, *where q, r, h, j are nonnegative constants satisfying $p \geq q$, $p \geq r$, $p \geq h$, $p \geq j$, $f_i, g_i \in \wp_+(\Omega^2)$, $i = 1,2$ and f_i, g_i are nondecreasing in the last two variables; then one has*

$$|u(m,n)| \leq \left\{ |a(m,n)| + \frac{J(M,N)}{1-\mu(M,N)} C(m,n) \right\}^{1/p}, \tag{3.2}$$

provided that $\mu(M,N) < 1$, where

$$\begin{aligned}
J(m,n) = {} & \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} \left\{ f_1(s,t,m,n) \left[\frac{q}{p} K_1^{(q-p)/p} |a(s,t)| + \frac{p-q}{p} K_1^{q/p} \right] \right. \\
& \left. + \sum_{\xi=m_0}^{s} \sum_{\eta=n_0}^{t} f_2(\xi,\eta,m,n) \left[\frac{r}{p} K_2^{(r-p)/p} |a(\xi,\eta)| + \frac{p-r}{p} K_2^{r/p} \right] \right\} \\
& + \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} \left\{ g_1(s,t,m,n) \left[\frac{h}{p} K_3^{(h-p)/p} |a(s,t)| + \frac{p-h}{p} K_3^{h/p} \right] \right. \\
& \left. + \sum_{\xi=m_0}^{s} \sum_{\eta=n_0}^{t} g_2(\xi,\eta,m,n) \left[\frac{j}{p} K_4^{(j-p)/p} |a(\xi,\eta)| + \frac{p-j}{p} K_4^{j/p} \right] \right\}, \\
\mu(m,n) = {} & \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} \left\{ g_1(s,t,m,n) \frac{h}{p} K_3^{(h-p)/p} C(s,t) \right. \\
& \left. + \sum_{\xi=m_0}^{s} \sum_{\eta=n_0}^{t} g_2(\xi,\eta,m,n) \frac{j}{p} K_4^{(j-p)/p} C(\xi,\eta) \right\}, \\
C(m,n) = {} & \exp\left\{ \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} B(s,t,m,n) \right\}, \\
B(s,t,m,n) = {} & f_1(s,t,m,n) \frac{q}{p} K_1^{(q-p)/p} + \sum_{\xi=m_0}^{s} \sum_{\eta=n_0}^{t} f_2(\xi,\eta,m,n) \frac{r}{p} K_2^{(r-p)/p}.
\end{aligned} \tag{3.3}$$

Proof. From (3.1) we have

$$\begin{aligned}
|u(m,n)|^p &\leq |a(m,n)| + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left[\left|F_1(s,t,m,n,u(s,t))\right| + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}\left|F_2(\xi,\eta,m,n,u(\xi,\eta))\right|\right] \\
&\quad + \sum_{s=m_0}^{M-1}\sum_{t=n_0}^{M-1}\left[\left|G_1(s,t,m,n,u(s,t))\right| + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}\left|G_2(\xi,\eta,m,n,u(\xi,\eta))\right|\right] \\
&\leq |a(m,n)| + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left[f_1(s,t,m,n)\left|u(s,t)\right|^q + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}f_2(\xi,\eta,m,n)\left|u(\xi,\eta)\right|^r\right] \\
&\quad + \sum_{s=m_0}^{M-1}\sum_{t=n_0}^{M-1}\left[g_1(s,t,m,n)\left|u(s,t)\right|^h + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}g_2(\xi,\eta,m,n)\left|u(\xi,\eta)\right|^j\right].
\end{aligned} \tag{3.4}$$

Then a suitable application of Theorem 2.5 (with $l_1 = l_2 = 1$) to (3.4) yields the desired result. □

The following theorem deals with the uniqueness of the solutions of (3.1).

Theorem 3.3. *Suppose that* $|F_i(s,t,m,n,u) - F_i(s,t,m,n,v)| \leq f_i(s,t,m,n)|u^p - v^p|$, $|G_i(s,t,m,n,u) - G_i(s,t,m,n,v)| \leq g_i(s,t,m,n)|u^p - v^p|$, $i = 1,2$ *hold for* $\forall u,v \in \mathbb{R}$, *where* $f_i, g_i \in \wp_+(\Omega^2)$, $i = 1,2$ *with* f_i, g_i *nondecreasing in the last two variables, and* $\mu(M,N) = \sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\{g_1(s,t,M,N)C(s,t) + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} g_2(\xi,\eta,M,N)C(\xi,\eta)\} < 1$, *where* $C(m,n) = \exp\{\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} B(s,t,m,n)\}$, *and* $B(s,t,m,n) = f_1(s,t,m,n) + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} f_2(\xi,\eta,m,n)$, *then* (3.1) *has at most one solution.*

Proof. Suppose that $u_1(m,n),\ u_2(m,n)$ are two solutions of (3.1). Then,

$$\begin{aligned}
\left|u_1^p(m,n) - u_2^p(m,n)\right| &\leq \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\Bigg[\left|F_1(s,t,m,n,u_1(s,t)) - F_1(s,t,m,n,u_2(s,t))\right| \\
&\qquad + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}\left|F_2(\xi,\eta,m,n,u_1(\xi,\eta)) - F_2(\xi,\eta,m,n,u_2(\xi,\eta))\right|\Bigg] \\
&\quad + \sum_{s=m_0}^{M-1}\sum_{t=n_0}^{M-1}\Bigg[\left|G_1(s,t,m,n,u_1(s,t)) - G_1(s,t,m,n,u_2(s,t))\right| \\
&\qquad + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}\left|G_2(\xi,\eta,m,n,u_1(\xi,\eta)) - G_2(\xi,\eta,m,n,u_2(\xi,\eta))\right|\Bigg]
\end{aligned}$$

$$\leq \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left[f_1(s,t,m,n)\left|u_1^p(s,t)-u_2^p(s,t)\right| + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} f_2(\xi,\eta,m,n)\left|u_1^p(\xi,\eta)-u_2^p(\xi,\eta)\right|\right]$$
$$+\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{M-1}\left[g_1(s,t,m,n)\left|u_1^p(s,t)-u_2^p(s,t)\right| + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} g_2(\xi,\eta,m,n)\left|u_1^p(\xi,\eta)-u_2^p(\xi,\eta)\right|\right]. \tag{3.5}$$

Treat $|u_1^p(m,n)-u_2^p(m,n)|$ as one variable, and a suitable application of Corollary 2.7 yields $|u_1^p(m,n)-u_2^p(m,n)| \leq 0$, which implies that $u_1^p(m,n) \equiv u_2^p(m,n)$. Since p is an odd number, then we have $u_1(m,n) \equiv u_2(m,n)$, and the proof is complete. □

Finally we study the continuous dependence of the solutions of (3.1) on the functions $a,\ F_1,\ F_2,\ G_1,\ G_2$.

Theorem 3.4. *Suppose that $u(m,n)$ is a solution of* (3.1), $|F_i(s,t,m,n,u)-F_i(s,t,m,n,v)| \leq f_i(s,t,m,n)|u^p-v^p|$, $|G_i(s,t,m,n,u)-G_i(s,t,m,n,v)| \leq g_i(s,t,m,n)|u^p-v^p|$, $i=1,2$ *hold for* $\forall u,v\in\mathbb{R}$, *where* $f_i, g_i \in \wp_+(\Omega^2)$, $i=1,2$ *with* f_i, g_i *nondecreasing in the last two variables, and, furthermore,*

$$|a(m,n)-\overline{a}(m,n)| + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left\{\left|F_1(s,t,\overline{u}(s,t))-\overline{F}_1(s,t,\overline{u}(s,t))\right| + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}\left|F_2(\xi,\eta,\overline{u}(\xi,\eta))-\overline{F}_2(\xi,\eta,\overline{u}(\xi,\eta))\right|\right\}$$
$$+\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left\{\left|G_1(s,t,\overline{u}(s,t))-\overline{G}_1(s,t,\overline{u}(s,t))\right| + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}\left|G_2(\xi,\eta,\overline{u}(\xi,\eta))-\overline{G}_2(\xi,\eta,\overline{u}(\xi,\eta))\right|\right\} \leq \varepsilon, \tag{3.6}$$

where $\varepsilon > 0$ is a constant, and $\overline{u}(m,n) \in \wp(\Omega)$ is the solution of the following difference equation:

$$\overline{u}^p(m,n) = \overline{a}(m,n) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left[\overline{F}_1(s,t,m,n,\overline{u}(s,t)) + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}\overline{F}_2(\xi,\eta,m,n,\overline{u}(\xi,\eta))\right]$$
$$+ \sum_{s=m_0}^{M-1}\sum_{t=n_0}^{M-1}\left[\overline{G}_1(s,t,m,n,\overline{u}(s,t)) + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}\overline{G}_2(\xi,\eta,m,n,\overline{u}(\xi,\eta))\right], \tag{3.7}$$

where $\overline{F}_i, \overline{G}_i : \Omega^2 \times \mathbb{R} \to \mathbb{R},\ i = 1,2$*; then one has*

$$\left|u^p(m,n) - \overline{u}^p(m,n)\right| \le \varepsilon\left[1 + \frac{J(M,N)}{1-\mu(M,N)}C(m,n)\right], \tag{3.8}$$

provided that $\mu(M,N) < 1$, *where*

$$J(m,n) = \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left\{f_1(s,t,m,n) + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}f_2(\xi,\eta,m,n)\right\}$$
$$+ \sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left\{g_1(s,t,m,n) + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}g_2(\xi,\eta,m,n)\right\},$$
$$\mu(m,n) = \sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left\{g_1(s,t,m,n)C(s,t) + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}g_2(\xi,\eta,m,n)C(\xi,\eta)\right\}, \tag{3.9}$$
$$C(m,n) = \exp\left\{\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}B(s,t,m,n)\right\},$$
$$B(s,t,m,n) = f_1(s,t,m,n) + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}f_2(\xi,\eta,m,n).$$

Proof. From (3.1) and (3.7) we deduce

$$\left|u^p(m,n) - \overline{u}^p(m,n)\right| \le |a(m,n) - \overline{a}(m,n)|$$
$$+ \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left\{\left|F_1(s,t,m,n,u(s,t)) - \overline{F}_1(s,t,m,n,\overline{u}(s,t))\right|\right.$$
$$\left. + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}\left|F_2(\xi,\eta,m,n,u(\xi,\eta)) - \overline{F}_2(\xi,\eta,m,n,\overline{u}(\xi,\eta))\right|\right\}$$

$$
\begin{aligned}
&+\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\Bigg\{\Big|G_1(s,t,m,n,u(s,t))-\overline{G}_1(s,t,m,n,\overline{u}(s,t))\Big| \\
&\qquad +\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}\Big|G_2(\xi,\eta,m,n,u(\xi,\eta))-\overline{G}_2(\xi,\eta,m,n,\overline{u}(\xi,\eta))\Big|\Bigg\} \\
&\le |a(m,n)-\overline{a}(m,n)| \\
&+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\Bigg\{\Big|F_1(s,t,m,n,u(s,t))-F_1(s,t,m,n,\overline{u}(s,t))\Big| \\
&\qquad +\Big|F_1(s,t,m,n,\overline{u}(s,t))-\overline{F}_1(s,t,m,n,\overline{u}(s,t))\Big| \\
&\qquad +\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}\big|F_2(\xi,\eta,m,n,u(\xi,\eta))-F_2(\xi,\eta,m,n,\overline{u}(\xi,\eta))\big| \\
&\qquad +\Big|F_2(\xi,\eta,m,n,\overline{u}(\xi,\eta))-\overline{F}_2(\xi,\eta,m,n,\overline{u}(\xi,\eta))\Big|\Bigg\} \\
&+\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\Bigg\{\Big|G_1(s,t,m,n,u(s,t))-G_1(s,t,m,n,\overline{u}(s,t))\Big| \\
&\qquad +\Big|G_1(s,t,m,n,\overline{u}(s,t))-\overline{G}_1(s,t,m,n,\overline{u}(s,t))\Big| \\
&\qquad +\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}\big|G_2(\xi,\eta,m,n,u(\xi,\eta))-G_2(\xi,\eta,m,n,\overline{u}(\xi,\eta))\big| \\
&\qquad +\Big|G_2(\xi,\eta,m,n,\overline{u}(\xi,\eta))-\overline{G}_2(\xi,\eta,m,n,\overline{u}(\xi,\eta))\Big|\Bigg\} \\
&\le \varepsilon+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\big\{\big|F_1(s,t,m,n,u(s,t))-F_1(s,t,m,n,\overline{u}(s,t))\big| \\
&\qquad +\big|F_2(\xi,\eta,m,n,u(\xi,\eta))-F_2(\xi,\eta,m,n,\overline{u}(\xi,\eta))\big|\big\} \\
&+\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\Bigg\{\big|G_1(s,t,m,n,u(s,t))-G_1(s,t,m,n,\overline{u}(s,t))\big| \\
&\qquad +\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}\big|G_2(\xi,\eta,m,n,u(\xi,\eta))-G_2(\xi,\eta,m,n,\overline{u}(\xi,\eta))\big|\Bigg\} \\
&\le \varepsilon+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\Bigg[f_1(s,t,m,n)\Big|u^p(s,t)-\overline{u}^p(s,t)\Big|
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} f_2(\xi,\eta,m,n)\left|u^p(\xi,\eta)-\overline{u}^p(\xi,\eta)\right| \Bigg] \\
&+\sum_{s=m_0}^{M-1}\sum_{t=\mathrm{n}_0}^{M-1}\Bigg[g_1(s,t,m,n)\left|u^p(s,t)-\overline{u}^p(s,t)\right| \\
&\qquad\qquad +\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} g_2(\xi,\eta,m,n)\left|u^p(\xi,\eta)-\overline{u}^p(\xi,\eta)\right| \Bigg].
\end{aligned}
\tag{3.10}
$$

Then a suitable application of Corollary 2.7 yields the desired result. □

Remark 3.5. We note that the results in [1–30] are not available here to establish the analysis above.

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