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Research Article

Essential Norm of Composition Operators on Banach Spaces of Hölder Functions

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Let (X, d) be a pointed compact metric space, let $0 < \alpha < 1$, and let $\varphi : X \to X$ be a base point preserving Lipschitz map. We prove that the essential norm of the composition operator C_φ induced by the symbol φ on the spaces $\mathrm{lip}_0(X, d^\alpha)$ and $\mathrm{Lip}_0(X, d^\alpha)$ is given by the formula $\|C_\varphi\|_e = \lim_{t\to 0} \sup_{0<d(x,y)<t}(d(\varphi(x), \varphi(y))^\alpha / d(x, y)^\alpha)$ whenever the dual space $\mathrm{lip}_0(X, d^\alpha)^*$ has the approximation property. This happens in particular when X is an infinite compact subset of a finite-dimensional normed linear space.

1. Introduction

Let (X, d) be a compact metric space with a distinguished point $e \in X$ and $0 < \alpha < 1$. The formula $d^\alpha(x, y) = d(x, y)^\alpha$ defines a new metric on X, and the metric space (X, d^α) is said to be a *Hölder metric space of order* α. As usual, $\mathbb{K}$ denotes the field of real or complex numbers.

The *Lipschitz space* $\mathrm{Lip}_0(X, d^\alpha)$ is the Banach space of all Lipschitz functions $f : X \to \mathbb{K}$ on the Hölder metric space (X, d^α) for which $f(e) = 0$ under the standard Lipschitz norm

$$L_\alpha(f) = \sup\left\{ \frac{|f(x) - f(y)|}{d(x, y)^\alpha} : x, y \in X,\ x \neq y \right\}. \tag{1.1}$$

Notice that the Lipschitz functions on (X, d^α) are precisely the *Hölder functions of order* α on (X, d).

The *little Lipschitz space* $\mathrm{lip}_0(X, d^\alpha)$ is the closed subspace consisting of those functions $f \in \mathrm{Lip}_0(X, d^\alpha)$ that satisfy the following *local flatness condition*:

$$\lim_{t \to 0} \sup_{0<d(x,y)<t} \frac{|f(x) - f(y)|}{d(x,y)^\alpha} = 0. \tag{1.2}$$

The Lipschitz space $\mathrm{Lip}_0(X, d^\alpha)$ has a canonical predual $\mathcal{F}(X, d^\alpha)$, the *free Lipschitz space* on (X, d^α), also known as the *Arens-Eells space* in [1], that can be defined as the closed linear span of the point evaluations

$$\delta_x(f) = f(x), \quad (x \in X,\ f \in \mathrm{Lip}_0(X, d^\alpha)), \tag{1.3}$$

in the dual space $\mathrm{Lip}_0(X, d^\alpha)^*$. As it turns out, $\mathcal{F}(X, d^\alpha)$ is itself the dual space of $\mathrm{lip}_0(X, d^\alpha)$. The structure of spaces of Lipschitz and Hölder functions and their preduals on general metric spaces was studied by Kalton in [2]. We refer to the book [1] by Weaver for a complete study of the spaces of Lipschitz functions.

We denote by $\mathcal{L}(E)$ the algebra of all bounded linear operators on a Banach space E, and by $\mathcal{K}(E)$, the closed ideal of all compact operators on E. The *essential norm* $\|T\|_e$ of an operator $T \in \mathcal{L}(E)$ is just the distance from T to $\mathcal{K}(E)$, that is,

$$\|T\|_e = \inf\{\|T - K\| : K \in \mathcal{K}(E)\}. \tag{1.4}$$

It is clear that an operator $T \in \mathcal{L}(E)$ is compact if and only if $\|T\|_e = 0$.

Recall that a Banach space E is said to have the *approximation property* if the identity operator on E can be approximated uniformly on every compact subset of E by operators of finite rank.

Let (X, d) be a pointed compact metric space with base point $e \in X$, let $0 < \alpha < 1$, and let $\varphi : X \to X$ be a Lipschitz mapping that preserves the base point, that is, $\varphi(e) = e$ and

$$L(\varphi) := \sup\left\{ \frac{d(\varphi(x), \varphi(y))}{d(x,y)} : x, y \in X,\ x \neq y \right\} < \infty. \tag{1.5}$$

The *composition operator* $C_\varphi : \mathrm{lip}_0(X, d^\alpha) \to \mathrm{lip}_0(X, d^\alpha)$ is defined by the expression

$$(C_\varphi f)(x) = f(\varphi(x)), \quad (x \in X,\ f \in \mathrm{lip}_0(X, d^\alpha)). \tag{1.6}$$

The aim of this paper is to give lower and upper estimates for the essential norm of the composition operator C_φ on $\mathrm{lip}_0(X, d^\alpha)$ in terms of φ. Results along these lines were obtained by Montes-Rodríguez [3, 4] and more recently by Galindo and Lindström [5], and also by Galindo et al. [6]. In Section 2, we compute the norm of C_φ. We show that

$$\|C_\varphi\| = \sup_{x \neq y} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x,y)^\alpha}. \tag{1.7}$$

The prototype of a formula as above with $\alpha = 1$ was provided by Weaver in [1, Proposition 1.8.2] for the composition operator C_φ on the space $\mathrm{Lip}_0(X,d)$. In Section 3, we give a lower bound for the essential norm of the operator C_φ, namely,

$$\lim_{t\to 0}\sup_{0<d(x,y)<t}\frac{d(\varphi(x),\varphi(y))^\alpha}{d(x,y)^\alpha}\le\|C_\varphi\|_e. \tag{1.8}$$

Section 4 contains our main result. When the dual space $\mathrm{lip}_0(X,d^\alpha)^*$ has the approximation property, we show that

$$\|C_\varphi\|_e\le\lim_{t\to 0}\sup_{0<d(x,y)<t}\frac{d(\varphi(x),\varphi(y))^\alpha}{d(x,y)^\alpha}. \tag{1.9}$$

The proof of this inequality depends on some results involving shrinking compact approximating sequences on $\mathrm{lip}_0(X,d^\alpha)$. Using the fact that the space $\mathrm{Lip}_0(X,d^\alpha)$ is isometrically isomorphic to the second dual of $\mathrm{lip}_0(X,d^\alpha)$, and the relationship between the essential norm of an operator and its adjoint, we derive in Section 5 the same formula for the essential norm of the operator C_φ on $\mathrm{Lip}_0(X,d^\alpha)$.

It is natural to ask for some examples where the dual space $\mathrm{lip}_0(X,d^\alpha)^*$ has the approximation property. For instance, this happens if X is uniformly discrete, that is, $\inf_{x\neq y} d(x,y)>0$ (see [2, Proposition 4.4]). Also, $\mathrm{lip}_0(X,d^\alpha)^*$ has the approximation property whenever the space $\mathrm{lip}_0(X,d^\alpha)$ is isomorphic to c_0 and hence $\mathrm{Lip}_0(X,d^\alpha)$ is isomorphic to ℓ_∞ and $\mathcal{F}(X,d^\alpha)$ is isomorphic to ℓ_1.

A classical result of Bonic et al. [7] ensures that $\mathrm{lip}_0(X,d^\alpha)$ is isomorphic to c_0 whenever X is an infinite compact subset of a finite-dimensional normed linear space. This result was corrected by Weaver, who asked whether such an isomorphism could be extended to any compact metric space [1, page 98]. Kalton answered this question negatively by proving that a compact convex subset X of a Hilbert space containing the origin has the property that $\mathrm{lip}_0(X,d^\alpha)$ is isomorphic to c_0 if and only if X is finite dimensional [2, Theorem 8.3]. In fact this statement is true for every general Banach space in place of a Hilbert space if $0<\alpha\le 1/2$ [2, Theorem 8.5] and for any Banach space that has nontrivial Rademacher type if $0<\alpha<1$ [2, Theorem 8.4]. Kalton conjectured that this holds in full generality for all Banach spaces.

Let us recall now that a metric space (X,d) satisfies the *doubling condition* (or has *finite Assouad dimension*) if there is an integer n such that for any $\delta>0$, every closed ball of radius δ can be covered by at most n closed balls of radius $\delta/2$. A theorem of Assouad [8] asserts that whenever a metric space (X,d) satisfies the doubling condition, every Hölder metric space (X,d^α) Lipschitz embeds in the euclidean space $\mathbb{R}^n$. Using this result, Kalton observed that if a compact metric space (X,d) satisfies the doubling condition, then the space $\mathrm{lip}_0(X,d^\alpha)$ is isomorphic to c_0 [2, Theorem 6.5]. Furthermore, he also showed that the converse is false by means of a counterexample [2, Proposition 6.8].

2. The Norm of C_φ on $\mathrm{Lip}_0(X, d^\alpha)$

The aim of this section is to derive a formula for the norm of the composition operator C_φ on $\mathrm{lip}_0(X, d^\alpha)$ in terms of the Lipschitz constant of φ. A similar expression was already provided by Weaver for the composition operator C_φ on the space $\mathrm{Lip}_0(X, d)$, obtaining in [1, Proposition 1.8.2] the following identity:

$$\|C_\varphi\| = \sup_{x \neq y} \frac{d(\varphi(x), \varphi(y))}{d(x, y)}. \tag{2.1}$$

Theorem 2.1. *Let X be a pointed compact metric space, $0 < \alpha < 1$ and $\varphi : X \to X$ a base point preserving Lipschitz mapping. Then the norm of the composition operator $C_\varphi : \mathrm{lip}_0(X, d^\alpha) \to \mathrm{lip}_0(X, d^\alpha)$ is given by the expression*

$$\|C_\varphi\| = \sup_{x \neq y} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha}. \tag{2.2}$$

Proof. We follow the steps of the proof of Weaver's formula. One inequality is formally identical, while the other inequality needs an adjustment of the suitable attaining functions. For any $f \in \mathrm{lip}_0(X, d^\alpha)$ with $L_\alpha(f) \leq 1$, we have

$$\begin{aligned} L_\alpha(C_\varphi f) &= \sup_{x \neq y} \frac{|f(\varphi(x)) - f(\varphi(y))|}{d(x, y)^\alpha} \\ &\leq \sup_{\varphi(x) \neq \varphi(y)} \frac{|f(\varphi(x)) - f(\varphi(y))|}{d(\varphi(x), \varphi(y))^\alpha} \cdot \sup_{x \neq y} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha} \\ &\leq L_\alpha(f) \cdot \sup_{x \neq y} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha} \\ &\leq \sup_{x \neq y} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha}, \end{aligned} \tag{2.3}$$

and so

$$\|C_\varphi\| = \sup_{L_\alpha(f) \leq 1} L_\alpha(C_\varphi f) \leq \sup_{x \neq y} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha}. \tag{2.4}$$

For the converse inequality, fix two points $x, y \in X$ such that $\varphi(x) \neq \varphi(y)$ and choose β strictly between α and 1. Define $h : X \to \mathbb{R}$ by

$$h(z) = \frac{d(z, \varphi(y))^\beta - d(z, \varphi(x))^\beta}{2d(\varphi(x), \varphi(y))^{\beta - \alpha}}, \tag{2.5}$$

and $f : X \to \mathbb{R}$ by

$$f(z) = h(z) - h(e). \tag{2.6}$$

It is not hard to show that $f \in \operatorname{lip}_0(X, d^\alpha)$ with $L_\alpha(f) = 1$ (see, for instance, [9]), so

$$\|C_\varphi\| \geq L_\alpha(C_\varphi f) \geq \frac{|f(\varphi(x)) - f(\varphi(y))|}{d(x,y)^\alpha} = \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x,y)^\alpha}. \tag{2.7}$$

Taking supremum over x and y, we conclude that

$$\|C_\varphi\| \geq \sup_{x \neq y} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x,y)^\alpha}. \tag{2.8}$$

□

3. The Lower Estimate of the Essential Norm of C_φ on $\operatorname{Lip}_0(X, d^\alpha)$

Next we bound from below the essential norm of C_φ on $\operatorname{lip}_0(X, d^\alpha)$ by means of an asymptotic quantity that measures the local flatness of φ.

Theorem 3.1. *Let X be a pointed compact metric space, $0 < \alpha < 1$ and $\varphi : X \to X$ a base point preserving Lipschitz mapping. Then the essential norm of the operator $C_\varphi : \operatorname{lip}_0(X, d^\alpha) \to \operatorname{lip}_0(X, d^\alpha)$ satisfies the lower estimate*

$$\lim_{t \to 0} \sup_{0 < d(x,y) < t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x,y)^\alpha} \leq \|C_\varphi\|_e. \tag{3.1}$$

We will need the following description of the weak convergence in $\operatorname{lip}_0(X, d^\alpha)$. This result is part of the folklore and it is immediate from the Banach-Steinhaus theorem since $\operatorname{lip}_0(X, d^\alpha)^* = \overline{\operatorname{span}}\{\delta_x : x \in X\}$ (see Weaver [1, Theorem 3.3.3]).

Lemma 3.2. *Let X be a pointed compact metric space, $0 < \alpha < 1$ and $\{f_n\}$ a sequence in $\operatorname{lip}_0(X, d^\alpha)$. Then $\{f_n\}$ converges to 0 weakly in $\operatorname{lip}_0(X, d^\alpha)$ if and only if $\{f_n\}$ is bounded in $\operatorname{lip}_0(X, d^\alpha)$ and converges to 0 pointwise on X.*

Proof of Theorem 3.1. Since the mapping $\varphi : X \to X$ is Lipschitz, the function

$$t \longmapsto \sup_{0 < d(x,y) < t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x,y)^\alpha}, \quad (t > 0) \tag{3.2}$$

is well defined. It is easy to check that

$$\lim_{t \to 0} \sup_{0 < d(x,y) < t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x,y)^\alpha} = \inf_{t > 0} \sup_{0 < d(x,y) < t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x,y)^\alpha}. \tag{3.3}$$

Now, for every natural number n we can take a real number t_n such that $0 < t_n < [2/n(1+L(\varphi)^\alpha)]^{1/\alpha}$ and two points $x_n, y_n \in X$ such that $0 < d(x_n, y_n) < t_n$, satisfying

$$\sup_{0<d(x,y)<t_n} \frac{d(\varphi(x),\varphi(y))^\alpha}{d(x,y)^\alpha} < \inf_{t>0} \sup_{0<d(x,y)<t} \frac{d(\varphi(x),\varphi(y))^\alpha}{d(x,y)^\alpha} + \frac{1}{n},$$
$$\sup_{0<d(x,y)<t_n} \frac{d(\varphi(x),\varphi(y))^\alpha}{d(x,y)^\alpha} - \frac{1}{n} < \frac{d(\varphi(x_n),\varphi(y_n))^\alpha}{d(x_n,y_n)^\alpha}. \tag{3.4}$$

In this way we obtain two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$0 < d(x_n, y_n) < \left[\frac{2}{n(1+L(\varphi)^\alpha)}\right]^{1/\alpha},$$
$$\left|\frac{d(\varphi(x_n),\varphi(y_n))^\alpha}{d(x_n,y_n)^\alpha} - \inf_{t>0} \sup_{0<d(x,y)<t} \frac{d(\varphi(x),\varphi(y))^\alpha}{d(x,y)^\alpha}\right| < \frac{1}{n}, \tag{3.5}$$

for all $n \in \mathbb{N}$, and this last inequality implies that

$$\inf_{t>0} \sup_{0<d(x,y)<t} \frac{d(\varphi(x),\varphi(y))^\alpha}{d(x,y)^\alpha} = \lim_{n\to\infty} \frac{d(\varphi(x_n),\varphi(y_n))^\alpha}{d(x_n,y_n)^\alpha}. \tag{3.6}$$

Now take $x_n, y_n \in X$ for which $\varphi(x_n) \neq \varphi(y_n)$ and choose $\beta \in]\alpha, 1[$. Define $h_n, f_n : X \to \mathbb{R}$ by

$$h_n(x) = \frac{d(x,\varphi(y_n))^\beta - d(x,\varphi(x_n))^\beta}{2d(\varphi(x_n),\varphi(y_n))^{\beta-\alpha}},$$
$$f_n(x) = h_n(x) - h_n(e). \tag{3.7}$$

Using (3.5), we have $\|h_n\|_\infty \le 1/n$. Clearly, $f_n \in \operatorname{lip}_0(X, d^\alpha)$ with $L_\alpha(f_n) = L_\alpha(h_n) = 1$ and $\|f_n\|_\infty \le 2/n$ for all $n \in \mathbb{N}$. Moreover, an easy calculation shows that $|f_n(\varphi(x_n)) - f_n(\varphi(y_n))| = d(\varphi(x_n), \varphi(y_n))^\alpha$. Since

$$\frac{d(\varphi(x_n),\varphi(y_n))^\alpha}{d(x_n,y_n)^\alpha} = \frac{|f_n(\varphi(x_n)) - f_n(\varphi(y_n))|}{d(x_n,y_n)^\alpha} \le L_\alpha(C_\varphi f_n), \tag{3.8}$$

for all $n \in \mathbb{N}$, we have

$$\lim_{n\to\infty} \frac{d(\varphi(x_n),\varphi(y_n))^\alpha}{d(x_n,y_n)^\alpha} \le \limsup_{n\to\infty} L_\alpha(C_\varphi f_n). \tag{3.9}$$

By Lemma 3.2, $\{f_n\} \to 0$ weakly in $\mathrm{lip}_0(X, d^\alpha)$. Thus, if K is any compact operator from $\mathrm{lip}_0(X, d^\alpha)$ into $\mathrm{lip}_0(X, d^\alpha)$, then $\lim_{n\to\infty} L_\alpha(Kf_n) = 0$ because compact operators map weakly convergent sequences into norm convergent sequences. It follows that

$$\begin{aligned}\limsup_{n\to\infty} L_\alpha(C_\varphi f_n) &= \limsup_{n\to\infty} \left(L_\alpha(C_\varphi f_n) - L_\alpha(Kf_n)\right)\\ &\le \limsup_{n\to\infty} L_\alpha((C_\varphi - K)f_n)\\ &\le \|C_\varphi - K\|.\end{aligned} \tag{3.10}$$

Combining (3.3), (3.6), (3.9), and (3.10), we conclude that

$$\lim_{t\to 0} \sup_{0<d(x,y)<t} \frac{d(\varphi(x),\varphi(y))^\alpha}{d(x,y)^\alpha} \le \|C_\varphi - K\|. \tag{3.11}$$

By taking the infimum on both sides of this inequality over all compact operators K on $\mathrm{lip}_0(X, d^\alpha)$, we obtain the lower estimate

$$\lim_{t\to 0} \sup_{0<d(x,y)<t} \frac{d(\varphi(x),\varphi(y))^\alpha}{d(x,y)^\alpha} \le \|C_\varphi\|_e. \tag{3.12}$$

□

4. The Upper Estimate of the Essential Norm of C_φ on $\mathrm{Lip}_0(X, d^\alpha)$

Now we prove that the lower bound of the essential norm of C_φ on $\mathrm{lip}_0(X, d^\alpha)$ obtained in Section 3 is also an upper bound whenever the dual space $\mathrm{lip}_0(X, d^\alpha)^*$ has the approximation property.

Theorem 4.1. *Let X be a pointed compact metric space and $0 < \alpha < 1$. Suppose that the dual space $\mathrm{lip}_0(X, d^\alpha)^*$ has the approximation property. Let $\varphi : X \to X$ be a base point preserving Lipschitz mapping. Then the essential norm of the composition operator $C_\varphi : \mathrm{lip}_0(X, d^\alpha) \to \mathrm{lip}_0(X, d^\alpha)$ satisfies*

$$\|C_\varphi\|_e \le \lim_{t\to 0} \sup_{0<d(x,y)<t} \frac{d(\varphi(x),\varphi(y))^\alpha}{d(x,y)^\alpha}. \tag{4.1}$$

The strategy for the proof of Theorem 4.1 is to work with a sequence $\{K_n\}$ of compact operators on $\mathrm{lip}_0(X, d^\alpha)$ that satisfies some prescribed conditions that are stated in Lemma 4.3 below. We borrow this technique from the work of Montes-Rodríguez [3].

First we recall some notions and results. A sequence $\{K_n\}$ is called a *compact approximating sequence* for a separable Banach space E if each $K_n : E \to E$ is a compact operator and $\lim_{n\to\infty}\|(I - K_n)f\| = 0$ for every $f \in E$, where I denotes the identity operator on E. Also, we say that $\{K_n\}$ is *shrinking* if $\lim_{n\to\infty}\|(I - K_n)^* f^*\| = 0$ for every $f^* \in E^*$.

Johnson [10, Theorem 2] showed that both the Banach space $\mathrm{lip}_0(X, d^\alpha)$ and its dual space $\mathrm{lip}_0(X, d^\alpha)^*$ are separable.

On the other hand, the *Banach-Mazur distance* between isomorphic Banach spaces E, F is defined by

$$d(E,F) = \inf\left\{\|T\| \cdot \left\|T^{-1}\right\| : T \text{ is an isomorphism of } E \text{ onto } F\right\}. \tag{4.2}$$

We say that E *embeds almost isometrically into* F provided that for every $\varepsilon > 0$, there is a subspace $F_\varepsilon \subset F$ such that $d(E, F_\varepsilon) < 1 + \varepsilon$.

The next proposition is immediate from a result of Kalton [11].

Proposition 4.2 (see [11, Corollary 3]). *Let $\{K_n\}$ be a sequence of compact operators between Banach spaces E and F and let us suppose that $\lim_{n\to\infty}\langle K_n^* f^*, f^{**}\rangle = 0$ for all $f^* \in F^*$ and $f^{**} \in E^{**}$. Then there exists a sequence $\{K_n^c\}$ of compact operators such that $K_n^c \in \operatorname{conv}\{K_m : m \geq n\}$ and $\lim_{n\to\infty}\|K_n^c\| = 0$.*

Lemma 4.3. *Let X be a pointed compact metric space and $0 < \alpha < 1$. Suppose that the dual space $\operatorname{lip}_0(X, d^\alpha)^*$ has the approximation property. Then there is a shrinking compact approximating sequence $\{K_n\}$ on $\operatorname{lip}_0(X, d^\alpha)$ such that $\limsup_{n\to\infty}\|I - K_n\| \leq 1$.*

Proof. Since the dual space $\operatorname{lip}_0(X, d^\alpha)^*$ is separable and has the approximation property, it has the metric approximation property and therefore there is a shrinking compact approximating sequence $\{S_n\}$ on $\operatorname{lip}_0(X, d^\alpha)$. We claim that for every $j \in \mathbb{N}$, there exist a natural $n_j \geq j$ and a compact operator K_j on $\operatorname{lip}_0(X, d^\alpha)$ in the convex hull of the set $\{S_m : m \geq n_j\}$ such that $\|I - \mathrm{K}_j\| < (1 + 1/j)^2$.

Fix $j \in \mathbb{N}$. Now, for the proof of our claim, we use a result that ensures that $\operatorname{lip}_0(X, d^\alpha)$ embeds almost isometrically into c_0. We refer to Kalton [2, Theorem 6.6] for a simple proof of this result due to Yoav Benyamini. Thus, there is a closed subspace $F_j \subset c_0$ and an isomorphism $T_j : \operatorname{lip}_0(X, d^\alpha) \to F_j$ such that $\|T_j\| \cdot \|T_j^{-1}\| < 1 + (1/j)$. Now consider $M_n := T_j S_n T_j^{-1}$, and notice that $\{M_n\}$ is a shrinking compact approximating sequence on F_j. Next, let $\{P_n\}$ be the sequence of projections on c_0 defined by

$$(P_n x)(k) = x(k), \quad (1 \leq k \leq n), \qquad (P_n x)(k) = 0, \quad (k > n), \tag{4.3}$$

for all $x \in c_0$, and let $J : F_j \hookrightarrow c_0$ be the inclusion map. Then consider the sequence of compact operators $D_n := P_n J - J M_n$ defined from F_j into c_0. Notice that $\lim_{n\to\infty}\langle D_n^* f^*, f^{**}\rangle = 0$ for all $f^* \in c_0^*$ and $f^{**} \in F_j^{**}$. It follows from Proposition 4.2 that there is a sequence of operators $\{D_n^c\}$ such that $D_n^c \in \operatorname{conv}\{D_m : m \geq n\}$ and $\lim_{n\to\infty}\|D_n^c\| = 0$. This gives rise to a pair of shrinking compact approximating sequences $\{P_n^c\}$ and $\{M_n^c\}$ such that, for each $n \in \mathbb{N}$, $P_n^c \in \operatorname{conv}\{P_m : m \geq n\}$, $M_n^c \in \operatorname{conv}\{M_m : m \geq n\}$, and $\lim_{n\to\infty}\|P_n^c J - J M_n^c\| = 0$. Now, consider $L_n := T_j^{-1} M_n^c T_j$. We have

$$\begin{aligned}\|I - L_n\| &= \left\|I - T_j^{-1} M_n^c T_j\right\| \leq \left\|T_j^{-1}\right\| \cdot \left\|T_j\right\| \cdot \|I - M_n^c\| \\ &\leq \left(1 + \frac{1}{j}\right) \cdot \|I - M_n^c\| = \left(1 + \frac{1}{j}\right) \cdot \|J(I - M_n^c)\|\end{aligned}$$

$$\leq \left(1+\frac{1}{j}\right)\cdot\left(\|(I-P_n^c)J\|+\|P_n^cJ-JM_n^c\|\right)$$
$$\leq \left(1+\frac{1}{j}\right)\cdot\left(1+\|P_n^cJ-JM_n^c\|\right), \tag{4.4}$$

for all $n\in\mathbb{N}$. Finally, choose $n_j\geq j$ large enough so that $\|P_{n_j}^cJ-JM_{n_j}^c\|<1/j$ and conclude that $\|I-L_{n_j}\|<(1+(1/j))^2$. The claim is proved if we take $K_j:=L_{n_j}$.

The proof of the lemma will be finished if we show that $\{K_j\}$ is a shrinking compact approximating sequence on $\mathrm{lip}_0(X,d^\alpha)$. Let $f\in\mathrm{lip}_0(X,d^\alpha)$ and $\varepsilon>0$. Since $\lim_{n\to\infty}L_\alpha(f-S_nf)=0$, there exists $m_0\in\mathbb{N}$ such that $L_\alpha(f-S_nf)<\varepsilon$ for $n\geq m_0$. If $j\geq m_0$, using that $K_j\in\mathrm{conv}\{S_m: m\geq n_j\}$ and $n_j\geq j$, we conclude that $L_\alpha(f-K_jf)<\varepsilon$. Hence $\lim_{j\to\infty}L_\alpha(f-K_jf)=0$. This shows that $\{K_j\}$ is approximating on $\mathrm{lip}_0(X,d^\alpha)$ and similarly it is seen that $\{K_j\}$ is shrinking. □

There is another preliminary result that is needed for the proof of Theorem 4.1 and that can be stated as follows.

Lemma 4.4. *Let X be a pointed compact metric space, $0<\alpha<1$ and $\varphi: X\to X$ a base point preserving Lipschitz mapping. Let $\{K_n\}$ be a shrinking compact approximating sequence on $\mathrm{lip}_0(X,d^\alpha)$.*

Then, for each $t>0$,

$$\lim_{n\to\infty}\sup_{L_\alpha(f)\leq 1}\ \sup_{d(x,y)\geq t}\frac{\left|[(I-K_n)f](\varphi(x))-[(I-K_n)f](\varphi(y))\right|}{d(x,y)^\alpha}=0. \tag{4.5}$$

Proof. Fix $t>0$. Since the inequality $\|f\|_\infty\leq \mathrm{diam}(X)^\alpha\cdot L_\alpha(f)$ is satisfied for all $f\in\mathrm{lip}_0(X,d^\alpha)$, there is a continuous injection $J:(\mathrm{lip}_0(X,d^\alpha),L_\alpha(\cdot))\hookrightarrow(\mathrm{lip}_0(X,d^\alpha),\|\cdot\|_\infty)$. Moreover, it follows from the Arzelà-Ascoli Theorem that J is a compact operator, and by Schauder's theorem, its adjoint J^* is a compact operator, too.

Let B be the unit ball of $(\mathrm{lip}_0(X,d^\alpha),\|\cdot\|_\infty)^*$. Since the bounded sequence of operators $\{(I-K_n)^*\}$ converges to zero pointwise on $\mathrm{lip}_0(X,\mathrm{d}^\alpha)^*$ and $J^*(B)$ is a relatively compact set in $(\mathrm{lip}_0(X,d^\alpha),L_\alpha(\cdot))^*$, it follows that $\lim_{n\to\infty}\|(I-K_n)^*J^*\|=0$. Thus, $\lim_{n\to\infty}\|J(I-K_n)\|=0$. Let $\varepsilon>0$ be given and choose $m\in\mathbb{N}$ such that if $n\geq m$, then $\|(I-K_n)f\|_\infty<\varepsilon t^\alpha/4$ for all $f\in\mathrm{lip}_0(X,d^\alpha)$ with $L_\alpha(f)\leq 1$. For $n\geq m$, we have

$$\frac{\left|[(I-K_n)f](\varphi(x))-[(I-K_n)f](\varphi(y))\right|}{d(x,y)^\alpha}\leq\frac{2\|(I-K_n)f\|_\infty}{t^\alpha}<\frac{\varepsilon}{2}, \tag{4.6}$$

whenever $f\in\mathrm{lip}_0(X,d^\alpha)$ with $L_\alpha(f)\leq 1$ and $x,y\in X$ such that $d(x,y)\geq t$. Finally, we get

$$\sup_{L_\alpha(f)\leq 1}\ \sup_{d(x,y)\geq t}\frac{\left|[(I-K_n)f](\varphi(x))-[(I-K_n)f](\varphi(y))\right|}{d(x,y)^\alpha}<\varepsilon, \tag{4.7}$$

as required. □

We now are ready to prove our main result.

Proof of Theorem 4.1. Let $\{K_n\}$ be the sequence of operators on $\mathrm{lip}_0(X, d^\alpha)$ provided by Lemma 4.3. Since each K_n is a compact operator, so is the product $C_\varphi K_n : \mathrm{lip}_0(X, d^\alpha) \to \mathrm{lip}_0(X, d^\alpha)$ and therefore

$$\|C_\varphi\|_e \le \|C_\varphi(I - K_n)\|. \tag{4.8}$$

Next, fix $t > 0$ and notice that

$$\begin{aligned}
\|C_\varphi(I - K_n)\| &= \sup_{L_\alpha(f)\le 1} L_\alpha(C_\varphi(I - K_n)f) \\
&= \sup_{L_\alpha(f)\le 1} \sup_{x\ne y} \frac{|[(I - K_n)f](\varphi(x)) - [(I - K_n)f](\varphi(y))|}{d(x,y)^\alpha} \\
&\le \sup_{L_\alpha(f)\le 1} \sup_{0<d(x,y)<t} \frac{|[(I - K_n)f](\varphi(x)) - [(I - K_n)f](\varphi(y))|}{d(x,y)^\alpha} \\
&\quad + \sup_{L_\alpha(f)\le 1} \sup_{d(x,y)\ge t} \frac{|[(I - K_n)f](\varphi(x)) - [(I - K_n)f](\varphi(y))|}{d(x,y)^\alpha}.
\end{aligned} \tag{4.9}$$

Then, for every $f \in \mathrm{lip}_0(X, d^\alpha)$, we have

$$\begin{aligned}
&\sup_{0<d(x,y)<t} \frac{|[(I - K_n)f](\varphi(x)) - [(I - K_n)f](\varphi(y))|}{d(x,y)^\alpha} \\
&\quad\le \sup_{\varphi(x)\ne\varphi(y)} \frac{|[(I - K_n)f](\varphi(x)) - [(I - K_n)f](\varphi(y))|}{d(\varphi(x), \varphi(y))^\alpha} \cdot \sup_{0<d(x,y)<t} \frac{d(\varphi(x),\varphi(y))^\alpha}{d(x,y)^\alpha} \\
&\quad\le L_\alpha[(I - K_n)f] \cdot \sup_{0<d(x,y)<t} \frac{d(\varphi(x),\varphi(y))^\alpha}{d(x,y)^\alpha},
\end{aligned} \tag{4.10}$$

so that

$$\sup_{L_\alpha(f)\le 1} \sup_{0<d(x,y)<t} \frac{|[(I - K_n)f](\varphi(x)) - [(I - K_n)f](\varphi(y))|}{d(x,y)^\alpha} \le \|I - K_n\| \sup_{0<d(x,y)<t} \frac{d(\varphi(x),\varphi(y))^\alpha}{d(x,y)^\alpha}. \tag{4.11}$$

Now, combining the above inequalities, we obtain

$$\|C_\varphi\|_e \le \|I - K_n\| \sup_{0<d(x,y)<t} \frac{d(\varphi(x),\varphi(y))^\alpha}{d(x,y)^\alpha} + \sup_{L_\alpha(f)\le 1} \sup_{d(x,y)\ge t} \frac{\left|[(I-K_n)f](\varphi(x)) - [(I-K_n)f](\varphi(y))\right|}{d(x,y)^\alpha}. \tag{4.12}$$

Letting $n \to \infty$ and using Lemmas 4.3 and 4.4, we conclude that

$$\|C_\varphi\|_e \le \sup_{0<d(x,y)<t} \frac{d(\varphi(x),\varphi(y))^\alpha}{d(x,y)^\alpha}. \tag{4.13}$$

Finally, taking limits as $t \to 0$ yields the desired inequality. □

5. The Essential Norm of C_φ on $\mathrm{Lip}_0(X, d^\alpha)$

Now we extend the estimates on the essential norm of a composition operator to the spaces $\mathrm{Lip}_0(X, d^\alpha)$. Recall that $\mathrm{lip}_0(X, d^\alpha)^{**}$ is isometrically isomorphic to $\mathrm{Lip}_0(X, d^\alpha)$ whenever (X, d) is a pointed compact metric space and $\alpha \in (0,1)$ (see [1, Theorem 3.3.3 and Proposition 3.2.2]). As a matter of fact, the mapping $\Delta : \mathrm{lip}_0(X, d^\alpha)^{**} \to \mathrm{Lip}_0(X, d^\alpha)$, defined by

$$\Delta(F)(x) = F(\delta_x), \quad \left(F \in \mathrm{lip}_0(X, d^\alpha)^{**},\ x \in X\right), \tag{5.1}$$

is an isometric isomorphism.

If T is a bounded linear operator on a Banach space, then $\|T^*\| = \|T\|$. However, this identity is no longer true for the essential norm. Since the adjoint of a compact operator is again a compact operator, we always have $\|T^*\|_e \le \|T\|_e$ and therefore $\|T^{**}\|_e \le \|T^*\|_e \le \|T\|_e$. Axler et al. [12] showed that in fact $\|T^{**}\|_e = \|T^*\|_e$, but they gave a counterexample where $\|T^*\|_e < \|T\|_e$.

Theorem 5.1. *Let X be a pointed compact metric space, $0 < \alpha < 1$ and $\varphi : X \to X$ a point preserving Lipschitz mapping. Then the essential norm of the operator $C_\varphi : \mathrm{Lip}_0(X, d^\alpha) \to \mathrm{Lip}_0(X, d^\alpha)$ satisfies the lower estimate*

$$\|C_\varphi\|_e \ge \lim_{t\to 0} \sup_{0<d(x,y)<t} \frac{d(\varphi(x),\varphi(y))^\alpha}{d(x,y)^\alpha}. \tag{5.2}$$

If, in addition, the space $\mathrm{lip}_0(X, d^\alpha)^*$ has the approximation property, then one has the upper estimate

$$\|C_\varphi\|_e \le \lim_{t\to 0} \sup_{0<d(x,y)<t} \frac{d(\varphi(x),\varphi(y))^\alpha}{d(x,y)^\alpha}. \tag{5.3}$$

Proof. Let us start with the lower estimate. Let $\{f_n\}$ be the weakly null sequence in $\mathrm{lip}_0(X, d^\alpha)$ that we constructed for the proof of Theorem 3.1. Then the sequence $\{f_n\}$ is weakly null in $\mathrm{Lip}_0(X, d^\alpha)$. Thus, if K is any compact operator on $\mathrm{Lip}_0(X, d^\alpha)$, we have $\lim_{n\to\infty} L_\alpha(Kf_n) = 0$. Hence, the same computation we performed in Theorem 3.1 yields the lower estimate

$$\lim_{t\to 0} \sup_{0<d(x,y)<t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x,y)^\alpha} \le \|C_\varphi\|_e. \tag{5.4}$$

Now, for the upper estimate, given $F \in \mathrm{lip}_0(X, d^\alpha)^{**}$ and $x \in X$, notice that

$$\begin{aligned}
\left(\Delta^{-1} C_\varphi \Delta\right)(F)(\delta_x) &= ((C_\varphi \Delta)(F))(x) = C_\varphi(\Delta(F))(x) = \Delta(F)(\varphi(x)) \\
&= F\left(\delta_{\varphi(x)}\right) = F\left(\delta_x \circ C_\varphi \mid_{\mathrm{lip}_0(X, d^\alpha)}\right) = \left(F \circ \left(C_\varphi \mid_{\mathrm{lip}_0(X, d^\alpha)}\right)^*\right)(\delta_x) \\
&= \left(C_\varphi \mid_{\mathrm{lip}_0(X, d^\alpha)}\right)^{**}(F)(\delta_x).
\end{aligned} \tag{5.5}$$

Since $\mathrm{lip}_0(X, d^\alpha)^* = \overline{\mathrm{span}}\{\delta_x : x \in X\}$, we conclude that $\Delta^{-1} C_\varphi \Delta = (C_\varphi|_{\mathrm{lip}_0(X, d^\alpha)})^{**}$. Finally, using the relationship between the essential norm of an operator and that of its second adjoint, and applying Theorem 4.1, we get

$$\|C_\varphi\|_e = \left\|\Delta^{-1} C_\varphi \Delta\right\|_e = \left\|\left(C_\varphi \mid_{\mathrm{lip}_0(X, d^\alpha)}\right)^{**}\right\|_e \le \left\|C_\varphi \mid_{\mathrm{lip}_0(X, d^\alpha)}\right\|_e \le \lim_{t\to 0} \sup_{0<d(x,y)<t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x,y)^\alpha}, \tag{5.6}$$

as we wanted. □

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