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Research Article

Positive Solutions for a Class of Fourth-Order Boundary Value Problems in Banach Spaces

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Using a specially constructed cone and the fixed point index theory, this work shows existence and nonexistence results of positive solutions for fourth-order boundary value problem with two different parameters in Banach spaces.

1. Introduction

In this paper, we study the existence of positive solutions for the following fourth-order boundary value problem (BVP) with singularity in Banach space $(E, \|\cdot\|)$:

$$\begin{gathered} x^{(4)}(t) + \beta x''(t) = \lambda f(t, x) + \mu g(t, x), \qquad t \in J, \\ x(0) = x(1) = \theta, \qquad x''(0) = x''(1) = \theta, \end{gathered} \tag{1.1}$$

where $J = [0, 1]$, $\beta < \pi^2$. $g, f \in C(J, [0, +\infty))$, $R^+ = [0, +\infty)$, and θ is the zero element of E.

Fourth-order boundary value problems are studied not only in mathematics but also in physics and many other fields. For example, some models for bridge, underground water flow, and plasma physics can be reduced to fourth-order boundary value problems. In the recent years, some authors (cf. [1–7]) studied fourth-order boundary value problems, but these papers mainly dealt with the problems in real space or the problems without singularity. As far as we know, the fourth-order ordinary differential equation with singularity has been seldom studied in Banach spaces. Also, not so much is known about the case that the nonlinear term has two different parameters or has two parts with different properties.

In [7], Yao studied the following two-point boundary value problem:

$$
\begin{gathered}
u^{(4)}(t) - \lambda h(t) f(u(t)) = 0, \\
u(0) = u(1) = u''(0) = u''(1) = 0,
\end{gathered}
\tag{1.2}
$$

where $\lambda > 0$ and $f \in C[0, +\infty)$. The author obtained positive solutions of BVP (1.2) in real space not abstract space.

In [4], the authors studied the following boundary value problem in real space:

$$
\begin{gathered}
x^{(4)}(t) + \beta x''(t) = \lambda f(t, x), \\
x(0) = x(1) = x''(0) = x''(1) = 0,
\end{gathered}
\tag{1.3}
$$

where $f \in C([0,1] \times [0, +\infty))$, $\beta < \pi^2$, and $\lambda > 0$. With the method of monotone iterative, the existence and uniqueness of positive solution of BVP (1.3) are obtained.

Comparing with the results in [5–7], this paper has the following features. Firstly, we discuss positive solutions of BVP (1.1) in abstract space, not $E = R$ as in [5, 7]. Secondly, BVP (1.1) has two parameters which may have different domains. Fourthly, f and g may have different properties, and the main tool we used is the fixed point theorem on cone. Thirdly, we talk about both the existence and nonexistence of positive solutions, but [5, 9] only study the existence of positive solutions.

Basic facts about ordered Banach space E can be found in [8]. Here we recall some of them. The cone P in E is said to be normal if there exists a positive constant N such that $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$. In this paper, we always suppose that P is normal in E, and without loss of generality, suppose that the normal constant $N = 1$. E^* is dual space of E. Denote $P^* = \{\psi \in E^* \mid \psi(x) \geq 0, \text{ for all } x \in P\}$, then P^* is a dual cone of cone P. The noncompactness of Kuratowski is denoted by $\alpha(\cdot)$.

We study BVP (1.1) in $C(J, E)$. Evidently, $(C(J, E), \|\cdot\|_c)$ is a Banach space under the norm $\|x\|_c = \max_{t\in J}\|x(t)\|$. $x \in C(J, E)$ is a positive solution of BVP(1.1) if it satisfies BVP (1.1) and $x(t) > \theta$, for all $t \in J$.

2. Preliminaries and Lemmas

In this paper, we make the following conditions.

(H_1) $f, g \in C(J' \times P, P)$, $f(t, P_r) = \{f(t, u),\ u \in P_r\}$ is relatively compact and $g(t, P_r)$ is also relatively compact, where $P_r = \{u \in P : \|u\| \leq r\}$. For any $u \in P$, there exist $a_i(t), b_i(t) \in L(J, R^+)$, $w_1(x), w_2(x) \in C(J, R^+)$ such that

$$
\|f(t,u)\| \leq a_1(t) + b_1(t) w_1(\|u\|), \quad \|g(t,u)\| \leq a_2(t) + b_2(t) w_2(\|u\|), \quad \text{a.e. } t \in J. \tag{2.1}
$$

(H_2) There exist a function $m \in L(J, (0, +\infty))$, such that $\psi(f(t,u)) \geq m(t)\|u\|$ or $\psi(g(t,u)) \geq m(t)\|u\|$, for $t \in J$, $u \in P$, where $\psi \in P^*$, $\|\psi\| = 1$.

Lemma 2.1 (see [8]). *Let $H = \{x \mid x : J \to E,\ x$ is strong measurable$\}$ be countable, and there exists $M \in L[J, R^+]$ such that $\|x(t)\| \le M(t)$, a.e. $t \in J$, $x \in H$, then $\alpha(H(t)) \in L[J, R^+]$ and*

$$\alpha\left(\left\{\int_J x(t)dt : x \in H\right\}\right) \le 2\int_J \alpha(H(t))dt. \tag{2.2}$$

Lemma 2.2 (see [8]). *Let P be a cone in Banach space E. For $r > 0$, define $P_r = \{x \in P : \|x\| < r\}$. Assume that $T : \overline{P}_r \to P$ is a completely continuous map such that $x \neq Tx$ for $x \in \partial P_r$.*

(i) *If $\|x\| \le \|Tx\|$ for $x \in \partial P_r$, then $i(T, P_r, P) = 0$.*

(ii) *If $\|x\| \ge \|Tx\|$ for $x \in \partial P_r$, then $i(T, P_r, P) = 1$.*

Lemma 2.3 (see [4]). *For any $\varphi \in C(J, P)$, the following fourth-order boundary value problem:*

$$\begin{gathered} x^{(4)}(t) + \beta x''(t) = \varphi(t), \\ x(0) = x(1) = x''(0) = x''(1) = \theta \end{gathered} \tag{2.3}$$

has the solution

$$x(t) = \int_0^1\int_0^1 G_1(t,\tau)G_2(\tau,s)\varphi(s)ds\,d\tau, \tag{2.4}$$

where $G_2(t,s)$ is defined in Proposition 2.4 and

$$G_1(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1. \end{cases} \tag{2.5}$$

Proposition 2.4. *For all $t, s \in [0,1] \times [0,1]$ one has the following properties:*

$$t(1-t)G_1(s,s) \le G_1(t,s) \le G_1(s,s) = s(1-s), \quad G_1(t,s) \le t(1-t); \quad G_1(t,s) \le \frac{1}{4}. \tag{2.6}$$

Let $\omega = \sqrt{\beta}$. $G_2(t,s)$ is explicitly given by the following

If $\beta < 0$, then

$$G_2(t,s) = \begin{cases} \dfrac{\sinh \omega t \sinh \omega(1-s)}{\omega \sinh \omega}, & 0 \le t \le s \le 1, \\[2ex] \dfrac{\sinh \omega s \sinh \omega(1-t)}{\omega \sinh \omega}, & 0 \le s \le t \le 1. \end{cases} \tag{2.7}$$

If $\beta = 0$, then $G_2(t,s) = G_1(t,s)$.

If $0 < \beta < \pi^2$, *then*

$$G_2(t,s) = \begin{cases} \dfrac{\sin \omega t \sin \omega(1-s)}{\omega \sin \omega}, & 0 \leq t \leq s \leq 1, \\ \dfrac{\sin \omega s \sin \omega(1-t)}{\omega \sin \omega} & 0 \leq s \leq t \leq 1. \end{cases} \tag{2.8}$$

Proposition 2.5. *For any* $t, s \in (0,1)$, *one has* $G_1(t,s) > 0$, $G_2(t,s) > 0$, *and* $G_1(t,s) \geq \delta G_1(u,s)$, *for* $t \in J_\delta$, $s, u \in [0,1]$, *where* $\delta \in (0, 1/2)$.

Proposition 2.6. *It is easy to see that*

$$G_2(t,s) \leq \begin{cases} \dfrac{1}{4}, & \beta = 0, \\ \dfrac{e^{2\omega}}{4}, & \beta < 0, \\ \dfrac{\sin \omega}{\omega}, & \beta > 0. \end{cases} \tag{2.9}$$

Let $d = \max\{1/4, e^{2\omega}/4, \sin\omega/\omega\}$, $Q = \{x \in C(J,E) : x(t) \geq \theta,\ t \in J\}$, and $K = \{x \in Q : x(t) \geq \delta x(s),\ t \in J_\delta,\ s \in [0,1]\}$. It is easy to know that K is a cone in $C(J,E)$. Let $K_r = \{x \in K : \|x\| \leq r\} \subset K$, $K \subset Q$. Define T as

$$(Tx)(t) = \int_0^1 \int_0^1 G_1(t,\tau) G_2(\tau,s) \left(\lambda f(s, x(s)) + \mu g(s, x(s))\right) ds\, d\tau, \tag{2.10}$$

then x is the solution of BVP(1.1) if and only if x is the fixed point of T.

Lemma 2.7. *Suppose that condition* (H_1) *holds, then* $T : K \to K$ *is completely continuous.*

Proof. By the continuity of $G_1(t,s)$ and (H_1), we have $Tx \in Q$. And for any $t \in J_\delta$, $x \in K$, by Proposition 2.5, we get

$$(Tx)(t) = \delta (Tx)(u). \tag{2.11}$$

Next we prove that T is compact. Let $V = \{x_n\}_{n=1}^{\infty} \subset K$ be any bounded set, and we suppose that $\|x_n\| \leq r$, for some $r > 0$. Let $M_r = \max_{0 \leq v \leq r}\{w_i(v),\ i = 1,2\}$, then by condition ($\mathrm{H}_1$),

$$f(t, x_n(t)) \leq a_1(t) + b_1(t) M_r, \quad g(t, x_n(t)) \leq a_2(t) + b_2(t) M_r, \quad \text{a.e. } t \in J,\ x_n \in V. \tag{2.12}$$

We obtain from Lemma 2.1

$$\begin{aligned}
&\alpha\{(Tx_n)(t) : x_n \in V\} \\
&\quad = \alpha\left\{\iint_0^1 G_1(t,\tau)G_2(\tau,s)\big(\lambda f(s,x_n(s)) + \mu g(s,x_n(s))\big)ds\,d\tau : x_n \in V\right\} \\
&\quad \le 2d\int_0^1 \alpha\big((\lambda f(s,x_n(s)) + \mu g(s,x_n(s))) : x_n \in V\big)ds \\
&\quad = 0.
\end{aligned} \tag{2.13}$$

Hence, TV is relatively compact, so there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{Tx_{n_i}\}$ converges to some $v \in C(J,E)$. So $T : K \to K$ is completely continuous. □

In this paper, for $u \in P$ we denote

$$(\psi f)_\nu = \liminf_{\|u\|\to\nu}\min_{t\in J}\frac{\psi(f(t,u))}{\|u\|}, \qquad (\psi g)_\nu = \liminf_{\|u\|\to\nu}\min_{t\in J}\frac{\psi(g(t,u))}{\|u\|}, \tag{2.14}$$

where ν denotes 0 or ∞, $\psi \in P^*$, $\|\psi\| = 1$.

3. Main Results

Theorem 3.1. *Assume that (H_1) holds and the following conditions are satisfied:*

$$(\psi f)_\infty = \infty, \qquad (\psi g)_0 = \infty. \tag{3.1}$$

Then BVP(1.1) *has at least two positive solutions when λ and μ are sufficiently small.*

Proof. T is completely continuous by Lemma 2.7. For any $l > 0$, set

$$F(l) = \frac{d}{4}\max_{x\in K,\ \|x\|_c=l}\int_0^1 \|f(s,x(s))\|ds, \qquad G(l) = \frac{d}{4}\max_{x\in K,\ \|x\|_c=l}\int_0^1 \|g(s,x(s))\|ds. \tag{3.2}$$

Since $(\psi f)_\infty = \infty$, there exists $r_1 > 0$ such that $F(r_1) > 0$. If $G(r_1) = 0$, let $\lambda_1 = r_1/F(r_1)$; then for any $\lambda \in (0,\lambda_1)$ and $x \in K, \|x\|_c = r_1$, we have

$$\begin{aligned}
\|(Tx)(t)\| &\le \frac{d}{4}\int_0^1 \|\lambda f(s,x(s)) + \mu g(s,x(s))\|ds \\
&\le \frac{d}{4}\int_0^1 \|\lambda f(s,x(s))\|ds + \frac{d}{4}\int_0^1 \|\mu g(s,x(s))\|ds \\
&\le \lambda_1 F(r_1) \\
&= r_1 = \|x\|_c.
\end{aligned} \tag{3.3}$$

If $G(r_1) \neq 0$, let $\mu_1 = r_1/2G(r_1)$ and $\lambda_1 = r_1/2F(r_1)$. For $\lambda \in (0,\lambda_1)$, $\mu \in (0,\mu_1)$, and $x \in K, \|x\|_c = r_1$, we get

$$\begin{aligned}\|(Tx)(t)\| &\leq \frac{d}{4}\int_0^1 \|\lambda f(s,x(s))\| ds + \frac{d}{4}\int_0^1 \|\mu g(s,x(s))\| ds \\ &\leq \frac{d}{4}\lambda_1 \int_0^1 \|\lambda f(s,x(s))\| ds + \frac{d}{4}\mu_1 \int_0^1 \|g(s,x(s))\| ds \\ &\leq r_1 F(r_1) + \mu_1 G(r_1) \\ &\leq \frac{r_1}{2} + \frac{r_1}{2} \\ &= r_1 = \|x\|_c,\end{aligned} \tag{3.4}$$

which means that $\|Tx\|_c < r_1 = \|x\|_c$, for $x \in K$, $\|x\|_c = r_1$, and Lemma 2.2 implies that $i(T, K_{r_1}, K) = 1$.

Since $(\psi f)_\infty = \infty$, there exists $\overline{r}_2 > 0$ such that $\psi(f(t,u)) \geq \varepsilon_1 \|u\|$, $u \in P$, $t \in J$, $\|u\| \geq \overline{r}_2$, where $\varepsilon_1 > 0$ satisfies $\varepsilon_1 \lambda \delta \int\int_\delta^{1-\delta} G_1(1/2,\tau) G_2(\tau,s) ds\, d\tau > 1$. Let $r_2 = \max\{2r_1, \overline{r}_2/\delta\}$, then for any $t \in J_\delta$, $x \in K$, $\|x\|_c = r_2$, by the definition of K we get $\|x(t)\| \geq \delta \|x\|_c \geq \overline{r}_2$, and we have

$$\begin{aligned}\left\|(Tx)\left(\frac{1}{2}\right)\right\| &\geq \psi\left((Tx)\left(\frac{1}{2}\right)\right) \\ &\geq \lambda \int_0^1 \int_0^1 G_1\left(\frac{1}{2},\tau\right) G_2(\tau,s) \psi(f(s,x(s))) ds\, d\tau \\ &\geq \lambda \int_0^1 \int_\delta^{1-\delta} G_1\left(\frac{1}{2},\tau\right) G_2(\tau,s) \psi(f(s,x(s))) ds\, d\tau \\ &\geq \lambda \varepsilon_1 \int_0^1 \int_\delta^{1-\delta} G_1\left(\frac{1}{2},\tau\right) G_2(\tau,s) \|x(s)\| ds\, d\tau \\ &\geq \lambda \varepsilon_1 \delta \|x\|_c \int_0^1 \int_\delta^{1-\delta} G_1\left(\frac{1}{2},\tau\right) G_2(\tau,s) ds\, d\tau \\ &> \|x\|_c,\end{aligned} \tag{3.5}$$

which implies that $\|Tx\|_c > r_2$, $x \in K_{r_2}$. By Lemma 2.2 we have $i(T, K_{r_2}, K) = 0$.

On the other hand, since $(\psi g)_0 = \infty$, there exists $\overline{r}_3 : 0 < \overline{r}_3 < r_1$ such that $\psi(g(t,u)) \geq \varepsilon_2 \|u\|$, for $t \in J$, $u \in P, \|u\| \leq \overline{r}_3$, where $\varepsilon_2 > 0$ and satisfies

$$\varepsilon_2 \mu \delta \int_0^1 \int_\delta^{1-\delta} G_1\left(\frac{1}{2},\tau\right) G_2(\tau,s) ds\, d\tau > 1. \tag{3.6}$$

Accordingly, $0 < r_3 < \overline{r}_3$. Then for $x \in K$, $\|x\|_c = r_3$, similar to (3.5) we get $\|Tx\|_c > r_3$. By Lemma 2.2 we have $i(T, K_{r_3}, K) = 0$. Hence, from the additivity of fixed point index, we obtain

$$i\left(T, \frac{K_{r_2}}{\overline{K}_{r_1}}, K\right) = -1, \qquad i\left(T, \frac{K_{r_1}}{\overline{K}_{r_3}}, K\right) = 1. \tag{3.7}$$

Hence, T has two fixed points $x_1 \in K_{r_2}/\overline{K}_{r_1}$ and $x_2 \in K_{r_1}/\overline{K}_{r_3}$. Therefore, BVP(1.1) has at least two fixed points x_1 and x_2. □

Corollary 3.2. *If* (H_1), $(\varphi g)_\infty = \infty$, *and* $(\varphi f)_0 = \infty$ *hold, then BVP*(1.1) *has at least two positive solutions when* λ *and* μ *are sufficiently small.*

Theorem 3.3. *Suppose that* (H_1) *and* (H_2) *hold, and if* λ *is sufficiently large, then BVP*(1.1) *has no positive solutions.*

Proof. First by (H_2) we can suppose that $\varphi(f(t,u)) \geq m(t)\|u\|$, for $t \in J$. suppose that BVP(1.1) has a positive solution x, and choose λ sufficiently large satisfying $\lambda\delta\int\int_\delta^{1-\delta} G_1(1/2,\tau)G_2(\tau,s)m(s)ds\,d\tau > 1$. Then we have

$$\begin{aligned}
\|x\|_c &\geq \left\|x\left(\frac{1}{2}\right)\right\| \geq \varphi\left(x\left(\frac{1}{2}\right)\right) \\
&\geq \lambda\int_0^1\int_0^1 G_1\left(\frac{1}{2},\tau\right)G_2(\tau,s)\varphi(f(s,x(s)))ds\,d\tau \\
&\geq \lambda\int_0^1\int_\delta^{1-\delta} G_1\left(\frac{1}{2},\tau\right)G_2(\tau,s)m(s)\|x(s)\|ds\,d\tau \\
&\geq \lambda\delta\int_0^1\int_\delta^{1-\delta} G_1\left(\frac{1}{2},\tau\right)G_2(\tau,s)m(s)ds\,d\tau\|x\|_c \\
&> \|x\|_c,
\end{aligned} \tag{3.8}$$

which is a contradiction; so BVP(1.1) has no positive solutions. Similarly, if $\varphi(g(t,u)) \geq m(t)\|u\|$, for $t \in J$, and μ is sufficiently large, then BVP(1.1) has no positive solutions. □

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