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Research Article

Possible Intervals for T- and M-Orders of Solutions of Linear Differential Equations in the Unit Disc

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In the case of the complex plane, it is known that there exists a finite set of rational numbers containing all possible growth orders of solutions of $f^{(k)} + a_{k-1}(z)f^{(k-1)} + \cdots + a_1(z)f' + a_0(z)f = 0$ with polynomial coefficients. In the present paper, it is shown by an example that a unit disc counterpart of such finite set does not contain all possible T- and M-orders of solutions, with respect to Nevanlinna characteristic and maximum modulus, if the coefficients are analytic functions belonging either to weighted Bergman spaces or to weighted Hardy spaces. In contrast to a finite set, possible intervals for T- and M-orders are introduced to give detailed information about the growth of solutions. Finally, these findings yield sharp lower bounds for the sums of T- and M-orders of functions in the solution bases.

1. Introduction

This research is a continuation of recent activity in the field of complex differential equations. In particular, the present paper concerns linear differential equations of the type

$$f^{(k)} + a_{k-1}(z)f^{(k-1)} + \cdots + a_1(z)f' + a_0(z)f = 0, \tag{1.1}$$

where the coefficients $a_0(z), \ldots, a_{k-1}(z)$ are analytic functions in the unit disc $\mathbb{D} := \{z : |z| < 1\}$ of the complex plane $\mathbb{C}$. A variety of publications in the existing literature illustrate that the connection between the growth of coefficient functions and the growth of solutions is relatively well understood. On the one hand, the growth estimates in [1] have been proven to be instrumental tools in estimating the growth of solutions when the growth of coefficients is known. On the other hand, proofs of the converse direction have taken advantage of the method of order reduction as well as different types of logarithmic derivative estimates.

For an analytic function in $\mathbb{D}$, it is known that T- and M-orders of growth, with respect to Nevanlinna characteristic and maximum modulus, are not equal in general. This is in contrast to the corresponding case in $\mathbb{C}$. Hence, there are two distinct cases in $\mathbb{D}$ to work with. First, if the growth of solutions is measured by using the T-order, then it is natural to express the other growth aspects by means of integration as well. In particular, it is reasonable to consider coefficient functions belonging to some weighted Bergman spaces and use integrated estimates for logarithmic derivatives [2]. Second, if the growth of solutions is measured by using the M-order, then it is natural to express the other growth aspects by means of the maximum modulus function. In particular, it is sensible to restrict the growth of the maximum modulus of coefficient functions, which leads to weighted Hardy spaces, and work with estimates for the maximum modulus of logarithmic derivatives involving exceptional sets [3].

The main focus of this paper is in improving the lower bounds for the growth of solutions of (1.1) given in [2, 3] and explore some consequences, which are motivated by the following observations.

By the classical results in $\mathbb{C}$ making use of Newton-Puiseux diagram, there is a finite set containing the possible growth orders of solutions of (1.1) assuming that coefficients are polynomials. In particular, Gundersen-Steinbart-Wang showed that this finite set consists of rational numbers obtained from simple arithmetic with the degrees of the polynomial coefficients in (1.1) [4, Theorem 1]. Their proof relies on classical Wiman-Valiron theory in $\mathbb{C}$. Even though a recent unit disc counterpart of Wiman-Valiron theory [5] has been successfully applied to differential equations, the possible orders of solutions of (1.1) in $\mathbb{D}$ have been obtained only by assuming that coefficients are α-polynomial regular. These α-polynomial regular functions have similar growth properties than polynomials in the sense that maximal growth is attained in every direction. However, they appear to be only a relatively small subset of the Korenblum space, which characterizes finite-order solutions of (1.1) in $\mathbb{D}$ [6, Theorem 6.1]. Note that in the case of $\mathbb{C}$, all solutions of (1.1) are of finite order if and only if coefficients are polynomials [7, Satz 1].

In the present paper, it is shown by an example that a unit disc counterpart of the finite set constructed by Gundersen-Steinbart-Wang does not contain all possible orders of solutions of (1.1), provided that the coefficients belong either to weighted Bergman spaces or to weighted Hardy spaces. In contrast to a finite set, we introduce possible intervals for T-orders and M-orders, giving detailed information about the growth of solutions. Finally, these findings are applied to estimate the sums of T- and M-orders of functions in the solution bases of (1.1) from below.

2. Results and Motivation

The results concerning T- and M-orders of solutions of (1.1) are given, respectively, in Sections 2.1-2.2 and 2.3-2.4. Due to the similarities of the assertions, we omit the proofs of results regarding M-orders of solutions of (1.1), excluding the sketched proof of Theorem 2.5 in Section 7.

Let $\mathcal{M}(\mathbb{D})$ and $\mathcal{H}(\mathbb{D})$ denote the sets of all meromorphic and analytic functions in $\mathbb{D}$. For simplicity, we write $\alpha^+ := \max\{\alpha, 0\}$ for any $\alpha \in \mathbb{R}$, $|f(z)| \lesssim |g(z)|$ if there exists a constant

$C > 0$ independent of z such that $|f(z)| \le C|g(z)|$, and $f(z) \sim g(z)$ if there exist constants $C_1 > 0$ and $C_2 > 0$ independent of z such that $C_1|g(z)| \le |f(z)| \le C_2|g(z)|$.

2.1. Growth of Solutions with Respect to Nevanlinna Characteristic

The *T-order of growth* of $f \in \mathcal{M}(\mathbb{D})$ is defined as

$$\sigma_T(f) := \limsup_{r \to 1^-} \frac{\log^+ T(r,f)}{-\log(1-r)}, \tag{2.1}$$

where $T(r,f)$ is the *Nevanlinna characteristic* of f. For $p > 0$ and $\alpha > -1$, the *weighted Bergman space* A^p_α consists of those $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{A^p_\alpha} := \left(\int_{\mathbb{D}} |f(z)|^p \left(1 - |z|^2\right)^\alpha dm(z) \right)^{1/p} < \infty. \tag{2.2}$$

Functions of maximal growth in $\bigcap_{q<\alpha<\infty} A^p_\alpha$ are distinguished by denoting $f \in \mathbb{A}^p_q$, if $q = \inf\{\alpha > -1 : f \in A^p_\alpha\}$.

If the growth of the coefficients is expressed by means of integration, then it is natural to consider the growth of solutions of (1.1) with respect to T-order.

Theorem A (see [2, Theorems 1 and 2]). *Suppose that $a_j \in \mathbb{A}^{1/(k-j)}_{\alpha_j}$, where $\alpha_j \ge 0$ for $j = 0, \ldots, k-1$, and denote $\alpha_k := 0$.*

(i) *Let $0 \le \alpha < \infty$. Then all solutions f of* (1.1) *satisfy $\sigma_T(f) \le \alpha$ if and only if $\max_{j=0,\ldots,k-1}\{\alpha_j\} \le \alpha$.*

(ii) *All nontrivial solutions f of* (1.1) *satisfy*

$$\min_{j=1,\ldots,k} \left\{ \frac{k(\alpha_0 - \alpha_j)}{j} + \alpha_j \right\} \le \sigma_T(f) \le \max_{j=0,\ldots,k-1} \{\alpha_j\}. \tag{2.3}$$

(iii) *If $q \in \{0, \ldots, k-1\}$ is the smallest index for which $\alpha_q = \max_{j=0,\ldots,k-1}\{\alpha_j\}$, then each solution base of* (1.1) *contains at least $k - q$ linearly independent solutions f such that $\sigma_T(f) = \alpha_q$.*

The assumption $a_j \in \mathbb{A}^{1/(k-j)}_{\alpha_j}$ in Theorem A(i) cannot be replaced by $a_j \in A^{1/(k-j)}_{\alpha_j}$; see [8]. We refine Theorem A and then further underscore its consequences.

Theorem 2.1. *Suppose that $a_j \in \mathbb{A}^{1/(k-j)}_{\alpha_j}$, where $\alpha_j \ge -1$ for $j = 0, \ldots, k-1$, and let $q \in \{0, \ldots, k-1\}$ be the smallest index for which $\alpha_q = \max_{j=0,\ldots,k-1}\{\alpha_j\}$. If $s \in \{0, \ldots, q\}$, then each solution base of* (1.1) *contains at least $k - s$ linearly independent solutions f such that*

$$\min_{j=s+1,\ldots,k} \left\{ \frac{(k-s)(\alpha_s - \alpha_j)}{j-s} + \alpha_j \right\} \le \sigma_T(f) \le \alpha_q^+, \tag{2.4}$$

where $\alpha_k := -1$.

The case $s = 0$ clearly reduces to Theorem A(ii). If $s = q$, then the condition $\alpha_q = \max_{j=0,\dots,k-1}\{\alpha_j\}$ implies that

$$\min_{j=q+1,\dots,k}\left\{\frac{(k-q)(\alpha_q-\alpha_j)}{j-q}+\alpha_j\right\} = \min_{j=q+1,\dots,k}\left\{\frac{(k-j)(\alpha_q-\alpha_j)}{j-q}+\alpha_q\right\} = \alpha_q, \tag{2.5}$$

where the minimum is attained for $j = k$. Hence the assertion of Theorem 2.1 for $s = q$ is contained in Theorem A(iii). Our contribution is to extend the first inequality in (2.4) for $s \in \{1,\dots,q-1\}$. Theorem 2.1 is proved in Section 4, and the sharpness and the special cases $k = 2$ and $k = 3$ are further discussed in Section 3.1.

Let $q \in \{0,\dots,k-1\}$ be the smallest index for which $\alpha_q = \max_{j=0,\dots,k-1}\{\alpha_j\}$. If $\alpha_q \le 0$, then all solutions in each solution base of (1.1) are of zero T-order by Theorem A(ii). Suppose that $\alpha_q > 0$. In order to state the following corollaries of Theorem 2.1, we denote

$$\beta_T(s) := \min_{j=s+1,\dots,k}\left\{\frac{(k-s)(\alpha_s-\alpha_j)}{j-s}+\alpha_j\right\}, \quad s = 0,\dots,q, \tag{2.6}$$

where $\alpha_k := -1$. Moreover, we define

$$s^\star := \min\{s \in \{0,\dots,q\} : \beta_T(s) > 0\}. \tag{2.7}$$

Remark that $\beta_T(q) > 0$, since (2.5) implies $\alpha_q = \beta_T(q)$.

Corollary 2.2. *Suppose that $a_j \in \mathbb{A}_{\alpha_j}^{1/(k-j)}$, where $\alpha_j \ge -1$ for $j = 0,\dots,k-1$, and let $q \in \{0,\dots,k-1\}$ be the smallest index for which $\alpha_q = \max_{j=0,\dots,k-1}\{\alpha_j\} > 0$. Then each solution base of* (1.1) *admits at most $s^\star \le q$ solutions f satisfying $\sigma_T(f) < \beta_T(s^\star)$. In particular, there are at most $s^\star \le q$ solutions f satisfying $\sigma_T(f) = 0$.*

To estimate the quantity $\sum_{j=1}^{k}\sigma_T(f_j)$ by using Theorem 2.1, we set

$$\gamma_T(j) := \max\{\beta_T(0),\dots,\beta_T(j)\}, \quad j = 0,\dots,q. \tag{2.8}$$

Evidently $\gamma_T(j) > 0$ for $j \in \{s^\star,\dots,q\}$, and $\gamma_T(j) \le 0$ for $j \in \{0,\dots,s^\star-1\}$.

Corollary 2.3. *Suppose that $a_j \in \mathbb{A}_{\alpha_j}^{1/(k-j)}$, where $\alpha_j \ge -1$ for $j = 0,\dots,k-1$, and let $q \in \{0,\dots,k-1\}$ be the smallest index for which $\alpha_q = \max_{j=0,\dots,k-1}\{\alpha_j\} > 0$. Let $\{f_1,\dots,f_k\}$ be a solution base of* (1.1)*. If $q = 0$, then $\sum_{j=1}^{k}\sigma_T(f_j) = k\alpha_0$, while if $q \ge 1$, then*

$$(k-q)\alpha_q + \sum_{j=s^\star}^{q-1}\gamma_T(j) \le \sum_{j=1}^{k}\sigma_T(f_j) \le k\alpha_q. \tag{2.9}$$

Note that the sum in (2.9) is considered to be empty, if $s^\star = q$.

2.2. Gundersen-Steinbart-Wang Method for T-Order

We proceed to give an alternative statement of Theorem 2.1 and its corollaries by modifying the key steps in [4]. This yields a natural way to define possible intervals for T-orders of solutions of (1.1). As a consequence, we get a useful estimate following from Corollary 2.3.

Set $\delta_j := (\alpha_j+1)(k-j)$ for all $j=0,\ldots,k-1$. Let $s_1 \in \{0,\ldots,k-1\}$ be the smallest index satisfying $\alpha_{s_1} = \max_{j=0,\ldots,k-1}\{\alpha_j\} > 0$, which is equivalent to

$$\frac{\delta_{s_1}}{k-s_1} = \max_{j=0,\ldots,k-1}\left\{\frac{\delta_j}{k-j}\right\} > 1. \tag{2.10}$$

If s_1 cannot be found, then all solutions of (1.1) are of zero T-order by Theorem A(ii). Otherwise, for a given s_m, $m\in\mathbb{N}$, let $s_{m+1}\in\{0,\ldots,s_m-1\}$ be the smallest index satisfying

$$\frac{\delta_{s_{m+1}}-\delta_{s_m}}{s_m-s_{m+1}} = \max_{j=0,\ldots,s_m-1}\left\{\frac{\delta_j-\delta_{s_m}}{s_m-j}\right\} > 1. \tag{2.11}$$

Eventually this process will stop, yielding a finite list of indices $s_1,\ldots,s_p$ such that $p\le k$ and $s_1>s_2>\cdots>s_p\ge 0$. Further, set

$$\mathcal{B}_T(t) := \frac{\delta_{s_t}-\delta_{s_{t-1}}}{s_{t-1}-s_t} - 1, \quad t=1,\ldots,p, \tag{2.12}$$

where $s_0 := k$ and $\delta_k := 0$. Due to resemblance between (2.12) and [4, Equation (2.4)], it seems plausible that the possible nonzero T-orders of solutions of (1.1) in the unit disc case could be found among the numbers $\mathcal{B}_T(t)$, where $t=1,\ldots,p$. However, Example 3.1 shows that this is not the case.

The following lemma allows us to view the results in Section 2.1 in a new perspective. In particular, Lemma 2.4 emphasizes the connection between $\mathcal{B}_T$ and γ_T.

Lemma 2.4. *One has the following:*

(i) $\mathcal{B}_T(1) > \mathcal{B}_T(2) > \cdots > \mathcal{B}_T(p) > 0$;

(ii) $\beta_T(s_t) = \mathcal{B}_T(t)$ *for all* $t\in\{1,\ldots,p\}$;

(iii) $\gamma_T(q) = \mathcal{B}_T(1)$, $\gamma_T(j) = \mathcal{B}_T(t)$ *for all* $s_t \le j < s_{t-1}$ *and* $t\in\{2,\ldots,p\}$, *and* $\gamma_T(j)\le 0$ *for all* $j<s_p$. *In particular,* $s_p = s^\star$.

By relying on Lemma 2.4, Theorem 2.1, and Corollary 2.2, we proceed to state *possible intervals for T-orders* of functions in solution bases of (1.1) in the case $a_j \in \mathbb{A}_{\alpha_j}^{1/(k-j)}$, where $\alpha_j \ge -1$ for $j=0,\ldots,k-1$. In fact, each solution base of (1.1) contains the following:

(i) at least $k-s_1$ solutions f satisfying $\sigma_T(f) = \mathcal{B}_T(1)$;

(ii) at least $k-s_t$ solutions f satisfying $\sigma_T(f) \in [\mathcal{B}_T(t), \mathcal{B}_T(1)]$ for $t=2,\ldots,p$;

(iii) at most s_p solutions f satisfying $\sigma_T(f) \in [0, \mathcal{B}_T(s_p))$.

For the following application, let $\{f_1,\dots,f_k\}$ be a solution base of (1.1). Knowing the possible intervals for T-orders, we get

$$\sum_{j=1}^{k}\sigma_T(f_j)\geq (k-s_1)\mathcal{B}_T(1)+\cdots+(s_{p-1}-s_p)\mathcal{B}_T(p)+s_p\cdot 0=\delta_{s_p}+s_p-k. \tag{2.13}$$

In view of Lemma 2.4, the lower estimates in (2.9) and (2.13) are equal.

Finally, we point out a useful consequence of (2.13). If $s_p=0$, then $\delta_{s_p}+s_p=\delta_0$. If $s_p>0$, then $(\delta_0-\delta_{s_p})/s_p\leq 1$ by (2.11), and $\delta_{s_p}+s_p\geq\delta_0$. Hence, in both cases we can state that

$$\sum_{j=1}^{k}\sigma_T(f_j)\geq\delta_{s_p}+s_p-k\geq\delta_0-k\geq\alpha_0 k, \tag{2.14}$$

where the equalities hold, if $\alpha_0=\max_{j=0,\dots,k-1}\{\alpha_j\}>0$.

2.3. Growth of Solutions with Respect to Maximum Modulus

Alongside of the T-order, we may also define the *M-order of growth* of $f\in\mathscr{H}(\mathbb{D})$ by

$$\sigma_M(f):=\limsup_{r\to 1^-}\frac{\log^+\log^+M(r,f)}{-\log(1-r)}, \tag{2.15}$$

where $M(r,f):=\max_{|z|=r}|f(z)|$ is the maximum modulus of f. It is well known that the inequalities

$$\sigma_T(f)\leq\sigma_M(f)\leq\sigma_T(f)+1 \tag{2.16}$$

are satisfied for all $f\in\mathscr{H}(\mathbb{D})$, and all possibilities allowed by (2.16) can be assumed [9, Theorems 3.5–3.7]. A function $f\in\mathscr{H}(\mathbb{D})$ is said to belong to the *weighted Hardy space H_α^∞*, if there exists $\alpha\geq 0$ such that

$$\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^{\alpha}|f(z)|<\infty. \tag{2.17}$$

Functions of maximal growth in $\bigcap_{\alpha>p}H_\alpha^\infty$ are distinguished by denoting $f\in\mathbb{H}_p^\infty$, if $p=\inf\{\alpha\geq 0: f\in H_\alpha^\infty\}$. Remark that $H_0^\infty=H^\infty$ is the space of all bounded analytic functions in $\mathbb{D}$. The union $\bigcup_{\alpha>0}H_\alpha^\infty$ is also known as the *Korenblum space* $\mathcal{A}^{-\infty}$ [10], and since [11] $\mathbb{H}_p^\infty$ is also known as G_p.

If the growth of coefficients is measured by means of maximum modulus estimates, then it is natural to consider the growth of solutions with respect to M-order.

Theorem B (see [3, Theorem 1.4]). *Suppose that $a_j\in\mathbb{H}^\infty_{(p_j+1)(k-j)}$, where $p_j\geq -1$ for $j=0,\dots,k-1$, and denote $p_k:=-1$.*

(i) *Suppose that*

$$\min_{j=1,\ldots,k}\left\{\frac{k(p_0-p_j)}{j}+p_j\right\}>1, \tag{2.18}$$

and let $1 \leq \alpha < \infty$. *Then all solutions* f *of* (1.1) *satisfy* $\sigma_M(f) \leq \alpha$ *if and only if* $\max_{j=0,\ldots,k-1}\{p_j\} \leq \alpha$.

(ii) *All nontrivial solutions* f *of* (1.1) *satisfy* $\sigma_M(f) \leq \max_{j=0,\ldots,k-1}\{p_j^+\}$, *and*

$$\min_{j=1,\ldots,k}\left\{\frac{k(p_0-p_j)}{j}+p_j\right\} \leq \max\{\sigma_M(f),1\}. \tag{2.19}$$

(iii) *Suppose that* (2.18) *holds. If* $q \in \{0,\ldots,k-1\}$ *is the smallest index for which* $p_q = \max_{j=0,\ldots,k-1}\{p_j\}$, *then each solution base of* (1.1) *contains at least* $k-q$ *linearly independent solutions* f *such that* $\sigma_M(f) = p_q$.

To conclude [3, Equation (4.17)] in the proof of Theorem B, the inequality [3, Equation (1.9)], corresponding to (2.18), must be strict. By a simple modification of the proof of Theorem B, the assumption (2.18) can be relaxed to

$$\max_{j=0,\ldots,k-1}\{p_j\} > 1, \tag{2.20}$$

which allows us to apply Theorem B(iii) also in the case that there are solutions f satisfying $\sigma_M(f) \leq 1$. To see that (2.20) is in fact a weaker assumption than (2.18), we refer to [2, Example 10], which is further considered in Section 3.2. In this case

$$\min_{j=1,\ldots,k}\left\{\frac{k(p_0-p_j)}{j}+p_j\right\}=-4, \qquad \max_{j=0,\ldots,k-1}\{p_j\}>1. \tag{2.21}$$

Note that by taking $j=k$ in (2.18), we obtain $p_0 > 1$. Hence (2.18) implies (2.20).

Theorem 2.5 corresponds to Theorem 2.1.

Theorem 2.5. *Suppose that* $a_j \in \mathbb{H}^{\infty}_{(p_j+1)(k-j)}$, *where* $p_j \geq -1$ *for* $j=0,\ldots,k-1$, *and let* $q \in \{0,\ldots,k-1\}$ *be the smallest index for which* $p_q = \max_{j=0,\ldots,k-1}\{p_j\}$. *If* $s \in \{0,\ldots,q\}$, *then each solution base of* (1.1) *contains at least* $k-s$ *linearly independent solutions* f *such that*

$$\min_{j=s+1,\ldots,k}\left\{\frac{(k-s)(p_s-p_j)}{j-s}+p_j\right\} \leq \max\{\sigma_M(f),1\}. \tag{2.22}$$

Note that (2.22) gives information on $\sigma_M(f)$ only in the case when the minimum in (2.22) is strictly greater than 1. The case $s=0$ in Theorem 2.5 reduces to Theorem B(ii), and the case $s=q$ reduces to Theorem B(iii) with the assumption (2.20), since now

$$\min_{j=q+1,\ldots,k}\left\{\frac{(k-q)(p_q-p_j)}{j-q}+p_j\right\}=p_q=\max_{j=0,\ldots,k-1}\{p_j\}, \tag{2.23}$$

where the minimum is attained for $j=k$. For a similar argumentation, see (2.5). Our contribution is to extend (2.22) for $s\in\{1,\ldots,q-1\}$. The proof of Theorem 2.5 is sketched in Section 7, and the sharpness and the the special cases $k=2$ and $k=3$ are further discussed in Section 3.2.

Let $q\in\{0,\ldots,k-1\}$ be the smallest index for which $p_q=\max_{j=0,\ldots,k-1}\{p_j\}$. If $p_q\le 1$, then all solutions f in each solution base of (1.1) satisfy $\sigma_M(f)\le 1$ by Theorem B(ii). Suppose that $p_q>1$. In order to state the following corollaries of Theorem 2.5, we denote

$$\beta_M(s):=\min_{j=s+1,\ldots,k}\left\{\frac{(k-s)(p_s-p_j)}{j-s}+p_j\right\},\quad s=0,\ldots,q, \tag{2.24}$$

where $p_k:=-1$. Moreover, we define

$$s^\star:=\min\{s\in\{0,\ldots,q\}:\beta_M(s)>1\}. \tag{2.25}$$

Remark that $\beta_M(q)>1$, since (2.23) implies $p_q=\beta_M(q)$.

Corollary 2.6. *Suppose that $a_j\in\mathbb{H}^\infty_{(p_j+1)(k-j)}$, where $p_j\ge -1$ for $j=0,\ldots,k-1$, and let $q\in\{0,\ldots,k-1\}$ be the smallest index for which $p_q=\max_{j=0,\ldots,k-1}\{p_j\}>1$. Then each solution base of* (1.1) *admits at most $s^\star\le q$ solutions f satisfying $\sigma_M(f)<\beta_M(s^\star)$. In particular, there are at most $s^\star\le q$ solutions f satisfying $\sigma_M(f)\le 1$.*

To estimate the quantity $\sum_{j=1}^k\sigma_M(f_j)$ by using Theorem 2.5, we set

$$\gamma_M(j):=\max\{\beta_M(0),\ldots,\beta_M(j)\},\quad j=0,\ldots,q. \tag{2.26}$$

Evidently $\gamma_M(j)>1$ for $j\in\{s^\star,\ldots,q\}$, and $\gamma_M(j)\le 1$ for $j\in\{0,\ldots,s^\star-1\}$.

Corollary 2.7. *Suppose that $a_j\in\mathbb{H}^\infty_{(p_j+1)(k-j)}$, where $p_j\ge -1$ for $j=0,\ldots,k-1$, and let $q\in\{0,\ldots,k-1\}$ be the smallest index for which $p_q=\max_{j=0,\ldots,k-1}\{p_j\}>1$. Let $\{f_1,\ldots,f_k\}$ be a solution base of* (1.1). *If $q=0$, then $\sum_{j=1}^k\sigma_M(f_j)=kp_0$, while if $q\ge 1$, then*

$$(k-q)p_q+\sum_{j=s^\star}^{q-1}\gamma_M(j)\le\sum_{j=1}^{k}\sigma_M(f_j)\le kp_q. \tag{2.27}$$

Note that the sum in (2.27) is considered to be empty, if $s^\star=q$.

2.4. Gundersen-Steinbart-Wang Method for M-Order

We proceed to give an alternative statement of Theorem 2.5 and its corollaries by modifying the key steps in [4]. This yields a natural way to define the possible intervals for M-orders of solutions of (1.1). As a consequence, we get a useful estimate following from Corollary 2.7.

Set $\delta_j := (p_j+1)(k-j)$ for all $j = 0, \ldots, k-1$. Let $s_1 \in \{0, \ldots, k-1\}$ be the smallest index satisfying $p_{s_1} = \max_{j=0,\ldots,k-1}\{p_j\} > 1$, which is equivalent to

$$\frac{\delta_{s_1}}{k-s_1} = \max_{j=0,\ldots,k-1}\left\{\frac{\delta_j}{k-j}\right\} > 2. \tag{2.28}$$

If s_1 cannot be found, then all solutions f of (1.1) satisfy $\sigma_M(f) \leq 1$ by Theorem B(ii). Otherwise, for a given s_m, $m \in \mathbb{N}$, let $s_{m+1} \in \{0, \ldots, s_m - 1\}$ be the smallest index satisfying

$$\frac{\delta_{s_{m+1}} - \delta_{s_m}}{s_m - s_{m+1}} = \max_{j=0,\ldots,s_m-1}\left\{\frac{\delta_j - \delta_{s_m}}{s_m - j}\right\} > 2. \tag{2.29}$$

Eventually this process will stop, yielding a finite list of indices $s_1, \ldots, s_p$ such that $p \leq k$ and $s_1 > s_2 > \cdots > s_p \geq 0$. Further, set

$$\mathcal{B}_M(t) := \frac{\delta_{s_t} - \delta_{s_{t-1}}}{s_{t-1} - s_t} - 1, \quad t = 1, \ldots, p, \tag{2.30}$$

where $s_0 := k$ and $\delta_k := 0$. By Example 3.1, it is possible that (1.1) possesses a solution f of M-order strictly greater than one such that $\sigma_M(f) \neq \mathcal{B}_M(t)$ for all $t = 1, \ldots, p$.

The following lemma, which can be proved similarly than Lemma 2.4, allows us to view the results in Section 2.3 in a new perspective.

Lemma 2.8. *One has the following:*

(i) $\mathcal{B}_M(1) > \mathcal{B}_M(2) > \cdots > \mathcal{B}_M(p) > 1$;

(ii) $\beta_M(s_t) = \mathcal{B}_M(t)$ *for all* $t \in \{1, \ldots, p\}$;

(iii) $\gamma_M(q) = \mathcal{B}_M(1)$, $\gamma_M(j) = \mathcal{B}_M(t)$ *for all* $s_t \leq j < s_{t-1}$ *and* $t \in \{2, \ldots, p\}$, *and* $\gamma_M(j) \leq 1$ *for all* $j < s_p$. *In particular,* $s_p = s^\star$.

By relying on Lemma 2.8, Theorem 2.5, and Corollary 2.6, we proceed to state *possible intervals for M-orders* of functions in solution bases of (1.1) in the case $a_j \in \mathbb{H}^\infty_{(p_j+1)(k-j)}$, where $p_j \geq -1$ for $j = 0, \ldots, k-1$. In fact, each solution base of (1.1) contains the following:

(i) at least $k - s_1$ solutions f satisfying $\sigma_M(f) = \mathcal{B}_M(1)$;

(ii) at least $k - s_t$ solutions f satisfying $\sigma_M(f) \in [\mathcal{B}_M(t), \mathcal{B}_M(1)]$ for $t = 2, \ldots, p$;

(iii) at most s_p solutions f satisfying $\sigma_M(f) \in [0, \mathcal{B}_M(s_p))$.

For results of the same type, we refer to [12, Theorem 1] and [13, Corollary 1]. To compare (i) and (ii) to the estimates given in [13, Corollary 1], note that there is −1 in (2.30) instead of −2 in [13, Equation (1.3)]. Evidently, assertions (i) and (ii) improve the estimates given for the M-orders of solutions in [13, Corollary 1]. Moreover, by means of (2.16) we see that (i) and (ii) reduce to [13, Corollary 1], if we consider the growth of solutions of (1.1) with respect to T-order.

For the following application, let $\{f_1, \dots, f_k\}$ be a solution base of (1.1). Knowing the possible intervals for M-orders, we get

$$\sum_{j=1}^{k} \sigma_M(f_j) \geq (k - s_1)\mathcal{B}_M(1) + \cdots + (s_{p-1} - s_p)\mathcal{B}_M(p) + s_p \cdot 0 = \delta_{s_p} + s_p - k. \tag{2.31}$$

Corresponding to the case in Section 2.2, by means of Lemma 2.8 we see that the lower estimates in (2.27) and (2.31) are equal.

Finally, we point out a practical estimate, which is a consequence of (2.31). If $s_p = 0$, then $\delta_{s_p} + s_p = \delta_0 - s_p$. If $s_p > 0$, then $(\delta_0 - \delta_{s_p})/s_p \leq 2$ by (2.29), and $\delta_{s_p} + s_p \geq \delta_0 - s_p$. Hence, in both cases we can state that

$$\sum_{j=1}^{k} \sigma_M(f_j) \geq \delta_0 - s_p - k \geq p_0 k - s_p. \tag{2.32}$$

We conclude that if $s_1 = 0$, then the equalities hold in (2.32), since in this case $s_p = s_1 = 0$. Note that if (2.18) holds, then we can conclude that $s_p = 0$.

3. Sharpness Discussion

3.1. Sharpness of Theorem 2.1

We may assume that $\max_{j=0,\dots,k-1}\{\alpha_j\} > 0$, for otherwise all solutions are of zero T-order. If $k = 2$, then the statement of Theorem 2.1 is contained in Theorem A, and all the assertions are sharp [2, Examples 3 and 6].

If $k = 3$, then we have three different cases to consider.

(A1) If $\alpha_1, \alpha_2 \leq \alpha_0$, then all nontrivial solutions f of (1.1) satisfy $\sigma_T(f) = \alpha_0$. In this case $s = 0 = q$.

(A2) If $\alpha_0 < \alpha_1$ and $\alpha_2 \leq \alpha_1$, then in every solution base $\{f_1, f_2, f_3\}$ of (1.1) there are at least two solutions f_1 and f_2 such that $\sigma_T(f_j) = \alpha_1$ for both $j = 1, 2$, and all solutions f_j satisfy

$$\sigma_T(f_j) \geq \min\left\{3\alpha_0 - 2\alpha_1, \frac{3}{2}\alpha_0 - \frac{1}{2}\alpha_2, \alpha_0\right\}, \quad j = 1, 2, 3. \tag{3.1}$$

In this case $s = 0$ or $s = 1 = q$.

(A3) If $\alpha_0, \alpha_1 < \alpha_2$, then in every solution base $\{f_1, f_2, f_3\}$ of (1.1) there is at least one solution f_1 such that $\sigma_T(f_1) = \alpha_2$, two solutions f_1 and f_2 such that

$$\sigma_T(f_j) \geq \min\{2\alpha_1 - \alpha_2, \alpha_1\}, \quad j = 1, 2, \tag{3.2}$$

and all solutions f_j satisfy (3.1). In this case $s = 0$, $s = 1$, or $s = 2 = q$.

It is clear that the assertion in (A1) is sharp, and so are the ones in (A2) by [2, Example 10]. Moreover, [2, Example 9] shows that the assertions in (A3) are sharp for $s = 0$ and $s = 2$. Example 3.2 shows the sharpness of the assertions in (A3) for $s = 0, 1, 2$. That is, in all cases there exists a solution for which the lower bound for the T-order of growth is attained.

3.2. Sharpness of Theorem 2.5

We may assume that $\max_{j=0,\ldots,k-1}\{p_j\} > 1$, for otherwise all solutions f of (1.1) satisfy $\max\{\sigma_M(f), 1\} = 1$, and we cannot conclude anything from (2.22). If $k = 2$, then the statement of Theorem 2.5 is contained in Theorem B, and all the assertions are sharp by [2, Examples 3 and 6]. In the case of [2, Examples 3 and 6], for $\beta > 1$, linearly independent solutions f_1 and f_2 satisfy $\sigma_M(f_1) = \beta$ and $\sigma_M(f_2) = \beta + 2$. Moreover, $a_j \in \mathbb{H}^\infty_{(p_j+1)(2-j)}$, where $p_0 = \beta + 1$ and $p_1 = \beta + 2$. Note that $\max\{p_0, p_1\} = p_1 = \beta + 2 > 1$, and hence $q = 1$. An easy computation shows the sharpness for $s = 0$ and for $s = q = 1$. In the case of [2, Example 6], for $\beta > 1$, linearly independent solutions f_1 and f_2 satisfy $\sigma_M(f_1) = \beta$ and $\sigma_M(f_2) = \beta$. Moreover, $a_j \in \mathbb{H}^\infty_{(p_j+1)(2-j)}$, where $p_0 = \beta$ and $p_1 = -1/2$. Note now that $\max\{p_0, p_1\} = p_0 = \beta > 1$, and hence $q = 0$. This example shows the sharpness for $s = q = 0$. For another example, see [3, Example 2].

If $k = 3$, then we have three different cases to consider.

(B1) If $p_1, p_2 \le p_0$, then $s = 0 = q$, and all nontrivial solutions f of (1.1) satisfy $\sigma_M(f) = p_0$ by (2.23).

(B2) If $p_0 < p_1$, and $p_2 \le p_1$, then in every solution base $\{f_1, f_2, f_3\}$ of (1.1) there are at least two solutions f_1 and f_2 such that $\sigma_M(f_j) = p_1$ for both $j = 1, 2$, and all solutions f_j satisfy

$$\max\{\sigma_M(f_j), 1\} \ge \min\left\{3p_0 - 2p_1, \frac{3}{2}p_0 - \frac{1}{2}p_2, p_0\right\}, \quad j = 1, 2, 3. \tag{3.3}$$

In this case $s = 0$ or $s = 1 = q$.

(B3) If $p_0, p_1 < p_2$, then in every solution base $\{f_1, f_2, f_3\}$ of (1.1) there is at least one solution f_1 such that $\sigma_M(f_1) = p_2$, two solutions f_1 and f_2 such that

$$\max\{\sigma_M(f_j), 1\} \ge \min\{2p_1 - p_2, p_1\}, \quad j = 1, 2, \tag{3.4}$$

and all solutions f_j satisfy (3.3). In this case $s = 0$, $s = 1$, or $s = 2 = q$.

It is clear that the assertion in (B1) is sharp. By [2, Example 10], we see that the assertion in (B2) corresponding to $s = 1$ is sharp. In this case, for $\beta > 1$, linearly independent solutions f_1, f_2, and f_3 satisfy $\sigma_M(f_1) = \sigma_M(f_2) = \beta$ and $\sigma_M(f_3) = 0$. Now $a_j \in \mathbb{H}^\infty_{(p_j+1)(3-j)}$, where $p_0 = (2/3)\beta$, $p_1 = \beta + 2$, and $p_2 = 0$. Moreover, by [2, Example 9], we see that the assertions in (B3) are sharp for $s = 0$ and $s = 2$. In this case for $\beta > 1$, linearly independent solutions f_1, f_2, and f_3 satisfy $\sigma_M(f_1) = \sigma_M(f_2) = \beta$ and $\sigma_M(f_3) = 2\beta$. Moreover, $a_j \in \mathbb{H}^\infty_{(p_j+1)(3-j)}$, where $p_0 = (4/3)\beta$, $p_1 = \beta$, and $p_2 = 2\beta$. Example 3.2 shows the sharpness of the assertions in (B3) for $s = 0, 1, 2$. That is, in all cases there exists a solution for which equality holds in (2.22).

3.3. Examples

Example 3.1 shows that a unit disc counterpart of the finite set constructed by Gundersen-Steinbart-Wang does not contain growth orders of solutions of

$$f'' + a_1(z)f' + a_0(z)f = 0, \tag{3.5}$$

if coefficients belong either to weighted Bergman spaces or to weighted Hardy spaces.

Example 3.1. Let $\alpha, \beta \in \mathbb{R}$ be constants satisfying $1 < \beta < \alpha < 2\beta - 1$. Then the functions

$$\begin{aligned} f_1(z) &= (1-z)^{\alpha+\beta} \exp\left(\left(\frac{1}{1-z}\right)^{\alpha} + \left(\frac{1}{1+z}\right)^{\beta}\right), \\ f_2(z) &= (1-z)^{\alpha+\beta} \exp\left(\frac{1}{1+z}\right)^{\beta} \end{aligned} \tag{3.6}$$

are linearly independent analytic solutions of (3.5), where

$$\begin{aligned} a_0(z) &= \frac{\beta^2}{(1+z)^{2+2\beta}} + \frac{\beta(\alpha+3\beta)(\gamma+z)}{(1-z)(1+z)^{2+\beta}} - \frac{\alpha(\alpha+\beta)}{(1-z)^{2+\alpha}} \\ &\quad - \frac{\alpha\beta}{(1-z)^{1+\alpha}(1+z)^{1+\beta}} + \frac{\beta(\alpha+\beta)}{(1-z)^2}, \\ a_1(z) &= \frac{2\beta}{(1+z)^{1+\beta}} - \frac{\alpha}{(1-z)^{1+\alpha}} + \frac{\alpha+2\beta-1}{1-z} \end{aligned} \tag{3.7}$$

belong to $\mathscr{H}(\mathbb{D})$, and $\gamma = (\alpha+\beta-2)/(\alpha+3\beta) \in (0,1/2)$.

It is clear that $a_j \in \mathbb{A}_{\alpha_j}^{1/(2-j)}$, where $\alpha_0 = \beta - 1$ and $\alpha_1 = \alpha - 1$. We calculate that $s_1 = 1$, $s_2 = 0$, $\mathcal{B}_T(1) = \alpha - 1$, and $\mathcal{B}_T(2) = 2\beta - \alpha - 1$. Hence $[2\beta - \alpha - 1, \alpha - 1]$ is the only possible interval for T-orders of solutions of (3.5). Since $\sigma_T(f_2) = \beta - 1$, we conclude that the T-order of a solution does not have to be one of the endpoints.

On the other hand, it is also clear that $a_j \in \mathbb{H}^{\infty}_{(p_j+1)(2-j)}$, where $p_0 = \beta$ and $p_1 = \alpha$. We calculate that $s_1 = 1$, $s_2 = 0$, $\mathcal{B}_M(1) = \alpha$, and $\mathcal{B}_M(2) = 2\beta - \alpha$. Hence $[2\beta - \alpha, \alpha]$ is the only possible interval for M-orders of solutions of (3.5). Since $\sigma_M(f_2) = \beta$, we conclude that the M-order of a solution does not have to be one of the endpoints.

The following example demonstrates the sharpness of Theorems 2.1 and 2.5 in the case that they do not reduce to known results.

Example 3.2. Let $\beta > 1$, and denote $g(z) = (5/(1-z))^{\beta}$. Then the functions

$$f_j(z) = (1-z)^{\beta} \exp\left(g(z)^j\right), \quad j = 1,2,3, \tag{3.8}$$

are linearly independent solutions of $f''' + a_2(z)f'' + a_1(z)f' + a_0(z)f = 0$, where

$$a_2(z) = \frac{P_2(g(z))}{(1-z)Q(g(z))}, \qquad a_1(z) = \frac{P_1(g(z))}{(1-z)^2 Q(g(z))}, \qquad a_0(z) = \frac{\beta^3 P_0(g(z))}{(1-z)^3 Q(g(z))} \tag{3.9}$$

are such that

$$\begin{aligned}
P_2(\zeta) &= 54\beta\zeta^8 - 27\beta\zeta^7 - 24\beta\zeta^6 + (108\beta+54)\zeta^5 - (82\beta+63)\zeta^4 + 3\beta\zeta^3 \\
&\quad + (22\beta+39)\zeta^2 - (6\beta+24)\zeta + 6, \\
P_1(\zeta) &= -108\beta^2\zeta^{10} + 72\beta^2\zeta^9 + \left(27\beta^2 - 54\beta\right)\zeta^8 + \left(27\beta - 135\beta^2\right)\zeta^7 \\
&\quad + \left(24\beta + 15\beta^2\right)\zeta^6 + \left(64\beta^2 - 108\beta - 18\right)\zeta^5 + \left(21 + 82\beta - 51\beta^2\right)\zeta^4 \\
&\quad - \left(3\beta + 15\beta^2\right)\zeta^3 + \left(31\beta^2 - 22\beta - 13\right)\zeta^2 + \left(8 + 6\beta - 14\beta^2\right)\zeta + 2\beta^2 - 2, \\
P_0(\zeta) &= 108\zeta^{11} - 234\zeta^{10} + 126\zeta^9 + 123\zeta^8 - 276\zeta^7 + 183\zeta^6 - 104\zeta^5 + 40\zeta^4 \\
&\quad - 6\zeta^3 - 4\zeta^2, \\
Q(\zeta) &= -18\zeta^5 + 21\zeta^4 - 13\zeta^2 + 8\zeta - 2.
\end{aligned} \tag{3.10}$$

The zeros of $Q(\zeta)$ lie in the open disc of radius $1 + 21/18$ centered at the origin by [14, Lemma 1.3.2]. Since $|g(z)| > |5/(1-z)| > 5/2 > 39/18$ for all $z \in \mathbb{D}$, we conclude that $a_0, a_1, a_2 \in \mathcal{H}(\mathbb{D})$. In fact, the coefficients $a_0, a_1,$ and a_2 satisfy

$$a_2(z) \sim \left(\frac{1}{1-z}\right)^{3\beta+1}, \qquad a_1(z) \sim \left(\frac{1}{1-z}\right)^{5\beta+2}, \qquad a_0(z) \sim \left(\frac{1}{1-z}\right)^{6\beta+3}, \tag{3.11}$$

in a neighborhood of $z = 1$, while they are bounded in a neighborhood of any boundary point in $\partial\mathbb{D} \setminus \{1\}$.

Note that $a_j \in \mathbb{A}_{\alpha_j}^{1/(3-j)}$, where $\alpha_2 = 3\beta - 1$, $\alpha_1 = (5/2)\beta - 1$, and $\alpha_0 = 2\beta - 1$. Evidently $\sigma_T(f_j) = \beta j - 1$ for $j = 1, 2, 3$. We deduce that there is one solution f_3 such that $\sigma_T(f_3) = \alpha_2 = 3\beta - 1$, two solutions f_2 and f_3 such that

$$\sigma_T(f_3) > \sigma_T(f_2) = \min\{2\alpha_1 - \alpha_2, \alpha_1\} = 2\beta - 1, \tag{3.12}$$

and three solutions f_1, f_2, and f_3 such that

$$\sigma_T(f_3) > \sigma_T(f_2) > \sigma_T(f_1) = \min\left\{3\alpha_0 - 2\alpha_1, \frac{3}{2}\alpha_0 - \frac{1}{2}\alpha_2, \alpha_0\right\} = \beta - 1. \tag{3.13}$$

That is, in all cases $s = 0, 1, 2$ there exists a solution for which the lower bound in (2.4) is attained. Further, this example is in line with Corollary 2.2, since all solutions f_1, f_2, and f_3 are of strictly positive T-order, and in this case $s^* = 0$.

Now $\gamma_T(0) = \beta_T(0) = \beta - 1$, $\gamma_T(1) = \beta_T(1) = 2\beta - 1$, and $\gamma_T(2) = \beta_T(2) = 3\beta - 1$. It follows that for the solution base $\{f_1, f_2, f_3\}$ equality holds in the first inequality in (2.9), and for the solution base $\{f_1 + f_3, f_2 + f_3, f_3\}$ equality holds in the last inequality in (2.9). This shows the sharpness of Corollary 2.3.

On the other hand, $a_j \in \mathbb{H}^{\infty}_{(p_j+1)(3-j)}$, where $p_2 = 3\beta$, $p_1 = 5\beta/2$, and $p_0 = 2\beta$. Evidently $\sigma_M(f_j) = \beta j$ for $j = 1, 2, 3$. We deduce that there is one solution f_3 such that $\sigma_M(f_3) = p_2 = 3\beta > 1$, two solutions f_2 and f_3 such that

$$\sigma_M(f_3) > \sigma_M(f_2) = \min\{2p_1 - p_2, p_1\} = 2\beta > 1, \tag{3.14}$$

and three solutions f_1, f_2, and f_3 such that

$$\sigma_M(f_3) > \sigma_M(f_2) > \sigma_M(f_1) = \min\left\{3p_0 - 2p_1, \frac{3}{2}p_0 - \frac{1}{2}p_2, p_0\right\} = \beta. \tag{3.15}$$

That is, in all cases $s = 0, 1, 2$ there exists a solution for which the lower bound in (2.22) is attained. Further, this example is in line with Corollary 2.6, since all solutions f_1, f_2, and f_3 are of M-order strictly greater than 1, and in this case $s^\star = 0$.

Now $\gamma_M(0) = \beta_M(0) = \beta$, $\gamma_M(1) = \beta_M(1) = 2\beta$, and $\gamma_M(2) = \beta_M(2) = 3\beta$. It follows that for the solution base $\{f_1, f_2, f_3\}$ equality holds in (2.27), and for the solution base $\{f_1 + f_3, f_2 + f_3, f_3\}$ upper bound for the sum of M-orders is attained. This shows the sharpness of Corollary 2.7.

4. Proof of Theorem 2.1

The following lemma on the order reduction procedure originates from $\mathbb{C}$.

Lemma C (see [4, Lemma 6.4]). *Let $f_{0,1}, f_{0,2}, \dots, f_{0,m}$ be $m \geq 2$ linearly independent meromorphic solutions of*

$$y^{(k)} + a_{0,k-1}(z)y^{(k-1)} + \cdots + a_{0,0}(z)y = 0, \quad k \geq m, \tag{4.1}$$

where $a_{0,0}(z), \dots, a_{0,k-1}(z)$ are meromorphic functions in $\mathbb{D}$. For $1 \leq p \leq m-1$, set

$$f_{p,j} = \left(\frac{f_{p-1,j+1}}{f_{p-1,1}}\right)', \quad j = 1, \dots, m-p. \tag{4.2}$$

Then $f_{p,1}, f_{p,2}, \dots, f_{p,m-p}$ are linearly independent meromorphic solutions of

$$y^{(k-p)} + a_{p,k-p-1}(z)y^{(k-p-1)} + \cdots + a_{p,0}(z)y = 0, \tag{4.3}$$

where

$$a_{p,j}(z) = \sum_{n=j+1}^{k-p+1} \binom{n}{j+1} a_{p-1,n}(z) \frac{f_{p-1,1}^{(n-j-1)}(z)}{f_{p-1,1}(z)} \tag{4.4}$$

for $j = 0, \dots, k-p-1$. *Here* $a_{n,k-n}(z) \equiv 1$ *for all* $n = 0, \dots, p$.

Lemma D (see [15, Lemma E(b)]). *Let* k *and* j *be integers satisfying* $k > j \geq 0$, *and let* $\varepsilon > 0$. *If* f *is meromorphic in* $\mathbb{D}$ *such that* $\sigma_T(f) < \infty$, *and* $f^{(j)} \not\equiv 0$, *then*

$$\int_{\mathbb{D}} \left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right|^{1/(k-j)} (1-|z|)^{\sigma_T(f)+\varepsilon} dm(z) < \infty. \tag{4.5}$$

4.1. *Case* $s = 1$

Let $k \geq 3$, $q \geq 2$, $s = 1$, and $\beta_T(1) > 0$, since otherwise there is nothing to prove. In particular, if $\alpha_1 \leq 0$, then (2.4) is trivial, since by taking $j = k$ in (2.6), we obtain $\beta_T(1) \leq \alpha_1 \leq 0$. Let $\{f_{0,1}, f_{0,2}, \dots, f_{0,k}\}$ be a solution base of (1.1), and assume on the contrary to the assertion that there exist $s+1 = 2$ linearly independent solutions $f_{0,1}$ and $f_{0,2}$ such that $\max\{\sigma_T(f_{0,1}), \sigma_T(f_{0,2})\} =: \sigma < \beta_T(1)$. Then the meromorphic function $g := (f_{0,1}/f_{0,2})'$ satisfies $\sigma_T(g) \leq \sigma$. Moreover, Lemma C implies that g satisfies

$$g^{(k-1)} + a_{1,k-2}(z) g^{(k-2)} + \cdots + a_{1,0}(z) g = 0, \tag{4.6}$$

where

$$a_{1,j}(z) = a_{0,j+1}(z) + \sum_{n=j+2}^{k} \binom{n}{j+1} a_{0,n}(z) \frac{f_{0,1}^{(n-j-1)}(z)}{f_{0,1}(z)} \tag{4.7}$$

for $j = 0, 1, \dots, k-2$, and $a_{0,k}(z) \equiv 1$. Therefore

$$|a_{0,1}(z)| \leq |a_{1,0}(z)| + \sum_{n=2}^{k} \binom{n}{1} |a_{0,n}(z)| \left| \frac{f_{0,1}^{(n-1)}(z)}{f_{0,1}(z)} \right|, \tag{4.8}$$

where

$$|a_{1,0}(z)| \leq \left| \frac{g^{(k-1)}(z)}{g(z)} \right| + |a_{1,k-2}(z)| \left| \frac{g^{(k-2)}(z)}{g(z)} \right| + \cdots + |a_{1,1}(z)| \left| \frac{g'(z)}{g(z)} \right|, \tag{4.9}$$

since g satisfies (4.6). Putting the last two inequalities together, we obtain

$$|a_{0,1}(z)| \lesssim \sum_{j=1}^{k-1} |a_{1,j}(z)| \left| \frac{g^{(j)}(z)}{g(z)} \right| + \sum_{n=2}^{k} |a_{0,n}(z)| \left| \frac{f_{0,1}^{(n-1)}(z)}{f_{0,1}(z)} \right|. \tag{4.10}$$

Let $\varepsilon > 0$. Raising both sides to the power $1/(k-1)$ and integrating over the disc $D(0,r)$ of radius $r \in (0,1)$ with respect to $(1-|z|^2)^{\alpha_1-\varepsilon}dm(z)$, we obtain

$$\begin{aligned}
&\int_{D(0,r)} |a_{0,1}(z)|^{1/(k-1)}\left(1-|z|^2\right)^{\alpha_1-\varepsilon} dm(z) \\
&\quad\lesssim \sum_{j=1}^{k-1}\int_{\mathbb{D}} \left|a_{1,j}(z)\right|^{1/(k-1)}\left|\frac{g^{(j)}(z)}{g(z)}\right|^{1/(k-1)}\left(1-|z|^2\right)^{\alpha_1-\varepsilon} dm(z) \\
&\qquad+\sum_{n=2}^{k}\int_{\mathbb{D}} |a_{0,n}(z)|^{1/(k-1)}\left|\frac{f_{0,1}^{(n-1)}(z)}{f_{0,1}(z)}\right|^{1/(k-1)}\left(1-|z|^2\right)^{\alpha_1-\varepsilon} dm(z).
\end{aligned}\tag{4.11}$$

To deal with the second sum in (4.11), consider

$$I_n := \int_{\mathbb{D}} |a_{0,n}(z)|^{1/(k-1)}\left|\frac{f_{0,1}^{(n-1)}(z)}{f_{0,1}(z)}\right|^{1/(k-1)}\left(1-|z|^2\right)^{\alpha_1-\varepsilon} dm(z), \quad n=2,\ldots,k. \tag{4.12}$$

By Lemma D,

$$I_k = \int_{\mathbb{D}} \left|\frac{f_{0,1}^{(k-1)}(z)}{f_{0,1}(z)}\right|^{1/(k-1)}\left(1-|z|^2\right)^{\alpha_1-\varepsilon} dm(z) < \infty \tag{4.13}$$

for $\varepsilon > 0$ being small enough since $\sigma_T(f_{0,1}) \leq \sigma < \beta_T(1) \leq \alpha_1$. Moreover, by Hölder's inequality, with indices $(k-1)/(k-n)$ and $(k-1)/(n-1)$, we have

$$\begin{aligned}
I_n \leq &\left(\int_{\mathbb{D}} |a_{0,n}(z)|^{1/(k-n)}\left(1-|z|^2\right)^{\alpha_n+\varepsilon} dm(z)\right)^{(k-n)/(k-1)} \\
&\cdot\left(\int_{\mathbb{D}} \left|\frac{f_{0,1}^{(n-1)}(z)}{f_{0,1}(z)}\right|^{1/(n-1)}\left(1-|z|^2\right)^{\omega_1(n)} dm(z)\right)^{(n-1)/(k-1)}
\end{aligned}\tag{4.14}$$

for all $n = 2,\ldots,k-1$, where

$$\omega_s(n) := \frac{(k-s)(\alpha_s-\alpha_n)}{n-s} + \alpha_n - \frac{2k-n-s}{n-s}\varepsilon. \tag{4.15}$$

The first member in the product is finite since $a_{0,n} \in \mathbb{A}_{\alpha_n}^{1/(k-n)}$ for all $n = 2,\ldots,k-1$ by the assumption, and so is the second one for $\varepsilon > 0$ small enough since

$$\sigma_T(f_{0,1}) \leq \sigma < \beta_T(1) \leq \frac{(k-1)(\alpha_1-\alpha_n)}{n-1} + \alpha_n, \quad n=2,\ldots,k-1, \tag{4.16}$$

by the antithesis. Thus $\sum_{n=2}^{k} I_n$ is finite for $\varepsilon > 0$ being small enough.

To deal with the first sum in (4.11), denote

$$J_j := \int_{\mathbb{D}} |a_{1,j}(z)|^{1/(k-1)} \left|\frac{g^{(j)}(z)}{g(z)}\right|^{1/(k-1)} \left(1-|z|^2\right)^{\alpha_1-\varepsilon} dm(z), \quad j = 1,\dots,k-1. \tag{4.17}$$

Lemma D implies that

$$J_{k-1} = \int_{\mathbb{D}} \left|\frac{g^{(k-1)}(z)}{g(z)}\right|^{1/(k-1)} \left(1-|z|^2\right)^{\alpha_1-\varepsilon} dm(z) < \infty \tag{4.18}$$

for $\varepsilon > 0$ being small enough since $\sigma_T(g) \le \sigma < \beta_T(1) \le \alpha_1$. Moreover, by (4.7) we have

$$\begin{aligned} J_j &\lesssim \int_{\mathbb{D}} |a_{0,j+1}(z)|^{1/(k-1)} \left|\frac{g^{(j)}(z)}{g(z)}\right|^{1/(k-1)} \left(1-|z|^2\right)^{\alpha_1-\varepsilon} dm(z) \\ &\quad + \sum_{n=j+2}^{k} \int_{\mathbb{D}} |a_{0,n}(z)|^{1/(k-1)} \left|\frac{f_{0,1}^{(n-j-1)}(z)}{f_{0,1}(z)}\right|^{1/(k-1)} \left|\frac{g^{(j)}(z)}{g(z)}\right|^{1/(k-1)} \left(1-|z|^2\right)^{\alpha_1-\varepsilon} dm(z) \\ &=: K_j + L_{j,k} + \sum_{n=j+2}^{k-1} L_{j,n} \end{aligned} \tag{4.19}$$

for all $j = 1,\dots,k-2$. Since $\max\{\sigma_T(g), \sigma_T(f_{0,1})\} \le \sigma < \beta_T(1)$, we deduce that K_j behaves like I_{j+1} and hence $\sum_{j=1}^{k-2} K_j < \infty$ for $\varepsilon > 0$ being small enough. Moreover, by Hölder's inequality, with indices $(k-1)/(k-j-1)$ and $(k-1)/j$, and Lemma D we have

$$\begin{aligned} L_{j,k} \le &\left(\int_{\mathbb{D}} \left|\frac{f_{0,1}^{(k-j-1)}(z)}{f_{0,1}(z)}\right|^{1/(k-j-1)} \left(1-|z|^2\right)^{\alpha_1-\varepsilon} dm(z)\right)^{(k-j-1)/(k-1)} \\ &\cdot \left(\int_{\mathbb{D}} \left|\frac{g^{(j)}(z)}{g(z)}\right|^{1/j} \left(1-|z|^2\right)^{\alpha_1-\varepsilon} dm(z)\right)^{j/(k-1)} < \infty \end{aligned} \tag{4.20}$$

for all $j = 1,\dots,k-2$ when $\varepsilon > 0$ is sufficiently small. It remains to consider the double sum $\sum_{j=1}^{k-3} \sum_{n=j+2}^{k-1} L_{j,n}$. By Hölder's inequality, with indices $(k-1)/(k-n)$ and $(k-1)/(n-1)$, we have

$$\begin{aligned} L_{j,n} \le &\left(\int_{\mathbb{D}} |a_{0,n}(z)|^{1/(k-n)} \left(1-|z|^2\right)^{\alpha_n+\varepsilon} dm(z)\right)^{(k-n)/(k-1)} \\ &\cdot \left(\int_{\mathbb{D}} \left|\frac{f_{0,1}^{(n-j-1)}(z)}{f_{0,1}(z)}\right|^{1/(n-1)} \left|\frac{g^{(j)}(z)}{g(z)}\right|^{1/(n-1)} \left(1-|z|^2\right)^{\omega_1(n)} dm(z)\right)^{(n-1)/(k-1)}, \end{aligned} \tag{4.21}$$

where $\omega_1(n)$ is defined in (4.15). The first term in the product is bounded for all $\varepsilon > 0$ since $a_{0,n} \in \mathbb{A}_{\alpha_n}^{1/(k-n)}$ for all $n = 3,\dots,k-1$ by the assumption. One more application of Hölder's inequality, with indices $(n-1)/(n-j-1)$ and $(n-1)/j$, together with Lemma D and the antithesis shows that also the second term in the product is bounded for $\varepsilon > 0$ being small enough, and thus $\sum_{j=1}^{k-3}\sum_{n=j+2}^{k-1} L_{j,n} < \infty$ for $\varepsilon > 0$ being small enough.

We have proved that the right-hand side of (4.11) is uniformly bounded for all $r \in (0,1)$, if $\varepsilon > 0$ is small enough. However, $a_{0,1} \in \mathbb{A}_{\alpha_1}^{1/(k-1)}$ by the assumption, and hence the left-hand side of (4.11) diverges as $r \to 1^-$. Contradiction follows.

4.2. Case $s > 1$

Let $k \geq 3$, $q \geq 2$, $s > 1$, and $\beta_T(s) > 0$, since otherwise there is nothing to prove. In particular, it follows that $\alpha_s > 0$. Let $\{f_{0,1}, f_{0,2},\dots,f_{0,k}\}$ be a solution base of (1.1), and assume on the contrary to the assertion that there exist $s+1$ linearly independent solutions $f_{0,1},\dots,f_{0,s+1}$ such that

$$\sigma := \max\{\sigma_T(f_{0,1}),\dots,\sigma_T(f_{0,s+1})\} < \beta_T(s). \tag{4.22}$$

Then the meromorphic functions $f_{1,j} = (f_{0,j+1}/f_{0,1})'$ satisfy $\sigma_T(f_{1,j}) \leq \sigma$ for all $j = 1,\dots,s$. This in turn implies that $f_{2,j} = (f_{1,j+1}/f_{1,1})'$ satisfy $\sigma_T(f_{2,j}) \leq \sigma$ for all $j = 1,\dots,s-1$. In general, $\sigma_T(f_{n,j}) \leq \sigma$ for all $j = 1,\dots,s-n+1$ and $n = 1,\dots,s$. Moreover, as in the case $s = 1$, Lemma C implies

$$\begin{aligned}
|a_{0,s}(z)| &\leq |a_{1,s-1}(z)| + \sum_{n=s+1}^{k}\binom{n}{s}|a_{0,n}(z)|\left|\frac{f_{0,1}^{(n-s)}(z)}{f_{0,1}(z)}\right| \\
&\leq |a_{2,s-2}(z)| + \sum_{n=s}^{k-1}\binom{n}{s-1}|a_{1,n}(z)|\left|\frac{f_{1,1}^{(n-s+1)}(z)}{f_{1,1}(z)}\right| \\
&\quad + \sum_{n=s+1}^{k}\binom{n}{s}|a_{0,n}(z)|\left|\frac{f_{0,1}^{(n-s)}(z)}{f_{0,1}(z)}\right| \\
&\leq \cdots \\
&\leq |a_{s,0}(z)| + \sum_{m=0}^{s-1}\sum_{n=s+1-m}^{k-m}\binom{n}{s-m}|a_{m,n}(z)|\left|\frac{f_{m,1}^{(n-s+m)}(z)}{f_{m,1}(z)}\right|,
\end{aligned} \tag{4.23}$$

where

$$|a_{s,0}(z)| \leq \left|\frac{f_{s,1}^{(k-s)}(z)}{f_{s,1}(z)}\right| + \sum_{m=1}^{k-s-1}|a_{s,m}(z)|\left|\frac{f_{s,1}^{(m)}(z)}{f_{s,1}(z)}\right|. \tag{4.24}$$

Putting these inequalities together, we obtain

$$|a_{0,s}(z)| \lesssim \sum_{m=0}^{s} \sum_{n=s+1-m}^{k-m-1} |a_{m,n}(z)| \left| \frac{f_{m,1}^{(n-s+m)}(z)}{f_{m,1}(z)} \right| + \sum_{m=0}^{s} \left| \frac{f_{m,1}^{(k-s)}(z)}{f_{m,1}(z)} \right|. \tag{4.25}$$

Let $\varepsilon > 0$. Raising both sides to the power $1/(k-s)$ and integrating over the disc $D(0,r)$ with respect to $(1-|z|^2)^{\alpha_s-\varepsilon}dm(z)$, we obtain

$$\begin{aligned}
&\int_{D(0,r)} |a_{0,s}(z)|^{1/(k-s)} \left(1-|z|^2\right)^{\alpha_s-\varepsilon} dm(z) \\
&\lesssim \sum_{m=0}^{s} \sum_{n=s+1-m}^{k-m-1} \int_{\mathbb{D}} |a_{m,n}(z)|^{1/(k-s)} \left| \frac{f_{m,1}^{(n-s+m)}(z)}{f_{m,1}(z)} \right|^{1/(k-s)} \left(1-|z|^2\right)^{\alpha_s-\varepsilon} dm(z) \\
&\quad + \sum_{m=0}^{s} \int_{\mathbb{D}} \left| \frac{f_{m,1}^{(k-s)}(z)}{f_{m,1}(z)} \right|^{1/(k-s)} \left(1-|z|^2\right)^{\alpha_s-\varepsilon} dm(z) \\
&=: \sum_{m=0}^{s} \sum_{n=s+1-m}^{k-m-1} I_{m,n} + \sum_{m=0}^{s} J_m.
\end{aligned} \tag{4.26}$$

Lemma D and the antithesis imply that $\sum_{m=0}^{s} J_m < \infty$ for $\varepsilon > 0$ being small enough. Hence, in order to obtain a contradiction with (4.26) and the assumption $a_{0,s} \in \mathbb{A}_{\alpha_s}^{1/(k-s)}$, it suffices to show that $I_{m,n} < \infty$ for all $m = 0, \dots, s$ and $n = s+1-m, \dots, k-m-1$ when $\varepsilon > 0$ is sufficiently small.

By Hölder's inequality, with indices $(k-s)/(k-n)$ and $(k-s)/(n-s)$, we have

$$\begin{aligned}
I_{0,n} \leq &\left(\int_{\mathbb{D}} |a_{0,n}(z)|^{1/(k-n)} \left(1-|z|^2\right)^{\alpha_n+\varepsilon} dm(z) \right)^{(k-n)/(k-s)} \\
&\cdot \left(\int_{\mathbb{D}} \left| \frac{f_{0,1}^{(n-s)}(z)}{f_{0,1}(z)} \right|^{1/(n-s)} \left(1-|z|^2\right)^{\omega_s(n)} dm(z) \right)^{(n-s)/(k-s)}
\end{aligned} \tag{4.27}$$

for all $n = s+1, \dots, k-1$, where $\omega_s(n)$ is defined in (4.15). The first member in the product is finite since $a_{0,n} \in \mathbb{A}_{\alpha_n}^{1/(k-n)}$ for all $n = s+1, \dots, k-1$ by the assumption, and so is the second one for $\varepsilon > 0$ being small enough, since

$$\sigma_T\left(f_{0,1}\right) \leq \sigma < \beta_T(s) \leq \frac{(k-s)(\alpha_s - \alpha_n)}{n-s} + \alpha_n, \quad n = s+1, \dots, k-1, \tag{4.28}$$

by the antithesis. In the general case Lemma C gives

$$\begin{aligned}
I_{m,n} &= \int_{\mathbb{D}} |a_{m,n}(z)|^{1/(k-s)} \left| \frac{f_{m,1}^{(n-s+m)}(z)}{f_{m,1}(z)} \right|^{1/(k-s)} \left(1-|z|^2\right)^{\alpha_s-\varepsilon} dm(z) \\
&\lesssim \sum_{n_1=n+1}^{k-m+1} \int_{\mathbb{D}} |a_{m-1,n_1}(z)|^{1/(k-s)} \left| \frac{f_{m-1,1}^{(n_1-n-1)}(z)}{f_{m-1,1}(z)} \right|^{1/(k-s)} \\
&\quad \cdot \left| \frac{f_{m,1}^{(n-s+m)}(z)}{f_{m,1}(z)} \right|^{1/(k-s)} \left(1-|z|^2\right)^{\alpha_s-\varepsilon} dm(z) \\
&\lesssim \sum_{n_1=n+1}^{k-m+1} \sum_{n_2=n_1+1}^{k-m+2} \int_{\mathbb{D}} |a_{m-2,n_2}(z)|^{1/(k-s)} \left| \frac{f_{m-2,1}^{(n_2-n_1-1)}(z)}{f_{m-2,1}(z)} \right|^{1/(k-s)} \\
&\quad \cdot \left| \frac{f_{m-1,1}^{(n_1-n-1)}(z)}{f_{m-1,1}(z)} \right|^{1/(k-s)} \left| \frac{f_{m,1}^{(n-s+m)}(z)}{f_{m,1}(z)} \right|^{1/(k-s)} \left(1-|z|^2\right)^{\alpha_s-\varepsilon} dm(z),
\end{aligned} \tag{4.29}$$

and finally

$$I_{m,n} \lesssim \sum_{n_1=n+1}^{k-m+1} \sum_{n_2=n_1+1}^{k-m+2} \cdots \sum_{n_m=n_{m-1}+1}^{k} K(n,n_1,\dots,n_m), \tag{4.30}$$

where

$$\begin{aligned}
K(n,n_1,\dots,n_m) &:= \int_{\mathbb{D}} |a_{0,n_m}(z)|^{1/(k-s)} \left| \frac{f_{0,1}^{(n_m-n_{m-1}-1)}(z)}{f_{0,1}(z)} \right|^{1/(k-s)} \cdots \left| \frac{f_{m-2,1}^{(n_2-n_1-1)}(z)}{f_{m-2,1}(z)} \right|^{1/(k-s)} \\
&\quad \cdot \left| \frac{f_{m-1,1}^{(n_1-n-1)}(z)}{f_{m-1,1}(z)} \right|^{1/(k-s)} \left| \frac{f_{m,1}^{(n-s+m)}(z)}{f_{m,1}(z)} \right|^{1/(k-s)} \left(1-|z|^2\right)^{\alpha_s-\varepsilon} dm(z).
\end{aligned} \tag{4.31}$$

If $n_m = k$, then $a_{0,n_m}(z) \equiv 1$, and general form of Hölder's inequality with indices

$$\frac{n_m - s}{n_m - n_{m-1} - 1}, \frac{n_m - s}{n_{m-1} - n_{m-2} - 1}, \dots, \frac{n_m - s}{n_1 - n - 1}, \frac{n_m - s}{n - s + m} \tag{4.32}$$

together with Lemma D shows that $K(n,n_1,\dots,n_m) < \infty$ for $\varepsilon > 0$ being small enough. If $n_m < k$, then an appropriate application of Hölder's inequality with indices $(k-s)/(k-n_m)$ and $(k-s)/(n_m-s)$ separates the coefficient from the solutions. The first term is finite by the assumption, and the second term can seen to be finite by another application of general form of Hölder's inequality with indices (4.32). This gives the desired contradiction, since the left-hand side of (4.26) diverges as $r \to 1^-$ and the right-hand side of (4.26) is uniformly bounded for all $r \in (0,1)$.

5. Proof of Corollary 2.3

The upper bound in (2.9) follows directly from Theorem 2.1. To conclude the lower bound in (2.9), assume that solutions $f_1, \ldots, f_k$ are given in increasing order with respect to T-order of growth; that is, $\sigma_T(f_1) \le \cdots \le \sigma_T(f_k)$. By applying Theorem 2.1 with $s = 0, \ldots, q$, we get the following sequence of successive statements. For all solutions f in the solution base, we have $\beta_T(0) \le \sigma_T(f)$; for $k-1$ solutions f in the solution base, we have $\beta_T(1) \le \sigma_T(f)$, ending up with the fact that $k-q$ solutions f in the solution base satisfy $\alpha_q = \beta_T(q) = \sigma_T(f)$. Hence we have $\gamma_T(0) = \beta_T(0) \le \sigma_T(f_1)$, $\gamma_T(1) = \max\{\beta_T(0), \beta_T(1)\} \le \sigma_T(f_2)$ continuing to

$$\gamma_T(q-1) = \max\{\beta_T(0), \ldots, \beta_T(q-1)\} \le \sigma_T(f_q). \tag{5.1}$$

Note that $\alpha_q = \gamma_T(q)$. To see this, note that $\beta_T(s) \le \alpha_s$ for every $s = 0, \ldots, q$, which follows by taking $j = k$ in (2.6), and hence

$$\alpha_q = \beta_T(q) \le \gamma_T(q) = \max\{\beta_T(0), \ldots, \beta_T(q)\} \le \max\{\alpha_0, \ldots, \alpha_q\} = \alpha_q. \tag{5.2}$$

The assertion follows by noting that if $j \in \{0, \ldots, s^\star - 1\}$, then $\gamma_T(j) \le 0$, and we only have the trivial estimate $\sigma_T(f_j) \ge 0$.

6. Proof of Lemma 2.4

Let $m \in \{1, \ldots, p\}$. By (2.11), we obtain

$$\frac{\delta_j - \delta_{s_{m-1}}}{s_{m-1} - j} \le \frac{\delta_{s_m} - \delta_{s_{m-1}}}{s_{m-1} - s_m} \tag{6.1}$$

for all $0 \le j < s_{m-1}$, and the inequality (6.1) is strict for all $0 \le j < s_m$. This estimate will be repeatedly needed later on.

6.1. Proof of Lemma 2.4(i)

Let $1 \le t \le p-1$. Note that $\mathcal{B}_T(t) > 0$ by definition. Since $s_{t+1} < s_t$,

$$\begin{aligned} \mathcal{B}_T(t) - \mathcal{B}_T(t+1) &= \frac{\delta_{s_t} - \delta_{s_{t-1}}}{s_{t-1} - s_t} - \frac{\delta_{s_{t+1}} - \delta_{s_t}}{s_t - s_{t+1}} \\ &= \frac{s_{t-1} - s_{t+1}}{s_t - s_{t+1}} \left(\frac{\delta_{s_t} - \delta_{s_{t-1}}}{s_{t-1} - s_t} - \frac{\delta_{s_{t+1}} - \delta_{s_{t-1}}}{s_{t-1} - s_{t+1}} \right) > 0 \end{aligned} \tag{6.2}$$

by (6.1), which proves the assertion of Lemma 2.4(i).

6.2. Proof of Lemma 2.4(ii)

Since $\max_{j=0,\ldots,k-1}\{\alpha_j\} = \alpha_{s_1} > 0$, we get

$$\beta_T(s_1) = \min_{j=s_1+1,\ldots,k}\left\{\frac{k-j}{j-s_1}(\alpha_{s_1}-\alpha_j)+\alpha_{s_1}\right\} = \alpha_{s_1} = \frac{\delta_{s_1}}{k-s_1} - 1 = \mathcal{B}_T(1), \tag{6.3}$$

where the minimum is obtained with $j = k$. This proves the claim for $t = 1$.

Assume that $t \in \{2,\ldots,p\}$. To prove the claim, we need the following observations. If $m \in \{1,\ldots,t-1\}$, then by (6.1) we get

$$\frac{\delta_{s_t}-\delta_{s_{m-1}}}{s_{m-1}-s_t} - \frac{\delta_{s_t}-\delta_{s_m}}{s_m-s_t} = \frac{s_{m-1}-s_m}{s_m-s_t}\left(\frac{\delta_{s_m}-\delta_{s_{m-1}}}{s_{m-1}-s_m} - \frac{\delta_{s_t}-\delta_{s_{m-1}}}{s_{m-1}-s_t}\right) > 0. \tag{6.4}$$

On the other hand, if $m \in \{1,\ldots,t\}$, then

$$\frac{\delta_{s_m}-\delta_j}{j-s_m} \geq \frac{\delta_{s_m}-\delta_{s_{m-1}}}{s_{m-1}-s_m} \tag{6.5}$$

for all $s_m < j \leq s_{m-1}$. To verify (6.5), we consider the following two cases. If $j = s_{m-1}$, then the equality in (6.5) holds. If $j < s_{m-1}$, then by using (6.1), we obtain

$$\begin{aligned}\frac{\delta_{s_m}-\delta_j}{j-s_m} &= \frac{\delta_{s_m}-\delta_{s_{m-1}}}{j-s_m} - \frac{\delta_j-\delta_{s_{m-1}}}{s_{m-1}-j}\frac{s_{m-1}-j}{j-s_m}\\ &\geq \frac{\delta_{s_m}-\delta_{s_{m-1}}}{j-s_m} - \frac{\delta_{s_m}-\delta_{s_{m-1}}}{s_{m-1}-s_m}\frac{s_{m-1}-j}{j-s_m} = \frac{\delta_{s_m}-\delta_{s_{m-1}}}{s_{m-1}-s_m},\end{aligned} \tag{6.6}$$

which proves (6.5).

To complete the proof of

$$\beta_T(s_t) = \min_{j=s_t+1,\ldots,k}\left\{\frac{\delta_{s_t}-\delta_j}{j-s_t} - 1\right\} = \frac{\delta_{s_t}-\delta_{s_{t-1}}}{s_{t-1}-s_t} - 1 = \mathcal{B}_T(t), \tag{6.7}$$

we argue as follows. First, we show that $\beta_T(s_t) \geq \mathcal{B}_T(t)$. If $s_t < j \leq s_{t-1}$, then (6.5) holds for $m = t$. If $j > s_{t-1}$, then let $m \in \{1,\ldots,t-1\}$ be the smallest index such that $s_m < j$. From (6.1), (6.4), and (6.5), we obtain

$$\begin{aligned}\frac{\delta_{s_t}-\delta_j}{j-s_t} - \frac{\delta_{s_t}-\delta_{s_m}}{s_m-s_t} &= \frac{j-s_m}{j-s_t}\left(\frac{\delta_{s_m}-\delta_j}{j-s_m} - \frac{\delta_{s_t}-\delta_{s_m}}{s_m-s_t}\right)\\ &> \frac{j-s_m}{j-s_t}\left(\frac{\delta_{s_m}-\delta_{s_{m-1}}}{s_{m-1}-s_m} - \frac{\delta_{s_t}-\delta_{s_{m-1}}}{s_{m-1}-s_t}\right) > 0,\end{aligned} \tag{6.8}$$

which together with (6.4) shows that

$$\frac{\delta_{s_t}-\delta_j}{j-s_t} > \frac{\delta_{s_t}-\delta_{s_m}}{s_m-s_t} > \cdots > \frac{\delta_{s_t}-\delta_{s_{t-1}}}{s_{t-1}-s_t}. \tag{6.9}$$

Second, we note that equality in (6.7) follows by taking $j = s_{t-1}$.

6.3. Proof of Lemma 2.4(iii)

Since $\alpha_{s_1} = \max_{j=0,\ldots,k-1}\{\alpha_j\} > 0$ and $s_1 = q$, we get by means of (5.2) that

$$\mathcal{B}_T(1) = \frac{\delta_q}{k-q} - 1 = \alpha_q = \gamma_T(q). \tag{6.10}$$

Let $t \in \{2,\ldots,p\}$. We proceed to prove that $\gamma_T(j) = \mathcal{B}_T(t)$ for all $s_t \le j < s_{t-1}$. Evidently, $\gamma_T(j) = \max\{\beta_T(0),\ldots,\beta_T(j)\} \ge \beta_T(s_t)$. By Lemma 2.4(ii), we conclude that $\gamma_T(j) \ge \mathcal{B}_T(t)$. To prove that $\gamma_T(j) \le \mathcal{B}_T(t)$, it is enough to show that $\beta_T(m) \le \mathcal{B}_T(t)$ for all $m \in \{0,\ldots,j\}$. Since $m+1 \le s_{t-1}$, we obtain by appealing to (6.1) that

$$\beta_T(m) = \min_{j=m+1,\ldots,k}\left\{\frac{\delta_m-\delta_j}{j-m} - 1\right\} \le \frac{\delta_m-\delta_{s_{t-1}}}{s_{t-1}-m} - 1 \le \frac{\delta_{s_t}-\delta_{s_{t-1}}}{s_{t-1}-s_t} - 1 = \mathcal{B}_T(t). \tag{6.11}$$

If $s_p > 0$, then for all $m \in \{0,\ldots,s_p-1\}$ we have

$$\beta_T(m) = \min_{j=m+1,\ldots,k}\left\{\frac{\delta_m-\delta_j}{j-m} - 1\right\} \le \frac{\delta_m-\delta_{s_p}}{s_p-m} - 1 \le 0 \tag{6.12}$$

by (2.11). Hence $\gamma_T(j) = \max\{\beta_T(0),\ldots,\beta_T(j)\} \le 0$ for all $j < s_p$. As a consequence we see that $s_p = s^\star$.

7. Proof of Theorem 2.5

Our proof of Theorem 2.5 is parallel to the proof of [3, Theorem 1.4], and hence we only outline the argumentation. We may assume that $k \ge 3$, $q \ge 2$, $s \in \{1,\ldots,q-1\}$, and $\beta_M(s) > 1$ for otherwise there is nothing to prove by Theorem B; see the discussion after Theorem 2.5. In particular, if $p_s \le 1$, then (2.22) is trivial, since by taking $j = k$ in (2.24), we obtain $\beta_M(s) \le p_s \le 1$. On the contrary to the claim, assume that (1.1) admits $s+1$ linearly independent solutions $f_{0,1},\ldots,f_{0,s+1}$ such that

$$\sigma_M(f_{0,t}) < \alpha := \beta_M(s), \quad t = 1,\ldots,s+1. \tag{7.1}$$

Remark that if $\alpha \le 1$, then there is nothing to prove in (2.22); so we may assume that $\alpha > 1$. Now $\max\{\beta, 1\} < \alpha$, where $\beta := \max_{t=1,\ldots,s+1}\{\sigma_M(f_{0,t})\} < \infty$.

Let $\varepsilon, \delta \in (0,1)$. Now [3, Lemma 4.3] for $m = s+1$ implies that there exists a solution $f_{s,1} \not\equiv 0$ of

$$f^{(k-s)} + a_{s,k-s-1}(z) f^{(k-s-1)} + \cdots + a_{s,1}(z) f' + a_{s,0}(z) f = 0 \tag{7.2}$$

of the from $f_{s,1} = g_{s,1}/h_{s,1}$, where $g_{s,1}, h_{s,1} \in \mathcal{H}(\mathbb{D})$ and

$$\max\{\sigma_M(g_{s,1}), \sigma_M(h_{s,1})\} \leq \max\{\beta, 1\} < \alpha. \tag{7.3}$$

It is easy to see that $\alpha = \beta_M(s)$ yields $(k-l)p_l \leq (k-s)p_s - (l-s)\alpha$ for all $l \in \{s+1, \ldots, k\}$. Hence, by [3, Lemma 4.3] and the assumption $a_{0,j} \in \mathbb{H}^\infty_{(p_j+1)(k-j)}$, we get

$$M(r, a_{s,j}) \leq \left(\frac{1}{1-r}\right)^{(p_s+1)(k-s)-j(\alpha+1)+\varepsilon}, \quad j = 1, \ldots, k-s-1, \tag{7.4}$$

for all $r \in [0,1) \setminus E$, where the set E satisfies the upper density condition:

$$\overline{D}(E) := \limsup_{r \to 1^-} \frac{m(E \cap [r,1))}{1-r} \leq \delta < 1. \tag{7.5}$$

Here $m(\Omega)$ is the Lebesque measure of the set Ω. We note that set E may not be the same at each occurrence; however, it always satisfies (7.5).

Let $\eta \in (\delta, 1)$. If we apply [3, Lemma 4.4] and use [3, Lemma 4.1] for the coefficient $a_{0,s} \in \mathbb{H}^\infty_{(p_s+1)(k-s)}$, we conclude that for $\varepsilon > 0$ being small enough, we have

$$M(r, a_{s,0}) \gtrsim \left(\frac{1}{1-r}\right)^{(p_s+1)(k-s)-\varepsilon} \tag{7.6}$$

for all $r \in F \setminus E$, where the set $F \subset [0,1)$ satisfies $\overline{D}(F) \geq \eta$. In particular, we have $\overline{D}(F \setminus E) \geq \eta - \delta > 0$.

On the other hand, by substituting $f = f_{s,1}$ in (7.2) and by applying [3, Corollary 4.2] to $f_{s,1}$, it follows that

$$|a_{s,0}(z)| \leq \left|\frac{f_{s,1}^{(k-s)}(z)}{f_{s,1}(z)}\right| + \sum_{j=1}^{k-s-1} |a_{s,j}(z)| \left|\frac{f_{s,1}^{(j)}(z)}{f_{s,1}(z)}\right| \lesssim \left(\frac{1}{1-|z|}\right)^{(p_s+1)(k-s)-2\varepsilon} \tag{7.7}$$

for all $z \in \mathbb{D}$, $|z| \notin E$. By comparing (7.6) to (7.7), we get a contradictory inequality:

$$(p_s+1)(k-s) - \varepsilon \leq (p_s+1)(k-s) - 2\varepsilon. \tag{7.8}$$

This shows that each solution base of (1.1) contains at least $k-s$ solutions f satisfying $\sigma_M(f) \geq \beta_M(s)$.

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