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Research Article

Neutral Operator and Neutral Differential Equation

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In this paper, we discuss the properties of the neutral operator $(Ax)(t) = x(t) - cx(t - \delta(t))$, and by applying coincidence degree theory and fixed point index theory, we obtain sufficient conditions for the existence, multiplicity, and nonexistence of (positive) periodic solutions to two kinds of second-order differential equations with the prescribed neutral operator.

1. Introduction

In [1], Zhang discussed the properties of the neutral operator $(A_1x)(t) = x(\mathrm{t}) - cx(t - \delta)$, which became an effective tool for the research on differential equations with this prescribed neutral operator, see, for example, [2–5]. Lu and Ge [6] investigated an extension of A_1, namely, the neutral operator $A_2x(t) = x(t) - \sum_{i=1}^{n} c_i x(t - \delta_i)$ and obtained the existence of periodic solutions for a corresponding neutral differential equation.

In this paper, we consider the neutral operator $(Ax)(t) = x(t) - cx(t - \delta(t))$, where c is constant and $|c| \neq 1$, $\delta \in C^1(\mathbb{R}, \mathbb{R})$, and δ is an ω-periodic function for some $\omega > 0$. Although A is a natural generalization of the operator A_1, the class of neutral differential equation with A typically possesses a more complicated nonlinearity than neutral differential equation with A_1 or A_2. For example, the neutral operators A_1 and A_2 are homogeneous in the following sense $(A_i x)'(t) = (A_i x')(t)$ for $i = 1, 2$, whereas the neutral operator A in general is inhomogeneous. As a consequence many of the new results for differential equations with the neutral operator A will not be a direct extension of known theorems for neutral differential equations.

The paper is organized as follows: in Section 2, we first analyze qualitative properties of the neutral operator A which will be helpful for further studies of differential equations with this neutral operator; in Section 3, by Mawhin's continuation theorem, we obtain the existence of periodic solutions for a second-order Rayleigh-type neutral differential equation; in Section 4, by an application of the fixed point index theorem we obtain sufficient conditions for the existence, multiplicity, and nonexistence of positive periodic solutions to second-order neutral differential equation. Several examples are also given to illustrate our results. Our results improve and extend the results in [1, 2, 4, 7].

2. Analysis of the Generalized Neutral Operator

Let $C_\omega = \{x \in C(\mathbb{R},\mathbb{R}) : x(t+\omega) = x(t),\ t \in \mathbb{R}\}$ with norm $\|x\| = \max_{t\in[0,\omega]}|x(t)|$. Then $(C_\omega, \|\cdot\|)$ is a Banach space. A cone K in C_ω is defined by $K = \{x \in C_\omega : x(t) \geq \alpha\|x\|, \text{for all } t \in \mathbb{R}\}$, where α is a fixed positive number with $\alpha < 1$. Moreover, define operators $A, B : C_\omega \to C_\omega$ by

$$(Ax)(t) = x(t) - cx(t-\delta(t)), \qquad (Bx)(t) = cx(t-\delta(t)). \tag{2.1}$$

Lemma 2.1. *If $|c| \neq 1$, then the operator A has a continuous inverse A^{-1} on C_ω, satisfying*

(1)

$$\left(A^{-1}f\right)(t) = \begin{cases} f(t) + \displaystyle\sum_{j=1}^{\infty} c^j f\left(s - \sum_{i=1}^{j-1}\delta(D_i)\right), & \text{for } |c| < 1,\ \forall\ f \in C_\omega, \\ -\dfrac{f(t+\delta(t))}{c} - \displaystyle\sum_{j=1}^{\infty}\left(1/c^{j+1}\right) f\left(s + \delta(t) + \sum_{i=1}^{j-1}\delta(D_i)\right), & \text{for } |c| > 1,\ \forall f \in C_\omega. \end{cases} \tag{2.2}$$

(2) $|(A^{-1}f)(t)| \leq \|f\|/|1-|c||$, *for all* $f \in C_\omega$.

(3) $\int_0^\omega |(A^{-1}f)(t)|dt \leq 1/|1-|c|| \int_0^\omega |f(t)|dt$, *for all* $f \in C_\omega$.

Proof. We have the following cases

Case 1 ($|c| < 1$). Let $t - \delta(t) = s$ and $D_j = s - \sum_{i=1}^{j-1}\delta(D_i)$, $j = 1, 2, \ldots$. Therefore,

$$\begin{aligned} B^j x(t) &= c^j x\left(s - \sum_{i=1}^{j-1}\delta(D_i)\right), \\ \sum_{j=0}^{\infty}\left(B^j f\right)(t) &= f(t) + \sum_{j=1}^{\infty} c^j f\left(s - \sum_{i=1}^{j-1}\delta(D_i)\right). \end{aligned} \tag{2.3}$$

Since $A = I - B$, we get from $\|B\| \leq |c| < 1$ that A has a continuous inverse $A^{-1} : C_\omega \to C_\omega$ with

$$A^{-1} = (I - B)^{-1} = I + \sum_{j=1}^{\infty} B^j = \sum_{j=0}^{\infty} B^j, \tag{2.4}$$

where $B^0 = I$. Then

$$\left(A^{-1} f(t)\right) = \sum_{j=0}^{\infty} \left[B^j f\right](t) = \sum_{j=0}^{\infty} c^j f\left(s - \sum_{i=1}^{j-1} \delta(D_i)\right), \tag{2.5}$$

and consequently

$$\left|\left(A^{-1} f\right)(t)\right| = \left|\sum_{j=0}^{\infty} \left[B^j f\right](t)\right| = \left|\sum_{j=0}^{\infty} c^j f\left(s - \sum_{i=1}^{j-1} \delta(D_i)\right)\right| \leq \frac{\|f\|}{1 - |c|}. \tag{2.6}$$

Moreover,

$$\begin{aligned}
\int_0^\omega \left|\left(A^{-1} f\right)(t)\right| dt &= \int_0^\omega \left|\sum_{j=0}^{\infty} \left(B^j f\right)(t)\right| dt \\
&\leq \sum_{j=0}^{\infty} \int_0^\omega \left|\left(B^j f\right)(t)\right| dt \\
&= \sum_{j=0}^{\infty} \int_0^\omega \left|c^j f\left(s - \sum_{i=1}^{j-1} \delta(D_i)\right)\right| dt \\
&\leq \frac{1}{1 - |c|} \int_0^\omega |f(t)| dt.
\end{aligned} \tag{2.7}$$

Case 2 ($|c| > 1$). Let

$$\begin{aligned}
E : C_\omega \longrightarrow C_\omega, &\qquad (Ex)(t) = x(t) - \frac{1}{c} x(t + \delta(t)), \\
B_1 : C_\omega \longrightarrow C_\omega, &\qquad (B_1 x)(t) = \frac{1}{c} x(t + \delta(t)).
\end{aligned} \tag{2.8}$$

By definition of the linear operator B_1, we have

$$\left(B_1^j f\right)(t) = \frac{1}{c^j} f\left(s + \sum_{i=1}^{j-1} \delta(D_i)\right), \tag{2.9}$$

where D_i is defined as in Case 1. Summing over j yields

$$\sum_{j=0}^{\infty}\left(B_1^j f\right)(t) = f(t) + \sum_{j=1}^{\infty}\frac{1}{c^j} f\left(s + \sum_{i=1}^{j-1}\delta(D_i)\right). \tag{2.10}$$

Since $\|B_1\| < 1$, we obtain that the operator E has a bounded inverse E^{-1},

$$E^{-1} : C_\omega \longrightarrow C_\omega, \qquad E^{-1} = (I - B_1)^{-1} = I + \sum_{j=1} B_1^j, \tag{2.11}$$

and for all $f \in C_\omega$ we get

$$\left(E^{-1} f\right)(t) = f(t) + \sum_{j=1}^{\infty}\left(B_1^j f\right)(t). \tag{2.12}$$

On the other hand, from $(Ax)(t) = x(t) - cx(t - \delta(t))$, we have

$$(Ax)(t) = x(t) - cx(t - \delta(t)) = -c\left[x(t - \delta(t)) - \frac{1}{c}x(t)\right], \tag{2.13}$$

that is,

$$(Ax)(t) = -c(Ex)(t - \delta(t)). \tag{2.14}$$

Let $f \in C_\omega$ be arbitrary. We are looking for x such that

$$(Ax)(t) = f(t). \tag{2.15}$$

that is,

$$-c(Ex)(t - \delta(t)) = f(t). \tag{2.16}$$

Therefore,

$$(Ex)(t) = -\frac{f(t + \delta(t))}{c} =: f_1(t), \tag{2.17}$$

and hence

$$x(t) = \left(E^{-1} f_1\right)(t) = f_1(t) + \sum_{j=1}^{\infty}\left(B_1^j f_1\right)(t) = -\frac{f(t + \delta(t))}{c} - \sum_{j=1}^{\infty} B_1^j \frac{f(t + \delta(t))}{c}, \tag{2.18}$$

proving that A^{-1} exists and satisfies

$$\left[A^{-1}f\right](t) = -\frac{f(t+\delta(t))}{c} - \sum_{j=1}^{\infty} B_1^j \frac{f(t+\delta(t))}{c} = -\frac{f(t+\delta(t))}{c} - \sum_{j=1}^{\infty} \frac{1}{c^{j+1}} f\left(s+\delta(t)+\sum_{i=1}^{j-1}\delta(D_i)\right),$$
$$\left|\left[A^{-1}f\right](t)\right| = \left|-\frac{f(t+\delta(t))}{c} - \sum_{j=1}^{\infty} \frac{1}{c^{j+1}} f\left(s+\delta(t)+\sum_{i=1}^{j-1}\delta(D_i)\right)\right| \leq \frac{\|f\|}{|c|-1}. \tag{2.19}$$

Statements (1) and (2) are proved. From the above proof, (3) can easily be deduced. □

Lemma 2.2. *If $c < 0$ and $|c| < \alpha$, one has for $y \in K$ that*

$$\frac{\alpha-|c|}{1-c^2}\|y\| \leq \left(A^{-1}y\right)(t) \leq \frac{1}{1-|c|}\|y\|. \tag{2.20}$$

Proof. Since $c < 0$ and $|c| < \alpha < 1$, by Lemma 2.1, we have for $y \in K$ that

$$\begin{aligned}
\left(A^{-1}y\right)(t) &= y(t) + \sum_{j=1}^{\infty} c^j y\left(s-\sum_{i=1}^{j-1}\delta(D_i)\right) \\
&= y(t) + \sum_{j\geq 1 \text{ even}} c^j y\left(s-\sum_{i=1}^{j-1}\delta(D_i)\right) - \sum_{j\geq 1 \text{ odd}} |c|^j y\left(s-\sum_{i=1}^{j-1}\delta(D_i)\right) \\
&\geq \alpha\|y\| + \alpha \sum_{j\geq 1 \text{ even}} c^j \|y\| - \|y\| \sum_{j\geq 1 \text{ odd}} |c|^j \\
&= \frac{\alpha}{1-c^2}\|y\| - \frac{|c|}{1-c^2}\|y\| \\
&= \frac{\alpha-|c|}{1-c^2}\|y\|.
\end{aligned} \tag{2.21}$$

□

Lemma 2.3. *If $c > 0$ and $c < 1$ then for $y \in K$ one has*

$$\frac{\alpha}{1-c}\|y\| \leq \left(A^{-1}y\right)(t) \leq \frac{1}{1-c}\|y\|. \tag{2.22}$$

Proof. Since $c > 0$ and $c < 1$, $\alpha < 1$, by Lemma 2.1, we have for $y \in K$ that

$$\begin{aligned}\left(A^{-1}y\right)(t) &= y(t) + \sum_{j\geq 1} c^j y\left(s - \sum_{i=1}^{j-1}\delta(D_i)\right) \\ &\geq \alpha\|y\| + \alpha\|y\|\sum_{j\geq 1} c^j \\ &= \frac{\alpha}{1-c}\|y\|.\end{aligned} \tag{2.23}$$

□

3. Periodic Solutions for Neutral Differential Equation

In this section, we consider the second-order neutral differential equation

$$(x(t) - cx(t-\delta(t)))'' = f\left(t, x'(t)\right) + g(t, x(t-\tau(t))) + e(t), \tag{3.1}$$

where $\tau, e \in C_\omega$ and $\int_0^\omega e(t)dt = 0$; f and g are continuous functions defined on $\mathbb{R}^2$ and periodic in t with $f(t,\cdot) = f(t+\omega,\cdot), g(t,\cdot) = g(t+\omega,\cdot)$, $f(t,0) = 0$, $f(t,u) \geq 0$, or $f(t,u) \leq 0$ for all $(t,u) \in \mathbb{R}^2$.

We first recall Mawhin's continuation theorem which our study is based upon. Let X and Y be real Banach spaces and $L : D(L) \subset X \to Y$ a Fredholm operator with index zero, where $D(L)$ denotes the domain of L. This means that $\operatorname{Im} L$ is closed in Y and $\dim \operatorname{Ker} L = \dim(Y/\operatorname{Im} L) < +\infty$. Consider supplementary subspaces X_1, Y_1, of X, Y respectively, such that $X = \operatorname{Ker} L \oplus X_1$, $Y = \operatorname{Im} L \oplus Y_1$, and let $P_1 : X \to \operatorname{Ker} L$ and $Q_1 : Y \to Y_1$ denote the natural projections. Clearly, $\operatorname{Ker} L \cap (D(L) \cap X_1) = \{0\}$, thus the restriction $L_{P_1} := L|_{D(L)\cap X_1}$ is invertible. Let $L_{P_1}^{-1}$ denote the inverse of L_{P_1}.

Let Ω be an open bounded subset of X with $D(L)\cap\Omega \neq \emptyset$. A map $N : \overline{\Omega} \to Y$ is said to be L-compact in $\overline{\Omega}$ if $Q_1 N(\overline{\Omega})$ is bounded and the operator $L_{P_1}^{-1}(I-Q_1)N : \overline{\Omega} \to X$ is compact.

Lemma 3.1 (Gaines and Mawhin [8]). *Suppose that X and Y are two Banach spaces and $L : D(L) \subset X \to Y$ is a Fredholm operator with index zero. Furthermore, $\Omega \subset X$ is an open bounded set, and $N : \overline{\Omega} \to Y$ is L-compact on $\overline{\Omega}$. Assume that the following conditions hold:*

(1) $Lx \neq \lambda Nx$, *for all* $x \in \partial\Omega \cap D(L)$, $\lambda \in (0,1)$;

(2) $Nx \notin \operatorname{Im} L$, *for all* $x \in \partial\Omega \cap \operatorname{Ker} L$;

(3) $\deg\{JQ_1N, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$, *where* $J : \operatorname{Im} \quad Q_1 \to \operatorname{Ker} L$ *is an isomorphism.*

Then the equation $Lx = Nx$ has a solution in $\overline{\Omega} \cap D(L)$.

In order to use Mawhin's continuation theorem to study the existence of ω-periodic solutions for (3.1), we rewrite (3.1) in the following form:

$$\begin{aligned}(Ax_1)'(t) &= x_2(t), \\ x_2'(t) &= f\left(t, x_1'(t)\right) + g(t, x_1(t-\tau(t))) + e(t).\end{aligned} \tag{3.2}$$

Clearly, if $x(t) = (x_1(t), x_2(t))^\top$ is an ω-periodic solution to (3.2), then $x_1(t)$ must be an ω-periodic solution to (3.1). Thus, the problem of finding an ω-periodic solution for (3.1) reduces to finding one for (3.2).

Recall that $C_\omega = \{\phi \in C(\mathbb{R}, \mathbb{R}) : \phi(t+\omega) \equiv \phi(t)\}$ with norm $\|\phi\| = \max_{t\in[0,\omega]}|\phi(t)|$. Define $X = Y = C_\omega \times C_\omega = \{x = (x_1(\cdot), x_2(\cdot)) \in C(\mathbb{R}, \mathbb{R}^2) : x(t) = x(t+\omega),\ t \in \mathbb{R}\}$ with norm $\|x\| = \max\{\|x_1\|, \|x_2\|\}$. Clearly, X and Y are Banach spaces. Moreover, define

$$L : D(L) = \left\{x \in C^1\left(\mathbb{R}, \mathbb{R}^2\right) : x(t+\omega) = x(t),\ t \in \mathbb{R}\right\} \subset X \longrightarrow Y \tag{3.3}$$

by

$$(Lx)(t) = \begin{pmatrix} (Ax_1)'(t) \\ x_2'(t) \end{pmatrix} \tag{3.4}$$

and $N : X \to Y$ by

$$(Nx)(t) = \begin{pmatrix} x_2(t) \\ f(t, x_1'(t)) + g(t, x_1(t-\tau(t))) + e(t) \end{pmatrix}. \tag{3.5}$$

Then (3.2) can be converted to the abstract equation $Lx = Nx$. From the definition of L, one can easily see that

$$\operatorname{Ker} L \cong \mathbb{R}^2, \qquad \operatorname{Im} L = \left\{ y \in Y : \int_0^\omega \begin{pmatrix} y_1(s) \\ y_2(s) \end{pmatrix} ds = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}. \tag{3.6}$$

So L is a Fredholm operator with index zero. Let $P_1 : X \to \operatorname{Ker} L$ and $Q_1 : Y \to \operatorname{Im} Q_1 \subset \mathbb{R}^2$ be defined by

$$P_1 x = \begin{pmatrix} (Ax_1)(0) \\ x_2(0) \end{pmatrix}; \qquad Q_1 y = \frac{1}{\omega}\int_0^\omega \begin{pmatrix} y_1(s) \\ y_2(s) \end{pmatrix} ds, \tag{3.7}$$

then $\operatorname{Im} P_1 = \operatorname{Ker} L$, $\operatorname{Ker} Q_1 = \operatorname{Im} L$. Setting $L_{P_1} = L|_{D(L)\cap \operatorname{Ker} P_1}$ and $L_{P_1}^{-1} : \operatorname{Im} L \to D(L)$ denotes the inverse of L_{P_1}, then

$$\begin{gathered} \left[L_{P_1}^{-1} y\right](t) = \begin{pmatrix} (A^{-1}Fy_1)(t) \\ (Fy_2)(t) \end{pmatrix}, \\ [Fy_1](t) = \int_0^t y_1(s)ds, \qquad [Fy_2](t) = \int_0^t y_2(s)ds. \end{gathered} \tag{3.8}$$

From (3.5) and (3.8), it is clear that $Q_1 N$ and $L_{P_1}^{-1}(I - Q_1)N$ are continuous and $Q_1 N(\overline{\Omega})$ is bounded, and then $L_{P_1}^{-1}(I - Q_1)N(\overline{\Omega})$ is compact for any open bounded $\Omega \subset X$ which means N is L-compact on $\overline{\Omega}$.

Now we give our main results on periodic solutions for (3.1).

Theorem 3.2. *Suppose there exist positive constants K_1, D, M, b with $M > \|e\|$ such that:*

(H_1) $|f(t,u)| \le K_1|u| + b$*, for* $(t,u) \in \mathbb{R} \times \mathbb{R}$*;*

(H_2) $\operatorname{sgn} x \cdot g(t,x) > \|e\|$*, for* $|x| > D$*;*

(H_3) $g(t,x) \ge -M$*, for* $x \le -D$ *and* $t \in \mathbb{R}$*.*

Then (3.1) *has at least one solution with period* ω *if* $0 < \omega^{1/2}(1+|c|)^{1/2}\sqrt{2K_1}/(|1-|c|| - |c|\delta_1) < 1$*, where* $\delta_1 = \max_{t\in[0,\omega]}|\delta'(t)|$*.*

Proof. By construction (3.2) has an ω-periodic solution if and only if the following operator equation

$$Lx = Nx \tag{3.9}$$

has an ω-periodic solution. From (3.8), we see that N is L-compact on $\overline{\Omega}$, where Ω is any open, bounded subset of C_ω. For $\lambda \in (0,1]$ define

$$\Omega_1 = \{x \in C_\omega : Lx = \lambda Nx\}. \tag{3.10}$$

Then $x = (x_1, x_2)^\top \in \Omega_1$ satisfies

$$\begin{aligned} (Ax_1)'(t) &= \lambda x_2(t), \\ x_2'(t) &= \lambda f\left(t, x_1'(t)\right) + \lambda g(t, x_1(t-\tau(t))) + \lambda e(t). \end{aligned} \tag{3.11}$$

We first claim that there is a constant $\xi \in \mathbb{R}$ such that

$$|x_1(\xi)| \le D. \tag{3.12}$$

In view of $\int_0^\omega (Ax_1)'(t)dt = 0$, we know that there exist two constants $t_1, t_2 \in [0,\omega]$ such that $(Ax_1)'(t_1) \ge 0, (Ax_1)'(t_2) \le 0$. From the first equation of (3.11), we have $x_2(t) = (1/\lambda)(Ax_1)'(t)$, so

$$\begin{aligned} x_2(t_1) &= \frac{1}{\lambda}(Ax_1)'(t_1) \ge 0, \\ x_2(t_2) &= \frac{1}{\lambda}(Ax_1)'(t_2) \le 0. \end{aligned} \tag{3.13}$$

Let $t_3, t_4 \in [0,\omega]$ be, respectively, a global maximum and minimum point of $x_2(t)$. Clearly, we have

$$\begin{aligned} x_2(t_3) &\ge 0, \qquad x_2'(t_3) = 0, \\ x_2(t_4) &\le 0, \qquad x_2'(t_4) = 0. \end{aligned} \tag{3.14}$$

Since $f(t,x_1') \geq 0$ or $f(t,x_1') \leq 0$, w.l.o.g., suppose $f(t,x_1') \geq 0$, for $(t,x_1') \in [0,\omega] \times \mathbb{R}$. Then

$$\begin{gathered} -g(t_3, x_1(t_3 - \tau(t_3))) - e(t_3) = f\left(t, x_1'(t_3)\right) \geq 0, \\ g(t_3, x_1(t_3 - \tau(t_3))) \leq -e(t_3) \leq \|e\|. \end{gathered} \tag{3.15}$$

From (H_2) we see that

$$x_1(t_3 - \tau(t_3)) < D. \tag{3.16}$$

Similarly, we have

$$g(t_4, x_1(t_4 - \tau(t_4))) \geq -e(t_4) \geq -\|e\|, \tag{3.17}$$

and again by (H_2),

$$x_1(t_4 - \tau(t_4)) < -D. \tag{3.18}$$

Case 1. If $x_1(t_3 - \tau(t_3)) \in (-D, D)$, define $\xi = t_3 - \tau(t_3)$, obviously $|x_1(\xi)| \leq D$.

Case 2. If $x_1(t_3 - \tau(t_3)) < -D$, from (3.18) and the fact that x is a continuous function in $\mathbb{R}$, there exists a constant ξ between $x_1(t_3 - \tau(t_3))$ and $x_1(t_4 - \tau(t_4))$ such that $|x_1(\xi)| = D$. This proves (3.12).

Choose an integer k and a constant $t_5 \in [0,\omega]$ such that $\xi = \omega k + t_5$, then $|x_1(\xi)| = |x_1(t_5)| \leq D$. Hence

$$|x_1(t)| \leq D + \int_0^{\omega} \left|x_1'(s)\right| ds. \tag{3.19}$$

Substituting $x_2(t) = (1/\lambda)(Ax_1)'(t)$ into the second equation of (3.11) yields

$$\left(\frac{1}{\lambda}(Ax_1)(t)\right)'' = \lambda f\left(t, x_1'(t)\right) + \lambda g(t, x_1(t - \tau(t))) + \lambda e(t), \tag{3.20}$$

that is,

$$((Ax_1)(t))'' = \lambda^2 f\left(t, x_1'(t)\right) + \lambda^2 g(t, x_1(t - \tau(t))) + \lambda^2 e(t). \tag{3.21}$$

Integrating both sides of (3.21) over $[0,\omega]$, we have

$$\int_0^{\omega} \left[f\left(t, x_1'(t)\right) + g(t, x_1(t - \tau(t)))\right] dt = 0. \tag{3.22}$$

On the other hand, multiplying both sides of (3.21) by $(Ax_1)(t)$ and integrating over $[0,\omega]$, we get

$$\begin{aligned}\int_0^\omega ((Ax_1)(t))''(Ax_1(t))dt &= -\int_0^\omega \left|(Ax_1)'(t)\right|^2 dt = -\lambda^2\int_0^\omega f\left(t,x_1'(t)\right)(Ax_1)(t)dt \\ &\quad - \lambda^2\int_0^\omega g(t,x_1(t-\tau(t)))(Ax_1)(t)dt - \lambda^2\int_0^\omega e(t)(Ax_1)(t)dt.\end{aligned} \tag{3.23}$$

Using (H_1), we have

$$\begin{aligned}\int_0^\omega \left|(Ax_1)'(t)\right|^2 dt &\le \int_0^\omega \left|f\left(t,x_1'(t)\right)\right| |[x_1(t)-cx_1(t-\delta(t))]| dt \\ &\quad + \int_0^\omega \left|g(t,x_1(t-\tau(t)))\right| |[x_1(t)-cx_1(t-\delta(t))]| dt \\ &\quad + \int_0^\omega |e(t)||[x_1(t)-cx_1(t-\delta(t))]| dt \\ &\le (1+|c|)\|x_1\|\left[K_1\int_0^\omega \left|x_1'(t)\right|dt + b\omega + \int_0^\omega \left|g(t,x_1(t-\tau(t)))\right|dt + \omega\|e\|\right].\end{aligned} \tag{3.24}$$

Besides, we can assert that there exists some positive constant N_1 such that

$$\int_0^\omega \left|g(t,x_1(t-\tau(t)))\right|dt \le 2\omega N_1 + \omega b + K_1\int_0^\omega \left|x_1'(t)\right|dt. \tag{3.25}$$

In fact, in view of condition (H_1) and (3.22) we have

$$\begin{aligned}\int_0^\omega \left\{g(t,x_1(t-\tau(t))) - K_1\left|x_1'(t)\right| - b\right\}dt &\le \int_0^\omega \left\{g(t,x_1(t-\tau(t))) - \left|f\left(t,x_1'(t)\right)\right|\right\}dt \\ &\le \int_0^\omega \left\{g(t,x_1(t-\tau(t))) + f\left(t,x_1'(t)\right)\right\}dt \\ &= 0.\end{aligned} \tag{3.26}$$

Define

$$\begin{aligned}&E_1 = \{t\in[0,\omega] : x_1(t-\tau(t)) > D\}; \\ &E_2 = \{t\in[0,\omega] : |x_1(t-\tau(t))| \le D\} \cup \{t\in[0,\omega] : x_1(t-\tau(t)) < -D\}.\end{aligned} \tag{3.27}$$

With these sets we get

$$\int_{E_2} \left|g(t, x_1(t-\tau(t)))\right| dt \leq \omega \max\left\{M, \sup_{t\in[0,\omega],|x_1(t-\tau(t))|\leq D} \left|g(t, x_1)\right|\right\}.$$

$$\begin{aligned}
\int_{E_1} &\left\{\left|g(t, x_1(t-\tau(t)))\right| - K_1\left|x_1'(t)\right| - b\right\} dt \\
&= \int_{E_1} \left\{g(t, x_1(t-\tau(t))) - K_1\left|x_1'(t)\right| - b\right\} dt \\
&\leq -\int_{E_2} \left\{g(t, x_1(t-\tau(t))) - K_1\left|x_1'(t)\right| - b\right\} dt \\
&\leq \int_{E_2} \left\{\left|g(t, x_1(t-\tau(t)))\right| + K_1\left|x_1'(t)\right| + b\right\} dt,
\end{aligned} \tag{3.28}$$

which yields

$$\begin{aligned}
\int_{E_1} \left|g(t, x_1(t-\tau(t)))\right| dt &\leq \int_{E_2} \left|g(t, x_1(t-\tau(t)))\right| dt + \int_{E_1\cup E_2} \left(K_1\left|x_1'(t)\right| + b\right) dt \\
&= \int_{E_2} \left|g(t, x_1(t-\tau(t)))\right| dt + \omega b + K_1 \int_0^{\omega} \left|x_1'(t)\right| dt.
\end{aligned} \tag{3.29}$$

That is,

$$\begin{aligned}
\int_0^{\omega} \left|g(t, x_1(t-\tau(t)))\right| dt &= \int_{E_1} \left|g(t, x_1(t-\tau(t)))\right| dt + \int_{E_2} \left|g(x_1(t-\tau(t)))\right| dt \\
&\leq 2\int_{E_2} \left|g(t, x_1(t-\tau(t)))\right| dt + \omega b + K_1 \int_0^{\omega} \left|x_1'(t)\right| dt \\
&\leq 2\omega \max\left\{M, \sup_{t\in[0,\omega],|x_1(t-\tau(t))|<D} \left|g(t, x_1)\right|\right\} + \omega b + K_1 \int_0^{\omega} \left|x_1'(t)\right| dt \\
&= 2\omega D_1 + \omega b + K_1 \int_0^{\omega} \left|x_1'(t)\right| dt,
\end{aligned} \tag{3.30}$$

where $N_1 = \max\{M, \sup_{t\in[0,\omega],|x_1(t-\tau(t))|<D} |g(t, x_1)|\}$, proving (3.25).

Substituting (3.25) into (3.24) and recalling (3.19), we get

$$
\begin{aligned}
\int_0^{\omega} \left|(Ax_1)'(t)\right|^2 dt &\le (1+|c|)|x_1|_0 \left(2K_1 \int_0^{\omega} \left|x_1'(t)\right| dt + 2\omega b + 2\omega N_1 + \omega \max_{t\in[0,\omega]} |e(t)|\right) \\
&= (1+|c|)\left(2K_1 |x_1|_0 \int_0^{\omega} \left|x_1'(t)\right| dt + 2\omega b |x_1|_0 + 2\omega N_1 |x_1|_0 + \omega |x_1|_0 \max_{t\in[0,\omega]} |e(t)|\right) \\
&\le (1+|c|)\left[2K_1\left(D + \int_0^{\omega} \left|x_1'(t)\right| dt\right) \int_0^{\omega} \left|x_1'(t)\right| dt \right. \\
&\qquad \left. + \left(2\omega b + 2\omega N_1 + \omega \max_{t\in[0,\omega]} |e(t)|\right)\left(D + \int_0^{\omega} \left|x_1'(t)\right| dt\right)\right] \\
&= (1+|c|)\left[2K_1 D \int_0^{\omega} \left|x_1'(t)\right| dt + 2K_1 \left(\int_0^{\omega} \left|x_1'(t)\right| dt\right)^2 + N_2 \int_0^{\omega} \left|x_1'(t)\right| dt + N_2 D\right] \\
&= 2K_1(1+|c|)\left(\int_0^{\omega} \left|x_1'(t)\right| dt\right)^2 + (1+|c|)(N_2 + 2K_1 D) \int_0^{\omega} \left|x_1'(t)\right| dt + (1+|c|) N_2 D,
\end{aligned}
\tag{3.31}
$$

where $N_2 = 2\omega b + 2\omega N_1 + \omega\|e\|$. Since $(Ax)(t) = x(t) - cx(t-\delta(t))$, we have

$$
\begin{aligned}
(Ax_1)'(t) &= (x_1(t) - cx_1(t-\delta(t)))' \\
&= x_1'(t) - cx_1'(t-\delta(t)) + cx_1'(t-\delta(t))\delta'(t) \\
&= \left(Ax_1'\right)(t) + cx_1'(t-\delta(t))\delta'(t), \\
\left(Ax_1'\right)(t) &= (Ax_1)'(t) - cx_1'(t-\delta(t))\delta'(t).
\end{aligned}
\tag{3.32}
$$

By applying Lemma 2.1, we have

$$
\begin{aligned}
\int_0^{\omega} \left|x_1'(t)\right| dt &= \int_0^{\omega} \left|\left(A^{-1} A x_1'\right)(t)\right| dt \\
&\le \frac{\int_0^{\omega} \left|\left(Ax_1'\right)(t)\right| dt}{|1-|c||} \\
&= \frac{\int_0^{\omega} \left|(Ax_1)'(t) - cx_1'(t-\delta(t))\delta'(t)\right| dt}{|1-|c||} \\
&\le \frac{\int_0^{\omega} \left|\left(Ax_1'\right)(t)\right| dt + |c|\delta_1 \int_0^{\omega} \left|x_1'(t)\right| dt}{|1-|c||},
\end{aligned}
\tag{3.33}
$$

where $\delta_1 = \max_{t\in[0,\omega]}|\delta'(t)|$. Since $0 < \omega^{1/2}(1+|c|)^{1/2}\sqrt{2K_1}/(|1-|c||-|c|\delta_1)$, then $|1-|c||-|c|\delta_1 > 0$, so we get

$$\int_0^{\omega}\left|x_1'(t)\right|dt \le \frac{\int_0^{\omega}\left|(Ax_1)'(t)\right|dt}{|1-|c||-|c|\delta_1} \le \frac{\omega^{1/2}\left(\int_0^{\omega}\left|(Ax_1)'(t)\right|^2dt\right)^{1/2}}{|1-|c||-|c|\delta_1}. \tag{3.34}$$

Applying the inequality $(a+b)^k \le a^k + b^k$ for $a, b > 0$, $0 < k < 1$, it follows from (3.31) and (3.34) that

$$\begin{aligned}
&\int_0^{\omega}\left|x_1'(t)\right|dt \\
&\quad\le \frac{\omega^{1/2}}{|1-|c||-|c|\delta_1}\Bigg[(1+|c|)^{1/2}\sqrt{2K_1}\int_0^{\omega}\left|x_1'(t)\right|dt + (1+|c|)^{1/2}\left(\int_0^{\omega}\left|x_1'(t)\right|dt\right)^{1/2} \\
&\qquad\times(N_2+2K_1D)^{1/2} + (1+|c|)^{1/2}N_2D^{1/2}\Bigg].
\end{aligned} \tag{3.35}$$

Since $\omega^{1/2}(1+|c|)^{1/2}\sqrt{2K_1}/(|1-|c||-|c|\delta_1) < 1$, it is easy to see that there exists a constant $M_1 > 0$ (independent of λ) such that

$$\int_0^{\omega}\left|x_1'(t)\right|dt \le M_1. \tag{3.36}$$

It follows from (3.19) that

$$\|x_1\| \le D + \int_0^{\omega}\left|x_1'(t)\right|dt \le D + M_1 := M_2. \tag{3.37}$$

By the first equation of (3.11) we have $\int_0^{\omega} x_2(t)dt = \int_0^{\omega}(Ax_1)'(t)dt = 0$, which implies that there is a constant $t_1 \in [0,\omega]$ such that $x_2(t_1) = 0$, hence $\|x_2\| \le \int_0^{\omega}|x_2'(t)|dt$. By the second equation of (3.11) we obtain

$$x_2'(t) = \lambda f\left(t, x_1'(t)\right) + \lambda g(x_1(t-\tau(t))) + \lambda e(t). \tag{3.38}$$

So, from (H_1) and (3.25), we have

$$\begin{aligned}
|x_2|_0 &\le \int_0^{\omega}\left|f\left(t, x_1'(t)\right)\right|dt + \int_0^{\omega}\left|g(t, x_1(t-\tau(t)))\right|dt + \int_0^{\omega}|e(t)|dt \\
&\le 2K_1M_1 + 2\omega b + 2\omega N_1 + \omega\|e\| := M_3.
\end{aligned} \tag{3.39}$$

Let $M_4 = \sqrt{M_2^2 + M_3^2} + 1$, $\Omega = \{x = (x_1, x_2)^\top : \|x_1\| < M_4,\ \|x_2\| < M_4\}$, then for all $x \in \partial\Omega \cap \operatorname{Ker} L$

$$Q_1 N x = \frac{1}{\omega}\int_0^{\omega}\begin{pmatrix} x_2(t) \\ f\left(t, x_1'(t)\right) + g(t, x_1(t-\tau(t))) + e(t)\end{pmatrix} dt. \tag{3.40}$$

If $Q_1 N x = 0$, then $x_2(t) = 0, x_1 = M_4$ or $-M_4$. But if $x_1(t) = M_4$, we know

$$0 = \int_0^{\omega} g(M_4)\,dt, \tag{3.41}$$

that is, $g(M_4) = 0$. From assumption (H_2), we know $M_4 \leq D$, which yields a contradiction, one can argue similarly if $x_1 = -M_4$. We also have $Q_1 N x \neq 0$, that is, for all $x \in \partial\Omega \cap \operatorname{Ker} L, x \notin \operatorname{Im} L$, so conditions (1) and (2) of Lemma 3.1 are both satisfied. Define the isomorphism $J : \operatorname{Im} Q_1 \to \operatorname{Ker} L$ as follows:

$$J(x_1, x_2)^\top = (x_2, x_1)^\top. \tag{3.42}$$

Let $H(\mu, x) = \mu x + (1-\mu) J Q_1 N x, (\mu, x) \in [0,1] \times \Omega$, then, for all $(\mu, x) \in (0,1) \times (\partial\Omega \cap \operatorname{Ker} L)$,

$$H(\mu, x) = \begin{pmatrix} \mu x_1(t) + \dfrac{1-\mu}{\omega}\displaystyle\int_0^{\omega}\left[f\left(t, x_1'(t)\right) + g(t, x_1(t-\tau(t))) + e(t)\right] dt \\ (\mu + (1-\mu)) x_2(t)\end{pmatrix}. \tag{3.43}$$

We have $\int_0^{\omega} e(t)\,dt = 0$. So, we can get

$$H(\mu, x) = \begin{pmatrix} \mu x_1(t) + \dfrac{1-\mu}{\omega}\displaystyle\int_0^{\omega}\left[f\left(t, x_1'(t)\right) + g(t, x_1(t-\tau(t)))\right] dt \\ (\mu + (1-\mu)) x_2(t)\end{pmatrix}, \quad \forall (\mu, x) \in (0,1) \times (\partial\Omega \cap \operatorname{Ker} L). \tag{3.44}$$

From (H_2), it is obvious that $x^\top H(\mu, x) > 0$, for all $(\mu, x) \in (0,1) \times (\partial\Omega \cap \operatorname{Ker} L)$. Hence

$$\begin{aligned} \deg\{J Q_1 N, \Omega \cap \operatorname{Ker} L, 0\} &= \deg\{H(0, x), \Omega \cap \operatorname{Ker} L, 0\} \\ &= \deg\{H(1, x), \Omega \cap \operatorname{Ker} L, 0\} \\ &= \deg\{I, \Omega \cap \operatorname{Ker} L, 0\} \neq 0. \end{aligned} \tag{3.45}$$

So condition (3) of Lemma 3.1 is satisfied. By applying Lemma 3.1, we conclude that equation $Lx = Nx$ has a solution $x = (x_1, x_2)^\top$ on $\overline{\Omega} \cap D(L)$, that is, (3.1) has an ω-periodic solution $x_1(t)$. □

By using a similar argument, we can obtain the following theorem.

Theorem 3.3. *Suppose there exist positive constants K_1, D, M, b with $M > \|e\|$ such that:*

(H_1) $|f(t,u)| \le K_1|u| + b$, *for* $(t,u) \in \mathbb{R} \times \mathbb{R}$;

(H_2) $\operatorname{sgn} x \cdot g(t,x) > \|e\|$, *for* $|x| > D$,

(H_3) $g(t,x) \le M$, *for* $x \ge D$ *and* $t \in \mathbb{R}$,

then (3.1) *has at least one solution with period* ω *if* $0 < \omega(1+|c|)^{1/2}\sqrt{2K_1}/(|1-|c|| - |c|\delta_1) < 1$.

Remark 3.4. If $\int_0^\omega e(t)dt \neq 0$ and $f(t,0) \neq 0$, the problem of existence of ω-periodic solutions to (3.1) can be converted to the existence of ω-periodic solutions to

$$(x(t) - cx(t-\delta(t)))'' = f_1(t, x'(t)) + g_1(t, x(t-\tau(t))) + e_1(t), \tag{3.46}$$

where $f_1(t,x) = f(t,x) - f(t,0)$, $g_1(t,x) = g(t,x) + (1/\omega)\int_0^\omega e(t)dt + f(t,0)$, and $e_1(t) = e(t) - (1/\omega)\int_0^\omega e(t)dt$. Clearly, $\int_0^\omega e_1(t)dt = 0$ and $f_1(t,0) = 0$, and (3.46) can be discussed by using Theorem 3.2 (or Theorem 3.3).

4. Positive Periodic Solutions for Neutral Equations

Consider the following second-order neutral functional differential equation:

$$(x(t) - cx(t-\delta(t)))'' = -a(t)x(t) + \lambda b(t) f(x(t-\tau(t))), \tag{4.1}$$

where λ is a positive parameter; $f \in C(\mathbb{R}, [0,\infty))$, and $f(x) > 0$ for $x > 0$; $a \in C(\mathbb{R}, (0,\infty))$ with $\max\{a(t) : t \in [0,\omega]\} < (\pi/\omega)^2$, $b \in C(\mathbb{R}, (0,\infty))$, $\tau \in C(\mathbb{R}, \mathbb{R})$, $a(t)$, $b(t)$, and $\tau(t)$ are ω-periodic functions.

Define the Banach space X as in Section 2, and let $C_\omega^+ = \{x \in C(\mathbb{R}, (0,\infty)) : x(t+\omega) = x(t)\}$. Denote

$$\begin{gathered} M = \max\{a(t) : t \in [0,\omega]\}, \qquad m = \min\{a(t) : t \in [0,\omega]\}, \qquad \beta = \sqrt{M}, \\ L = \frac{1}{2\beta \sin(\beta\omega/2)}, \qquad l = \frac{\cos(\beta\omega/2)}{2\beta\sin(\beta\omega/2)}, \qquad k = l(M+m) + LM, \\ k_1 = \frac{k - \sqrt{k^2 - 4LlMm}}{2LM}, \qquad \alpha = \frac{l[m - (M+m)|c|]}{LM(1-|c|)}. \end{gathered} \tag{4.2}$$

It is easy to see that $M, m, \beta, L, l, k, k_1 > 0$.

Now we consider (4.1). First let

$$\overline{f}_0 = \overline{\lim_{x\to 0}} \frac{f(x)}{x}, \qquad \overline{f}_\infty = \overline{\lim_{x\to\infty}} \frac{f(x)}{x}, \qquad \underline{f}_0 = \underline{\lim_{x\to 0}} \frac{f(x)}{x}, \qquad \underline{f}_\infty = \underline{\lim_{x\to\infty}} \frac{f(x)}{x}, \tag{4.3}$$

and denote

$$\begin{aligned} \bar{i}_0 &= \text{number of } 0\text{'s in} \left(\bar{f}_0, \bar{f}_\infty \right), \qquad \underline{i}_0 = \text{number of } 0\text{'s in} \left(\underline{f}_0, \underline{f}_\infty \right); \\ \bar{i}_\infty &= \text{number of } \infty\text{'s in} \left(\bar{f}_0, \bar{f}_\infty \right), \qquad \underline{i}_\infty = \text{number of } \infty\text{'s in} \left(\underline{f}_0, \underline{f}_\infty \right). \end{aligned} \tag{4.4}$$

It is clear that $\bar{i}_0, \underline{i}_0, \bar{i}_\infty, \underline{i}_\infty \in \{0,1,2\}$. We will show that (4.1) has $\bar{i}_0$ or $\underline{i}_\infty$ positive w-periodic solutions for sufficiently large or small λ, respectively.

In the following we discuss (4.1) in two cases, namely, the case where $c < 0$ and $c > -\min\{k_1, m/(M+m)\}$ (note that $c > -m/(M+m)$ implies $\alpha > 0$; $c > -k_1$ implies $|c| < \alpha$) and the case where $c > 0$ and $c < min\{m/(M+m), (LM - lm)/((L-l)M - lm)\}$ (note that $c < m/(M+m)$ implies $\alpha > 0$; $c < (LM - lm)/((L-l)M - lm)$ implies $\alpha < 1$). Obviously, we have $|c| < 1$ which makes Lemma 2.1 applicable for both cases and also Lemmas 2.2 or 2.3, respectively.

Let $K = \{x \in X : x(t) \geq \alpha\|x\|\}$ denote the cone in X as defined in Section 2, where α is just as defined above. We also use $K_r = \{x \in K : \|x\| < r\}$ and $\partial K_r = \{x \in K : \|x\| = r\}$.

Let $y(t) = (Ax)(t)$, then from Lemma 2.1 we have $x(t) = (A^{-1}y)(t)$. Hence (4.1) can be transformed into

$$y''(t) + a(t)\left(A^{-1}y\right)(t) = \lambda b(t) f\left(\left(A^{-1}y\right)(t - \tau(t))\right), \tag{4.5}$$

which can be further rewritten as

$$y''(t) + a(t)y(t) - a(t)H(y(t)) = \lambda b(t) f\left(\left(A^{-1}y\right)(t - \tau(t))\right), \tag{4.6}$$

where $H(y(t)) = y(t) - (A^{-1}y)(t) = -c(A^{-1}y)(t - \delta(t))$.

Now we discuss the two cases separately.

4.1. Case I

Assume $c < 0$ and $c > -\min\{k_1, m/(M+m)\}$.

Lemma 4.1 (see [7]). *The equation*

$$y''(t) + My(t) = h(t), \quad h \in C_\omega^+, \tag{4.7}$$

has a unique ω-periodic solution

$$y(t) = \int_t^{t+\omega} G(t,s)h(s)ds, \tag{4.8}$$

where

$$G(t,s) = \frac{\cos\beta((\omega/2) + t - s)}{2\beta\sin(\beta\omega/2)}, \quad s \in [t, t+\omega]. \tag{4.9}$$

Lemma 4.2 (see [7]). *One has* $\int_t^{t+\omega} G(t,s)ds = 1/M$. *Furthermore, if* $\max\{a(t) : t \in [0,\omega]\} < (\pi/\omega)^2$, *then* $0 < l \leq G(t,s) \leq L$ *for all* $t \in [0,\omega]$ *and* $s \in [t, t+\omega]$.

Now we consider

$$y''(t) + a(t)y(t) - a(t)H(y(t)) = h(t), \quad h \in C_\omega^+, \tag{4.10}$$

and define operators $T, \widehat{H} : X \to X$ by

$$(Th)(t) = \int_t^{t+\omega} G(t,s)h(s)ds, \qquad \left(\widehat{H}y\right)(t) = M - a(t)y(t) + a(t)H(y(t)). \tag{4.11}$$

Clearly $T, \widehat{H}$ are completely continuous $(Th)(t) > 0$ for $h(t) > 0$ and $\|\widehat{H}\| \leq (M-m+M(|c|/(1-|c|)))$.

By Lemma 4.1, the solution of (4.10) can be written in the form

$$y(t) = (Th)(t) + \left(T\widehat{H}y\right)(t). \tag{4.12}$$

In view of $c < 0$ and $c > -\min\{k_1, m/(M+m)\}$, we have

$$\left\|T\widehat{H}\right\| \leq \|T\|\left\|\widehat{H}\right\| \leq \frac{M-m+m|c|}{M(1-|c|)} < 1, \tag{4.13}$$

and hence

$$y(t) = \left(I - T\widehat{H}\right)^{-1}(Th)(t). \tag{4.14}$$

Define an operator $P : X \to X$ by

$$(Ph)(t) = \left(I - T\widehat{H}\right)^{-1}(Th)(t). \tag{4.15}$$

Obviously, for any $h \in C_\omega^+$, if $\max\{a(t) : t \in [0,\omega]\} < (\pi/\omega)^2$, $y(t) = (Ph)(t)$ is the unique positive ω-periodic solution of (4.10).

Lemma 4.3. *P is completely continuous and*

$$(Th)(t) \leq (Ph)(t) \leq \frac{M(1-|c|)}{m-(M+m)|c|}\|Th\|, \quad \forall h \in C_\omega^+. \tag{4.16}$$

Proof. By the Neumann expansion of P, we have

$$P = \left(I - T\widehat{H}\right)^{-1} T = \left(I + T\widehat{H} + \left(T\widehat{H}\right)^2 + \cdots + \left(T\widehat{H}\right)^n + \cdots\right) T = T + T\widehat{H}T + \left(T\widehat{H}\right)^2 T + \cdots + \left(T\widehat{H}\right)^n T + \cdots. \tag{4.17}$$

Since T and $\widehat{H}$ are completely continuous, so is P. Moreover, by (4.17), and recalling that $\|T\widehat{H}\| \le (M - m + m|c|)/M(1 - |c|) < 1$, we get

$$(Th)(t) \le (Ph)(t) \le \frac{M(1-|c|)}{m - (M+m)|c|}\|Th\|. \tag{4.18}$$

□

Define an operator $Q : X \to X$ by

$$Qy(t) = P\left(\lambda b(t) f\left(\left(A^{-1}y\right)(t - \tau(t))\right)\right). \tag{4.19}$$

Lemma 4.4. *One has $Q(K) \subset K$.*

Proof. From the definition of Q, it is easy to verify that $Qy(t+\omega) = Qy(t)$. For $y \in K$, we have from Lemma 4.3 that

$$\begin{aligned} Qy(t) &= P\left(\lambda b(t) f\left(\left(A^{-1}y\right)(t - \tau(t))\right)\right) \\ &\ge T\left(\lambda b(t) f\left(\left(A^{-1}y\right)(t - \tau(t))\right)\right) \\ &= \lambda \int_t^{t+\omega} G(t,s) b(s) f\left[\left(A^{-1}y\right)(s - \tau(s))\right] ds \\ &\ge \lambda l \int_0^{\omega} b(s) f\left[\left(A^{-1}y\right)(s - \tau(s))\right] ds. \end{aligned} \tag{4.20}$$

On the other hand,

$$\begin{aligned} Qy(t) &= P\left(\lambda b(t) f\left(\left(A^{-1}y\right)(t - \tau(t))\right)\right) \\ &\le \frac{M(1-|c|)}{m - (M+m)|c|}\left\|T\left(\lambda b(t) f\left(\left(A^{-1}y\right)(t - \tau(t))\right)\right)\right\| \end{aligned}$$

$$
\begin{aligned}
&= \lambda \frac{M(1-|c|)}{m-(M+m)|c|} \max_{t\in[0,\omega]} \int_t^{t+\omega} G(t,s)b(s)f\Big(\Big(A^{-1}y\Big)(s-\tau(s))\Big)ds \\
&\le \lambda \frac{M(1-|c|)}{m-(M+m)|c|} L \int_0^{\omega} b(s)f\Big(\Big(A^{-1}y\Big)(s-\tau(s))\Big)ds.
\end{aligned} \tag{4.21}
$$

Therefore,

$$
Qy(t) \ge \frac{l[m-(M+m)|c|]}{LM(1-|c|)} \|Qy\| = \alpha \|Qy\|, \tag{4.22}
$$

that is, $Q(K) \subset K$. □

From the continuity of P, it is easy to verify that Q is completely continuous in X. Comparing (4.6) to (4.10), it is obvious that the existence of periodic solutions for (4.6) is equivalent to the existence of fixed points for the operator Q in X. Recalling Lemma 4.4, the existence of positive periodic solutions for (4.6) is equivalent to the existence of fixed points of Q in K. Furthermore, if Q has a fixed point y in K, it means that $(A^{-1}y)(t)$ is a positive ω-periodic solutions of (4.1).

Lemma 4.5. *If there exists $\eta > 0$ such that*

$$
f\Big(\Big(A^{-1}y\Big)(t-\tau(t))\Big) \ge \Big(A^{-1}y\Big)(t-\tau(t))\eta, \quad \text{for } t \in [0,\omega],\ y \in K, \tag{4.23}
$$

then

$$
\|Qy\| \ge \lambda l \eta \frac{\alpha - |c|}{1-c^2} \int_0^{\omega} b(s)ds \|y\|, \quad y \in K. \tag{4.24}
$$

Proof. By Lemmas 2.2, 4.2, and 4.3, we have for $y \in K$ that

$$
\begin{aligned}
Qy(t) &= P\Big(\lambda b(t)f\Big(\Big(A^{-1}y\Big)(t-\tau(t))\Big)\Big) \\
&\ge T\Big(\lambda b(t)f\Big(\Big(A^{-1}y\Big)(t-\tau(t))\Big)\Big) \\
&= \lambda \int_t^{t+\omega} G(t,s)b(s)f\Big(\Big(A^{-1}y\Big)(s-\tau(s))\Big)ds \\
&\ge \lambda l \eta \int_0^{\omega} b(s)\Big(A^{-1}y\Big)(s-\tau(s))ds \\
&\ge \lambda l \eta \frac{\alpha-|c|}{1-c^2} \int_0^{\omega} b(s)ds \|y\|.
\end{aligned} \tag{4.25}
$$

Hence

$$\|Qy\| \geq \lambda l\eta \frac{\alpha - |c|}{1 - c^2} \int_0^\omega b(s)ds\|y\|, \quad y \in K. \tag{4.26}$$

□

Lemma 4.6. *If there exists $\varepsilon > 0$ such that*

$$f\left(\left(A^{-1}y\right)(t - \tau(t))\right) \leq \left(A^{-1}y\right)(t - \tau(t))\varepsilon, \quad \text{for } t \in [0, \omega],\ y \in K, \tag{4.27}$$

then

$$\|Qy\| \leq \lambda\varepsilon \frac{LM \int_0^\omega b(s)ds}{m - (M + m)|c|}\|y\|, \quad y \in K. \tag{4.28}$$

Proof. By Lemmas 2.2, 4.2, and 4.3, we have

$$\begin{aligned}
\|Qy(t)\| &\leq \lambda \frac{M(1 - |c|)}{m - (M + m)|c|} L \int_0^\omega b(s) f\left(\left(A^{-1}y\right)(s - \tau(s))\right)ds \\
&\leq \lambda \frac{M(1 - |c|)}{m - (M + m)|c|} L\varepsilon \int_0^\omega b(s)\left(A^{-1}y\right)(s - \tau(s))ds \\
&\leq \lambda\varepsilon \frac{LM \int_0^\omega b(s)ds}{m - (M + m)|c|}\|y\|.
\end{aligned} \tag{4.29}$$

□

Define

$$\begin{aligned}
F(r) &= \max\left\{ f(t) : 0 \leq t \leq \frac{r}{1 - |c|} \right\}, \\
f_1(r) &= \min\left\{ f(t) : \frac{\alpha - |c|}{1 - c^2} r \leq t \leq \frac{r}{1 - |c|} \right\}.
\end{aligned} \tag{4.30}$$

Lemma 4.7. *If $y \in \partial K_r$, then*

$$\|Qy\| \geq \lambda l f_1(r) \int_0^\omega b(s)ds. \tag{4.31}$$

Proof. By Lemma 2.2, we obtain $((\alpha - |c|)/(1 - c^2))r \leq (A^{-1}y)(t - \tau(t)) \leq r/(1 - |c|)$ for $y \in \partial K_r$, which yields $f((A^{-1}y)(t - \tau(t))) \geq f_1(r)$. The lemma now follows analog to the proof of Lemma 4.5. □

Lemma 4.8. *If $y \in \partial K_r$, then*

$$\|Qy\| \le \lambda \frac{LM(1-|c|)F(r)}{m-(M+m)|c|} \int_0^{\omega} b(s)ds. \tag{4.32}$$

Proof. By Lemma 2.2, we can have $0 \le (A^{-1}y)(t-\tau(t)) \le r/(1-|c|)$ for $y \in \partial K_r$, which yields $f((A^{-1}y)(t-\tau(t))) \le F(r)$. Similar to the proof of Lemma 4.6, we get the conclusion. □

We quote the fixed point theorem which our results will be based on.

Lemma 4.9 (see [9]). *Let X be a Banach space and K a cone in X. For $r > 0$, define $K_r = \{u \in K : \|u\| < r\}$. Assume that $T : \overline{K}_r \to K$ is completely continuous such that $Tx \ne x$ for $x \in \partial K_r = \{u \in K : \|u\| = r\}$.*

(i) *If $\|Tx\| \ge \|x\|$ for $x \in \partial K_r$, then $i(T, K_r, K) = 0$.*

(ii) *If $\|Tx\| \le \|x\|$ for $x \in \partial K_r$, then $i(T, K_r, K) = 1$.*

Now we give our main results on positive periodic solutions for (4.1).

Theorem 4.10. (a) *If $\bar{i}_0 = 1$ or 2, then* (4.1) *has $\bar{i}_0$ positive ω-periodic solutions for $\lambda > 1/f_1(1)l\int_0^{\omega} b(s)ds > 0$;*

(b) *If $\underline{i}_{\infty} = 1$ or 2, then* (4.1) *has $\underline{i}_{\infty}$ positive ω-periodic solutions for $0 < \lambda < (m-(M+m)|c|)/LM(1-|c|)F(1)\int_0^{\omega} b(s)ds$;*

(c) *If $\bar{i}_{\infty} = 0$ or $\underline{i}_0 = 0$, then* (4.1) *has no positive ω-periodic solutions for sufficiently small or sufficiently large $\lambda > 0$, respectively.*

Proof. (a) Choose $r_1 = 1$. Take $\lambda_0 = 1/f_1(r_1)l\int_0^{\omega} b(s)ds > 0$, then for all $\lambda > \lambda_0$, we have from Lemma 4.7 that

$$\|Qy\| > \|y\|, \quad \text{for } y \in \partial K_{r_1}. \tag{4.33}$$

Case 1. If $\overline{f}_0 = 0$, we can choose $0 < \bar{r}_2 < r_1$, so that $f(u) \le \varepsilon u$ for $0 \le u \le \bar{r}_2$, where the constant $\varepsilon > 0$ satisfies

$$\lambda\varepsilon \frac{LM\int_0^{\omega} b(s)ds}{m-(M+m)|c|} < 1. \tag{4.34}$$

Letting $r_2 = (1-|c|)\bar{r}_2$, we have $f((A^{-1}y)(t-\tau(t))) \le \varepsilon(A^{-1}y)(t-\tau(t))$ for $y \in K_{r_2}$. By Lemma 2.2, we have $0 \le (A^{-1}y)(t-\tau(t)) \le \|y\|/(1-|c|) \le \bar{r}_2$ for $y \in \partial K_{r_2}$. In view of Lemma 4.6 and (4.34), we have for $y \in \partial K_{r_2}$ that

$$\|Qy\| \le \lambda\varepsilon \frac{LM\int_0^{\omega} b(s)ds}{m-(M+m)|c|}\|y\| < \|y\|. \tag{4.35}$$

It follows from Lemma 4.9 and (4.33) that

$$i(Q, K_{r_2}, K) = 1, \qquad i(Q, K_{r_1}, K) = 0, \tag{4.36}$$

thus $i(Q, K_{r_1} \setminus \overline{K}_{r_2}, K) = -1$ and Q has a fixed point y in $K_{r_1} \setminus \overline{K}_{r_2}$, which means $(A^{-1}y)(t)$ is a positive ω-positive solution of (4.1) for $\lambda > \lambda_0$.

Case 2. If $\overline{f}_{\infty} = 0$, there exists a constant $\widetilde{H} > 0$ such that $f(u) \leq \varepsilon u$ for $u \geq \widetilde{H}$, where the constant $\varepsilon > 0$ satisfies

$$\lambda \varepsilon \frac{LM \int_0^{\omega} b(s)ds}{m - (M+m)|c|} < 1. \tag{4.37}$$

Letting $r_3 = \max\{2r_1, \widetilde{H}(1-c^2)/(\alpha - |c|)\}$, we have $f((A^{-1}y)(t-\tau(t))) \leq \varepsilon (A^{-1}y)(t-\tau(t))$ for $y \in K_{r_3}$. By Lemma 2.2, we have $(A^{-1}y)(t-\tau(t)) \geq ((\alpha - |c|)/(1-c^2))\|y\| \geq \widetilde{H}$ for $y \in \partial K_{r_3}$. Thus by Lemma 4.6 and (4.37), we have for $y \in \partial K_{r_3}$ that

$$\|Qy\| \leq \lambda \varepsilon \frac{LM \int_0^{\omega} b(s)ds}{m - (M+m)|c|} \|y\| < \|y\|. \tag{4.38}$$

Recalling from Lemma 4.9 and (4.33) that

$$i(Q, K_{r_3}, K) = 1, \qquad i(Q, K_{r_1}, K) = 0, \tag{4.39}$$

then $i(Q, K_{r_3} \setminus \overline{K}_{r_1}, K) = 1$ and Q has a fixed point y in $K_{r_3} \setminus \overline{K}_{r_1}$, which means $(A^{-1}y)(t)$ is a positive ω-positive solution of (4.1) for $\lambda > \lambda_0$.

Case 3. If $\overline{f}_0 = \overline{f}_{\infty} = 0$, from the above arguments, there exist $0 < r_2 < r_1 < r_3$ such that Q has a fixed point $y_1(t)$ in $K_{r_1} \setminus \overline{K}_{r_2}$ and a fixed point $y_2(t)$ in $K_{r_3} \setminus \overline{K}_{r_1}$. Consequently, $(A^{-1}y_1)(t)$ and $(A^{-1}y_2)(t)$ are two positive ω-periodic solutions of (4.1) for $\lambda > \lambda_0$.

(b) Let $r_1 = 1$. Take $\lambda_0 = (m - (M+m)|c|)/LM(1-|c|)F(r_1)\int_0^{\omega} b(s)ds > 0$; then by Lemma 4.8 we know if $\lambda < \lambda_0$ then

$$\|Qy\| < \|y\|, \quad y \in \partial K_{r_1}. \tag{4.40}$$

Case 1. If $\underline{f}_0 = \infty$, we can choose $0 < \overline{r}_2 < r_1$ so that $f(u) \geq \eta u$ for $0 \leq u \leq \overline{r}_2$, where the constant $\eta > 0$ satisfies

$$\lambda l \eta \frac{\alpha - |c|}{1-c^2} \int_0^{\omega} b(s)ds > 1. \tag{4.41}$$

Letting $r_2 = (1-|c|)\overline{r}_2$, we have $f((A^{-1}y)(t-\tau(t))) \geq \eta (A^{-1}y)(t-\tau(t))$ for $y \in K_{r_2}$. By Lemma 2.2, we have $0 \leq (A^{-1}y)(t-\tau(t)) \leq \|y\|/(1-|c|) \leq \overline{r}_2$ for $y \in \partial K_{r_2}$. Thus by Lemma 4.5 and (4.41),

$$\|Qy\| \geq \lambda l \eta \frac{\alpha - |c|}{1-c^2} \int_0^{\omega} b(s)ds \|y\| > \|y\|. \tag{4.42}$$

It follows from Lemma 4.9 and (4.40) that

$$i(Q, K_{r_2}, K) = 0, \qquad i(Q, K_{r_1}, K) = 1, \tag{4.43}$$

which implies $i(Q, K_{r_1} \setminus \overline{K}_{r_2}, K) = 1$ and Q has a fixed point y in $K_{r_1} \setminus \overline{K}_{r_2}$. Therefore, $(A^{-1}y)(t)$ is a positive ω-periodic solution of (4.1) for $0 < \lambda < \lambda_0$.

Case 2. If $\underline{f}_{\infty} = \infty$, there exists a constant $\widetilde{H} > 0$ such that $f(u) \geq \eta u$ for $u \geq \widetilde{H}$, where the constant $\eta > 0$ satisfies

$$\lambda l \eta \frac{\alpha - |c|}{1 - c^2} \int_0^{\omega} b(s) ds > 1. \tag{4.44}$$

Letting $r_3 = \max\{2r_1, \widetilde{H}(1 - c^2)/(\alpha - |c|)\}$, we have $f((A^{-1}y)(t - \tau(t))) \geq \eta(A^{-1}y)(t - \tau(t))$ for $y \in K_{r_3}$. By Lemma 2.2, we have $(A^{-1}y)(t - \tau(t)) \geq ((\alpha - |c|)/(1 - c^2))\|y\| \geq \widetilde{H}$ for $y \in \partial K_{r_3}$. Thus by Lemma 4.5 and (4.44), we have for $y \in \partial K_{r_3}$ that

$$\|Qy\| \geq \lambda l \eta \frac{\alpha - |c|}{1 - c^2} \int_0^{\omega} b(s) ds \|y\| > \|y\|. \tag{4.45}$$

It follows from Lemma 4.9 and (4.40) that

$$i(Q, K_{r_3}, K) = 0, \qquad i(Q, K_{r_1}, K) = 1. \tag{4.46}$$

that is, $i(Q, K_{r_3} \setminus \overline{K}_{r_1}, K) = -1$ and Q has a fixed point y in $K_{r_3} \setminus \overline{K}_{r_1}$. That means $(A^{-1}y)(t)$ is a positive ω-periodic solution of (4.1) for $0 < \lambda < \lambda_0$.

Case 3. If $\underline{f}_0 = \underline{f}_{\infty} = \infty$, from the above arguments, Q has a fixed point y_1 in $K_{r_1} \setminus \overline{K}_{r_2}$ and a fixed point y_2 in $K_{r_3} \setminus \overline{K}_{r_1}$. Consequently, $(A^{-1}y_1)(t)$ and $(A^{-1}y_2)(t)$ are two positive ω-periodic solutions of (4.1) for $0 < \lambda < \lambda_0$.

(c) By Lemma 2.2, if $y \in K$, then $(A^{-1}y)(t - \tau(t)) \geq ((\alpha - |c|)/(1 - c^2))\|y\| > 0$ for $t \in [0, \omega]$.

Case 1. If $\underline{i}_0 = 0$, we have $\underline{f}_0 > 0$ and $\underline{f}_{\infty} > 0$. Let $b_1 = \min\{f(u)/u;\ u > 0\} > 0$, then we obtain

$$f(u) \geq b_1 u, \quad u \in [0, +\infty). \tag{4.47}$$

Assume $y(t)$ is a positive ω-periodic solution of (4.1) for $\lambda > \lambda_0$, where $\lambda_0 = (1 - c^2)/l b_1 (\alpha - |c|) \int_0^{\omega} b(s) ds > 0$. Since $Qy(t) = y(t)$ for $t \in [0, \omega]$, then by Lemma 4.5, if $\lambda > \lambda_0$ we have

$$\|y\| = \|Qy\| \geq \lambda l b_1 \frac{\alpha - |c|}{1 - c^2} \int_0^{\omega} b(s) ds \|y\| > \|y\|, \tag{4.48}$$

which is a contradiction.

Case 2. If $\bar{i}_\infty = 0$, we have $\overline{f}_0 < \infty$ and $\overline{f}_\infty < \infty$. Let $b_2 = \max\{f(u)/u : u > 0\} > 0$, then we obtain

$$f(u) \le b_2 u, \quad u \in [0, \infty). \tag{4.49}$$

Assume $y(t)$ is a positive ω-periodic solution of (4.1) for $0 < \lambda < \lambda_0$, where $\lambda_0 = (m - (M + m)|c|)/b_2 LM \int_0^\omega b(s)ds$. Since $Qy(t) = y(t)$ for $t \in [0, \omega]$, it follows from Lemma 4.6 that

$$\|y\| = \|Qy\| \le \lambda b_2 \frac{LM \int_0^\omega b(s)ds}{m - (M+m)|c|} \|y\| < \|y\|, \tag{4.50}$$

which is a contradiction. □

Theorem 4.11. (a) *If there exists a constant $b_1 > 0$ such that $f(u) \ge b_1 u$ for $u \in [0, +\infty)$, then* (4.1) *has no positive ω-periodic solution for $\lambda > (1 - c^2)/l b_1(\alpha - |c|) \int_0^\omega b(s)ds$.*

(b) *If there exists a constant $b_2 > 0$ such that $f(u) \le b_2 u$ for $u \in [0, +\infty)$, then* (4.1) *has no positive ω-periodic solution for $0 < \lambda < (m - (M+m)|c|)/b_2 LM \int_0^\omega b(s)ds$.*

Proof. From the proof of (c) in Theorem 4.10, we obtain this theorem immediately. □

Theorem 4.12. *Assume $\underline{i}_0 = \bar{i}_0 = \underline{i}_\infty = \bar{i}_\infty = 0$ and that one of the following conditions holds:*

(1) $\overline{f}_0 \le \underline{f}_\infty$;

(2) $\underline{f}_0 > \overline{f}_\infty$;

(3) $\underline{f}_0 \le \underline{f}_\infty \le \overline{f}_0 \le \overline{f}_\infty$;

(4) $\underline{f}_\infty \le \underline{f}_0 \le \overline{f}_\infty \le \overline{f}_0$.

If

$$\frac{1 - c^2}{l(\alpha - |c|) \int_0^\omega b(s)ds \, \max\left\{\underline{f}_0, \overline{f}_0, \underline{f}_\infty, \overline{f}_\infty\right\}} < \lambda < \frac{m - (M+m)|c|}{LM \int_0^\omega b(s)ds \, \min\left\{\underline{f}_0, \overline{f}_0, \underline{f}_\infty, \overline{f}_\infty\right\}}, \tag{4.51}$$

then (4.1) *has one positive ω-periodic solution.*

Proof. We have the following cases.

Case 1. If $\overline{f}_0 \le \underline{f}_\infty$, then

$$\frac{1 - c^2}{\overline{f}_\infty l(\alpha - |c|) \int_0^\omega b(s)ds} < \lambda < \frac{m - (M+m)|c|}{\underline{f}_0 LM \int_0^\omega b(s)ds}. \tag{4.52}$$

It is easy to see that there exists an $0 < \varepsilon < f_{\infty}$ such that

$$\frac{1-c^2}{\left(\overline{f}_{\infty}-\varepsilon\right) l(\alpha-|c|) \int_0^{\omega} b(s)ds} < \lambda < \frac{m-(M+m)|c|}{\left(\underline{f}_0+\varepsilon\right) LM \int_0^{\omega} b(s)ds}. \tag{4.53}$$

For the above ε, we choose $\overline{r}_1 > 0$ such that $f(u) \le (\underline{f}_0 + \varepsilon)u$ for $0 \le u \le \overline{r}_1$. Letting $r_1 = (1-|c|)\overline{r}_1$, we have $f((A^{-1}y)(t-\tau(t))) \le (\underline{f}_0 + \varepsilon)(A^{-1}y)(t-\tau(t))$ for $y \in K_{r_1}$. By Lemma 2.2, we have $0 \le (A^{-1}y)(t-\tau(t)) \le \|y\|/(1-|c|) \le \overline{r}_1$ for $K \in \partial K_{r_1}$. Thus by Lemma 4.6 we have for $y \in \partial K_{r_1}$ that

$$\|Qy\| \le \lambda\left(\underline{f}_0+\varepsilon\right)\frac{LM\int_0^{\omega} b(s)ds}{m-(M+m)|c|}\|y\| < \|y\|. \tag{4.54}$$

On the other hand, there exists a constant $\widetilde{H} > 0$ such that $f(u) \ge (\overline{f}_{\infty} - \varepsilon)u$ for $u \ge \widetilde{H}$. Letting $r_2 = \max\{2r_1, \widetilde{H}(1-c^2)/(\alpha-|c|)\}$, we have $f((A^{-1}y)(t-\tau(t))) \ge (\overline{f}_{\infty}-\varepsilon)(A^{-1}y)(t-\tau(t))$ for $y \in K_{r_2}$. By Lemma 2.2, we have $(A^{-1}y)(t-\tau(t)) \ge ((\alpha-|c|)/(1-c^2))\|y\| \ge \widetilde{H}$ for $y \in \partial K_{r_2}$. Thus by Lemma 4.5, for $y \in \partial K_{r_2}$

$$\|Qy\| \ge \lambda l\left(\overline{f}_{\infty}-\varepsilon\right)\frac{\alpha-|c|}{1-c^2}\int_0^{\omega} b(s)ds\|y\| > \|y\|. \tag{4.55}$$

It follows from Lemma 4.9 that

$$i(Q, K_{r_1}, K) = 1, \qquad i(Q, K_{r_2}, K) = 0, \tag{4.56}$$

thus $i(Q, K_{r_2} \setminus \overline{K}_{r_1}, K) = -1$ and Q has a fixed point y in $K_{r_2} \setminus \overline{K}_{r_1}$. So $(A^{-1}y)(t)$ is a positive ω-periodic solution of (4.1).

Case 2. If $\underline{f}_0 > \overline{f}_{\infty}$, in this case, we have

$$\frac{1-c^2}{\overline{f}_0 l(\alpha-|c|) \int_0^{\omega} b(s)ds} < \lambda < \frac{m-(M+m)|c|}{\underline{f}_{\infty} LM \int_0^{\omega} b(s)ds}. \tag{4.57}$$

It is easy to see that there exists an $0 < \varepsilon < f_0$ such that

$$\frac{1-c^2}{\left(\overline{f}_0-\varepsilon\right) l(\alpha-|c|) \int_0^{\omega} b(s)ds} < \lambda < \frac{m-(M+m)|c|}{\left(\underline{f}_{\infty}+\varepsilon\right) LM \int_0^{\omega} b(s)ds}. \tag{4.58}$$

For the above ε, we choose $\overline{r}_1 > 0$ such that $f(u) \ge (\overline{f}_0 - \varepsilon)u$ for $0 \le u \le \overline{r}_1$. Letting $r_1 = (1-|c|)\overline{r}_1$, we have $f((A^{-1}y)(t-\tau(t))) \ge (\overline{f}_0 - \varepsilon)(A^{-1}y)(t-\tau(t))$ for $y \in K_{r_1}$. By Lemma 2.2, we have $0 \le (A^{-1}y)(t-\tau(t)) \le \|y\|/(1-|c|) \le \overline{r}_1$ for $y \in \partial K_{r_1}$. Thus we have by Lemma 4.5 that for $y \in \partial K_{r_1}$

$$\|Qy\| \ge \lambda l\left(\overline{f}_0 - \varepsilon\right)\frac{\alpha - |c|}{1-c^2}\int_0^{\omega} b(s)ds\|y\| > \|y\|. \tag{4.59}$$

On the other hand, there exists a constant $\widetilde{H} > 0$ such that $f(u) \le (\underline{f}_{\infty} + \varepsilon)u$ for $u \ge \widetilde{H}$. Letting $r_2 = \max\{2r_1, \widetilde{H}(1-c^2)/(\alpha - |c|)\}$, we have $f((A^{-1}y)(t-\tau(t))) \le (\underline{f}_{\infty} + \varepsilon)(A^{-1}y)(t-\tau(t))$ for $y \in K_{r_2}$. By Lemma 2.2 we have $(A^{-1}y)(t-\tau(t)) \ge ((\alpha-|c|)/(1-c^2))\|y\| \ge \widetilde{H}$ for $y \in \partial K_{r_2}$. Thus by Lemma 4.6, for $y \in \partial K_{r_2}$

$$\|Qy\| \le \lambda\left(\underline{f}_{\infty} + \varepsilon\right)\frac{LM\int_0^{\omega} b(s)ds}{m-(M+m)|c|}\|y\|. \tag{4.60}$$

It follows from Lemma 4.9 that

$$i(Q, K_{r_1}, K) = 0, \qquad i(Q, K_{r_2}, K) = 1. \tag{4.61}$$

Thus $i(Q, K_{r_2} \setminus \overline{K}_{r_1}, K) = -1$ and Q has a fixed point y in $K_{r_2} \setminus \overline{K}_{r_1}$, proving that $(A^{-1}y)(t)$ is a positive ω-periodic solution of (4.1).

Case 3. One has $\underline{f}_0 \le \underline{f}_{\infty} \le \overline{f}_0 \le \overline{f}_{\infty}$. The proof is the same as in Case 1.

Case 4. One has $\underline{f}_{\infty} \le \underline{f}_0 \le \overline{f}_{\infty} \le \overline{f}_0$. The proof is the same as in Case 2. □

4.2. Case II

Assume $c > 0$ and $c < \min\{m/(M+m), (LM-lm)/(L-l)M - lm\}$.

Define

$$f_2(r) = \min\left\{f(t) : \frac{\alpha}{1-c}r \le t \le \frac{r}{1-c}\right\}. \tag{4.62}$$

Similarly as in Section 4.1, we get the following results.

Theorem 4.13. (a) *If* $\overline{i}_0 = 1$ *or 2, then* (4.1) *has* i_0 *positive* ω*-periodic solutions for* $\lambda > 1/f_2(1)l\int_0^{\omega} b(s)ds > 0$.

(b) *If* $\underline{i}_{\infty} = 1$ *or 2, then* (4.1) *has* i_{∞} *positive* ω*-periodic solutions for* $0 < \lambda < (m-(M+m)c)/LM(1-c)F(1)\int_0^{\omega} b(s)ds$.

(c) *If* $\overline{i}_{\infty} = 0$ *or* $\underline{i}_0 = 0$*, then* (4.1) *has no positive* ω*-periodic solution for sufficiently small or large* $\lambda > 0$*, respectively.*

Theorem 4.14. (a) *If there exists a constant* $b_1 > 0$ *such that* $f(u) \ge b_1 u$ *for* $u \in [0, +\infty)$*, then* (4.1) *has no positive* ω*-periodic solution for* $\lambda > (1-c)/l\alpha b_1\int_0^{\omega} b(s)ds$.

(b) *If there exists a constant* $b_2 > 0$ *such that* $f(u) \le b_2 u$ *for* $u \in [0, +\infty)$, *then* (4.1) *has no positive* ω*-periodic solution for* $0 < \lambda < (m - (M+m)c)/b_2 LM \int_0^\omega b(s)ds$.

Theorem 4.15. *Assume* $\underline{i}_0 = \overline{i}_0 = \underline{i}_\infty = \overline{i}_\infty = 0$ *hold and that one of the following conditions holds:*

(1) $\overline{f}_0 \le \underline{f}_\infty$;

(2) $\underline{f}_0 > \overline{f}_\infty$;

(3) $\underline{f}_0 \le \underline{f}_\infty \le \overline{f}_0 \le \overline{f}_\infty$;

(4) $\underline{f}_\infty \le \underline{f}_0 \le \overline{f}_\infty \le \overline{f}_0$.

If

$$\frac{1-c}{l\alpha \int_0^\omega b(s)ds \, \max\left\{\underline{f}_0, \overline{f}_0, \underline{f}_\infty, \overline{f}_\infty\right\}} < \lambda < \frac{m-(M+m)c}{LM \int_0^\omega b(s)ds \, \min\left\{\underline{f}_0, \overline{f}_0, \underline{f}_\infty, \overline{f}_\infty\right\}}, \tag{4.63}$$

then (4.1) *has one positive* ω*-periodic solution.*

Remark 4.16. In a similar way, one can consider the second-order neutral functional differential equation $(x(t) - cx(t - \delta(t)))'' - a(t)x(t) = -\lambda b(t) f(x(t - \tau(t)))$.

5. Examples

Example 5.1. Consider the following equation:

$$\left(x(t) - 15x\left(t - \frac{1}{60}\sin 4t\right)\right)'' = x'(t)\sin 4t + \arctan\left(\frac{x(t - \sin 4t)}{1 + \cos^3(4t)}\right) + \cos 4t. \tag{5.1}$$

Comparing (5.1) to (3.1), we have $\omega = \pi/2$, $f(t,x) = x(t)\sin 4t$, $g(t,x) = \arctan(x/(1 + \cos^3(4t)))$, $c = 15$, $\delta(t) = (1/60)\sin 4t$, $\tau(t) = \sin 4t$, $e(t) = \cos 4t$ and $\delta_1 = \max_{t\in[0,\omega]}|(1/15)\cos 4t| = 1/15$, and we can easily choose $D > \pi/2$ and $M = \pi/2$ such that (H_2) and (H_3) holds. Regarding assumption (H_1) note that

$$\left|f(t, x'(t))\right| \le \left|x'(t)\right|, \tag{5.2}$$

that is, (H_1) holds with $K_1 = 1, b = 0$, and

$$\frac{\omega^{1/2}(1+|c|)^{1/2}\sqrt{2K_1}}{|1-|c|| - |c|\delta_1} = \frac{\sqrt{\pi/2}(1+15)^{1/2}\sqrt{2}}{|1-15| - (1/15)\cdot 15} = \frac{4\sqrt{\pi}}{13} < 1. \tag{5.3}$$

Hence by Theorem 3.2, (5.1) has at least one $\pi/2$-periodic solution.

Example 5.2. Consider the following neutral functional differential equation:

$$\left(u(t)+\frac{7}{30}u(t-\sin t)\right)''+\frac{1}{16}u(t)=\lambda(1-\sin t)u^2(t-\tau(t))a^{u(t-\tau(t))}, \tag{5.4}$$

where λ and $0<a<1$ are two positive parameters, $\tau(t+2\pi)=\tau(t)$.

Comparing (5.4) to (4.1), we see that $\delta(t)=\sin t$, $c=-7/30$, $a(t)\equiv 1/16$, $b(t)=1-\sin t$, $\omega=2\pi$, $f(u)=u^2a^u$. Clearly, $M=1/16<(\pi/2\pi)^2=1/4$, $\overline{f}_0=0$, $\overline{f}_\infty=0$, $\bar{i}_0=2$. By Theorem 4.10, we easily get the following conclusion: (5.4) has two positive ω-periodic solutions for $\lambda>1/4\pi r_1$, where $r_1=\min\{f(0.27),f(30/23)\}$.

In fact, by simple computations, we have

$$M=m=\frac{1}{16},\qquad \beta=\frac{1}{4},\qquad L=\frac{1}{2\beta\sin(\beta 2\pi/2)}=2\sqrt{2},\qquad l=\frac{\cos(\beta 2\pi/2)}{(2\beta\sin(\beta 2\pi/2))}=2,$$
$$k=\frac{2+\sqrt{2}}{8},\qquad k_1=\frac{\sqrt{2}+1-\sqrt{3}}{2},\qquad \alpha=\frac{8}{23}\sqrt{2},$$
$$|c|=\frac{7}{30}<\min\left\{k_1,\frac{m}{M+m}\right\}=\frac{\sqrt{2}+1-\sqrt{3}}{2},\qquad |c|=\frac{7}{30}<\frac{8}{23}\sqrt{2}=\alpha,$$
$$f_1(1)=\min\left\{f(t):0.27\approx\frac{(8/23)\sqrt{2}-(7/30)}{1-(7/30)^2}\le t\le\frac{30}{23}\right\}=\min\left\{f(0.27),f\left(\frac{30}{23}\right)\right\}=r_1,$$
$$\frac{1}{f_1(1)l\int_0^\omega b(s)ds}=\frac{1}{4\pi r_1}. \tag{5.5}$$

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