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Research Article

Hopf Bifurcation Analysis on General Gause-Type Predator-Prey Models with Delay

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A class of three-dimensional Gause-type predator-prey model with delay is considered. Firstly, a group of sufficient conditions for the existence of Hopf bifurcation is obtained via employing the polynomial theorem by analyzing the distribution of the roots of the associated characteristic equation. Secondly, the direction of the Hopf bifurcation and the stability of the bifurcated periodic solutions are determined by applying the normal form method and the center manifold theorem. Finally, some numerical simulations are carried out to illustrate the obtained results.

1. Introduction

Multispecies predator-prey models arise frequently on various ecosystems [1–4]. They usually exhibit more rich and complex dynamics as the number of species increases. Take three-species models, for example. There might be a food chain, or two predators or preys, whose relations might be cooperative or competitive. It has attracted extensive studies on the dynamics of various multispecies predator-prey models. Cheng et al. [5] derived some results to ensure the global stability of a predator-prey system with Holling's type III functional response. klebanoff and Hastings [6] investigated a three-species food chain model. It showed that the system exhibits rich complexity features such as stable, periodic, and chaotic dynamics. Hsu and Huang [7] deals with the question of global stability of the positive locally asymptotically stable equilibrium in a class of predator-prey systems by using the Dulac's criterion and constructing Lyapunov functions. See [8–14].

In 1977, Freedman and Waltman [12] investigated the following general Gause-type predator-prey model:

$$\begin{aligned}\frac{dx(t)}{dt} &= xg(x) - yp(x),\\ \frac{dy(t)}{dt} &= y\left[-h + ep(x)\right] - zq(y),\\ \frac{dz(t)}{dt} &= z\left[-s + mq(y)\right],\end{aligned} \tag{1.1}$$

where $x(t)$, $y(t)$, and $z(t)$ are the population densities of prey, predator, and top predator at time t, respectively. $g(x)$ is the intrinsic growth rate of prey; $p(x)$ and $q(y)$ are the specific growth rates of predator and top predator; $h, s > 0$ are the death rates of $y(t)$ and $z(t)$; $e, m > 0$ are the conversion rates for prey and predator. They established the stability criteria and argued that the unique interior equilibrium exists and is locally asymptotically stable. Ginoux et al. [13] highlighted that this model has several Hopf bifurcations and a period-doubling cascade generating a snail shell-shaped chaotic attractor. They compared the quantity and property of the equilibria, bifurcation structure, and shape of attractors among this model and the so-called Rosenzweig-MacArthur and Hastings-Powell models and also gave the bifurcation analysis for each model. Hastings and Powell [14] discussed a continuous time model of a food chain incorporating nonlinear functional responses, and the model exhibits chaotic in long-term behavior when appropriate biologically reasonable parameter values are chosen. They found that, for different values of the key parameter, the system exhibits several types of asymptotic motions, namely, stable equilibrium point, limit cycles, change in periodicity of these cycles, and the so called teacup chaos.

We know that the population outbreak may happen for the species with periodic fluctuation. The outbreaks of pests and mice are famous. For example, Finerty discovered in Canada the populations of polar rabbits, lynxs and Ondatrazibethicas have 10-year cycle fluctuations, and the populations of lemmings and some murine experience 4-year cycle fluctuations. When the peak arrived, the vegetation would be severely damaged. Some cyclical fluctuations are consistent with the periodicity of intensity of certain infectious diseases. Thus, it is of great significance to study periodic solutions of biological systems for controlling insect. It is known that the delay differential equations (DDEs) will exhibit much more complicated dynamics than ordinary differential equations (ODEs) such as the existence of Bogdanov-Takens bifurcation and even chaos. See [15–17]. Many authors have researched the different delays Gause-type predator-prey systems [18–22].

For two-dimensional Gause-type predator-prey model,

$$\begin{aligned}\frac{dx(t)}{dt} &= x(t)g(x(t)) - y(t)p(x(t)),\\ \frac{dy(t)}{dt} &= y(t)\left[-d + \mu p(x(t))\right],\end{aligned} \tag{1.2}$$

the time delay can be incorporated into (1.2) at three different terms [4], including the prey specific growth term $g(x(t))$, the predator response term $p(x(t))$, or the interaction term $y(t)p(x(t))$. The detailed analysis on stability and bifurcation can be found in [23]. Because the prey usually has a gestation or reaction time of predator, introducing time delay is necessary and hence changes the model into a DDE. It is well known that delay could cause a stable equilibrium to be unstable and induce bifurcations as well as periodic oscillations. Under the hypothesis that prey $x(t)$ has a gestation in (1.1), we modify it to be the following one:

$$\begin{aligned}\frac{dx(t)}{dt} &= xg(x) - yp(x-\tau),\\ \frac{dy(t)}{dt} &= y[-h+ep(x)] - zq(y),\\ \frac{dz(t)}{dt} &= z[-s+mq(y)],\end{aligned} \tag{1.3}$$

where τ is the time of gestation.

The purpose of current work is to analyze the effect of delay on the dynamics for (1.3), so we plan to employ bifurcation analysis approach with delay τ being the parameter. In particular, we choose $g(x) = \alpha(1-(x/K))$, $p(x) = \beta x/(1+px)$, and $q(y) = ry$. So the following delay model will be obtained:

$$\begin{aligned}\frac{dx(t)}{dt} &= \alpha x\left(1-\frac{x}{k}\right) - \frac{\beta y x(t-\tau)}{1+px(t-\tau)},\\ \frac{dy(t)}{dt} &= y\left[-h+\frac{e\beta x}{1+px}\right] - rzy,\\ \frac{dz(t)}{dt} &= z[-s+mry],\end{aligned} \tag{1.4}$$

where $\alpha, \beta, k, p, h, e, r, s, m$ are all positive parameters.

Our results reveal that Hopf bifurcation can occur as the delay crosses some critical values which leads to the existence of periodic solution that may conform to certain phenomena in ecosystem system.

The rest of the paper is organized as follows. In Section 2, we first investigate the stability of coexisting equilibrium and the existence of the Hopf bifurcation of (1.4) by analyzing the characteristic equation of the associated linearized system. In Section 3, we derive an explicit formula for determining the stability and the direction of bifurcating periodic solutions by the normal form method and the center manifold theory. In Section 4, we carry out some numerical simulations to illustrate the results obtained and forecast the change of population quantity.

2. Stability and Hopf Bifurcation of Coexisting Equilibrium

For the sake of convenience, we nondimensionalized (1.4) with the following: scaling

$$x \longrightarrow \frac{x}{k}, \qquad t \longrightarrow \alpha t, \qquad y \longrightarrow y, \qquad z \longrightarrow z, \tag{2.1}$$

then (1.4) takes the form

$$\begin{aligned} \frac{dx(t)}{dt} &= x(1-x) - ay\frac{x(t-\tau)}{x(t-\tau)+b}, \\ \frac{dy(t)}{dt} &= y\left(-l + \frac{cx}{x+b} - rz\right), \\ \frac{dz(t)}{dt} &= z(-s+dy), \end{aligned} \tag{2.2}$$

where

$$a = \frac{\beta}{pk\alpha}, \qquad b = \frac{1}{pk}, \qquad r = \frac{h}{\alpha}, \qquad c = \frac{e\beta}{p\alpha}, \qquad d = mr. \tag{2.3}$$

Obviously, the delay cannot change the number of equilibria and non-dimensionalizations cannot change the properties of system. Through simple analysis, we know (2.2) has four equilibria: $E_1(0,0,0)$, $E_2(1,0,0)$, $E_3(x_0, y_0, 0)$, and $\overline{E}(x^*, y^*, z^*)$ with

$$\begin{aligned} x_0 &= \frac{bl}{c-l}, \\ y_0 &= \frac{(1-x_0)(x_0+b)}{a}, \\ x^* &= \frac{(1-b) + \sqrt{(1-b)^2 + 4b - (4as/d)}}{2}, \\ y^* &= \frac{s}{d}, \\ z^* &= -\frac{l}{r} + \frac{cx^*}{r(x^*+b)}. \end{aligned} \tag{2.4}$$

There is no obvious biological significance for $E_1(0,0,0)$ and $E_2(1,0,0)$. In this paper, we mainly study the change of stability of coexisting equilibrium $\overline{E}(x^*, y^*, z^*)$ with the variation of time delay. If $as/d \le b < 1$, then $\overline{E}(x^*, y^*, z^*)$ is the uniqueness equilibrium of (2.2). We consider the linearized system of (2.2) at $\overline{E}$. The characteristic equation at $\overline{E}$ is given by

$$\begin{vmatrix} \lambda - m_{11} - n_{11}e^{-\lambda\tau} & -m_{12} & 0 \\ -m_{21} & \lambda & -m_{23} \\ 0 & -m_{32} & \lambda \end{vmatrix} = 0, \tag{2.5}$$

where

$$\begin{aligned} &m_{11} = 1 - 2x^*, \qquad m_{12} = -\frac{ax^*}{x^*+b} < 0, \qquad m_{21} = \frac{bcy^*}{(x^*+b)^2} > 0, \\ &m_{23} = -ry^* < 0, \qquad m_{32} = dz^* > 0, \qquad n_{11} = -\frac{aby^*}{(x^*+b)^2} < 0. \end{aligned} \tag{2.6}$$

The eigenvalue λ satisfies the characteristic equation

$$D(\lambda,\tau) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 + \left(b_2\lambda^2 + b_0\right)e^{-\lambda\tau} = 0, \tag{2.7}$$

where $a_2 = -m_{11}$, $a_1 = -m_{12}m_{21} - m_{32}m_{23} > 0$, $a_0 = m_{32}m_{23}m_{11}$, $b_2 = -n_{11} > 0$, and $b_0 = m_{32}m_{23}n_{11} > 0$.

If $m_{11} < 0$, then all the eigenvalues of (2.7) have negative real parts when $\tau = 0$ by the Routh-Hurwitz criterion. So we have the Lemma 2.1.

Lemma 2.1. *If $as/d \le b < 1$ and $m_{11} < 0$, then the coexisting equilibrium $\overline{E}(x^*, y^*, z^*)$ of* (2.2) *is locally asymptotically stable.*

It is known that $\overline{E}(x^*, y^*, z^*)$ is asymptotically stable if all roots of the corresponding characteristic equation (2.7) have negative real parts. We shall study the distribution of the roots of the (2.7) when $\tau \neq 0$. We assume that the equilibrium $\overline{E}(x^*, y^*, z^*)$ of the ODE model is stable, then we derive some conditions to ensure that the steady state of the delay model is still stable.

We compute the eigenvalues of the Jacobian matrix at the coexisting equilibrium $\overline{E}(x^*, y^*, z^*)$. Substituting $\lambda = i\omega$ into (2.7) yields,

(1) when $\omega = 0$, $D(0,\tau) = a_0 + b_0 = m_{23}m_{32}(n_{11} + m_{11}) \neq 0$,

(2) when $\omega \neq 0$, $D(i\omega,\tau) = (i\omega)^3 + a_2(i\omega)^2 + a_1 i\omega + a_0 + (b_0 - b_2\omega^2)e^{-i\omega\tau} = 0$. Separating the real and imaginary parts, we get

$$\begin{aligned} -a_2\omega^2 + a_0 + \left(b_0 - b_2\omega^2\right)\cos\omega\tau &= 0, \\ -\omega^3 + a_1\omega - \left(b_0 - b_2\omega^2\right)\sin\omega\tau &= 0. \end{aligned} \tag{2.8}$$

Consequently, we get

$$\omega^6 + \left(a_2^2 - 2a_1 - b_2^2\right)\omega^4 + \left(a_1^2 - 2a_0a_2 + 2b_0b_2\right)\omega^2 + a_0^2 - b_0^2 = 0. \tag{2.9}$$

Let $\omega^2 = l$, $A = a_2^2 - 2a_1 - b_2^2$, $B = a_1^2 - 2a_0a_2 + 2b_0b_2$ and $C = a_0^2 - b_0^2$, then (2.9) becomes

$$l^3 + Al^2 + Bl + C = 0. \tag{2.10}$$

From Ruan and Wei [24] or Li and Wei [25], we have the following results on the distribution of roots of (2.10).

Lemma 2.2. *Denote*

$$l_1 = \frac{-A + \sqrt{\Delta}}{3}, \qquad l_2 = \frac{-A - \sqrt{\Delta}}{3}, \qquad \Delta = A^2 - 3B, \qquad h(l) = l^3 + Al^2 + Bl + C. \tag{2.11}$$

(1) *If* $C \geq 0$, *then* (2.10) *has at least one positive root.*

(2) *If* $C \geq 0$ *and* $\Delta < 0$, *then* (2.10) *has no positive root.*

(3) *If* $C \geq 0$, *then* (2.10) *has positive roots* $\Leftrightarrow$ *if* $l_1 > 0$ *and* $h'(l_1) \leq 0$.

Suppose (2.10) has root with positive real part. Without loss of generality, we assume that it has three positive roots, denoted by l_1, l_2, l_3, respectively. Then, (2.10) has three positive roots $\omega_i = \sqrt{l_i}$ $(i = 1, 2, 3)$. Let $\tau_k^{(0)}$ be the unique root of (2.8) such that $\tau_k^{(0)}\omega_k \in [0, 2\pi)$. Also denote by

$$\tau_k^{(j)} = \tau_k^{(0)} + \frac{2j\pi}{\omega_k}, \tag{2.12}$$

for $k = 1, 2, 3$.

So $(\pm\omega_k, \tau_k^{(j)})$ is the solution of (2.7). Clearly,

$$\lim \tau_k^{(j)} = \infty, \quad k = 1, 2, 3. \tag{2.13}$$

We can define

$$\tau_0 = \tau_{k_0} = \min \tau_k^{(0)}, \omega_0 = \omega_{k_0}, \tag{2.14}$$

that is, $\pm i\omega_0$ is the purely imaginary roots of (2.7) for $\tau = \tau_0$. So we have the following.

Lemma 2.3. *τ_0 is defined by* (2.14), *then*

(1) if one of the following holds (i) $C < 0$; (ii) $C \geq 0$, $\Delta \geq 0$; *(iii)* $C \geq 0, l_1 > 0$; $h'(l_1) > 0$, *then all roots of* (2.7) *have negative real parts for* $\tau \in [0, \tau_0)$;

(2) if none of the conditions (i)–(iii) *are satisfied, then all toots of* (2.7) *have negative real parts for all* $\tau \geq 0$.

So we have the following theorem.

Theorem 2.4. *Supposed that*

(a) $as/d \leq b < 1$ *and* $m_{11} < 0$,

(b) *either* $C \geq 0, A^2 - 3B < 0$, *or* $C \geq 0, B > 0$,

then the equilibrium $\overline{E}(x^*, y^*, z^*)$ *of the delay model* (2.2) *is absolutely stable; that is,* $\overline{E}(x^*, y^*, z^*)$ *is asymptotically stable for all* $\tau \geq 0$.

Denoting $\lambda(\tau) = \alpha(\tau) + \beta(\tau)$ to be the root of (2.7) satisfying $\alpha(\tau_0) = 0, \omega(\tau_0) = \omega_0$, then we have the following Lemma 2.5.

Lemma 2.5. *Suppose* $A^2 - 3B < 0$,

(a) $\pm i\omega_0$ *is a pair of simple purely imaginary roots of* (2.7) *when* $\tau = \tau_0$,

(b) $[d \operatorname{Re} \lambda(\tau)/d\tau]|_{\tau=\tau_0} > 0$.

Proof. If $i\omega_0$ is not simple, then ω_0 must satisfy

$$\frac{d}{d\tau}\left[\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 + \left(b_2\lambda^2 + b_0\right)e^{-\lambda\tau}\right]|_{\lambda=i\omega_0} = 0, \tag{2.15}$$

that is, ω_0 and τ_0 must satisfy

$$\begin{aligned} \tau\left(b_2\omega^2 - b_0\right)\cos\omega\tau + 2b_2\omega\sin\omega\tau = 3\omega^2 - a_1, \\ \tau\left(b_2\omega^2 - b_0\right)\sin\omega\tau - 2b_2\omega\cos\omega\tau = 2a_2\omega. \end{aligned} \tag{2.16}$$

Eliminating τ, we have

$$\left(a_1 - 3\omega^2\right)\sin\omega\tau + 2a_2\omega\cos\omega\tau = -2b_2\omega, \tag{2.17}$$

substituting $\sin\omega\tau$, $\cos\omega\tau$ from (2.8) into (2.17). We have

$$3\omega^4 + 2\left(a_2^2 - 2a_1 - b_2^2\right)\omega^2 + \left(a_1^2 - 2a_2a_0 + 2b_2b_0\right) = 0. \tag{2.18}$$

Recall $\omega_0^2 = l_0$, which implies

$$3l_0^2 + 2Al_0 + B = 0. \tag{2.19}$$

However if $A^2 - 3B < 0$, we have

$$h'(l_0) = 3l_0^2 + 2Al_0 + B > 0, \tag{2.20}$$

a contradiction. So $\pm i\omega_0$ is a pair of simple purely imaginary roots of (2.7).

Note that

$$\begin{aligned} M &= \left(a_1 - 3\omega_0^2\right)\cos\omega_0\tau_0 - 2a_2\omega_0\sin\omega_0\tau_0 + \left(b_2\omega_0^2 - b_0\right)\tau_0, \\ N &= \left(a_1 - 3\omega_0^2\right)\sin\omega_0\tau_0 + 2a_2\omega_0\cos\omega_0\tau_0 + 2b_2\omega_0, \\ Q &= b_0\omega_0 - b_2\omega_0^3. \end{aligned} \tag{2.21}$$

So

$$\frac{d\lambda(\tau)}{d\tau} = \frac{iQ}{M+iN} = \frac{NQ+iMQ}{M^2+N^2}. \tag{2.22}$$

Then,

$$\frac{dRe\lambda(\tau_0)}{d\tau} = \frac{NQ}{M^2+N^2} \tag{2.23}$$

has the same sign as that of NQ.

Now,

$$\begin{aligned} NQ &= \left[\left(a_1 - 3\omega_0^2\right)\sin\omega_0\tau_0 + 2a_2\omega_0\cos\omega_0\tau_0 + 2b_2\omega_0\right]\left(b_0\omega - b_2\omega^3\right) \\ &= \omega_0^2\left[3\omega_0^4 + 2\left(a_2^2 - 2a_1 - b_2^2\right)\omega_0^2 + \left(a_1^2 - 2a_2a_0 + 2b_2b_0\right)\right] \\ &= \omega_0^2\left[3\omega_0^4 + 2A\omega_0^2 + B\right]. \end{aligned} \tag{2.24}$$

When $A^2 - 3B < 0$, NQ is greater than zero, that is $[d\ \mathrm{Re}\,\lambda(\tau)/d\tau]|_{\tau=\tau_0} > 0$. □

By Lemma 2.5, we have the Theorem 2.6.

Theorem 2.6. *Supposed that Lemma 2.3 is right, then, when $A^2 - 3B < 0$, the equilibrium $\overline{E}(x^*, y^*, z^*)$ of the delay model* (2.2) *is asymptotically stable when $\tau < \tau_0$ and unstable when $\tau > \tau_0$, where τ_0 is defined by* (2.14). *When $\tau = \tau_0$, a Hopf bifurcation occurs.*

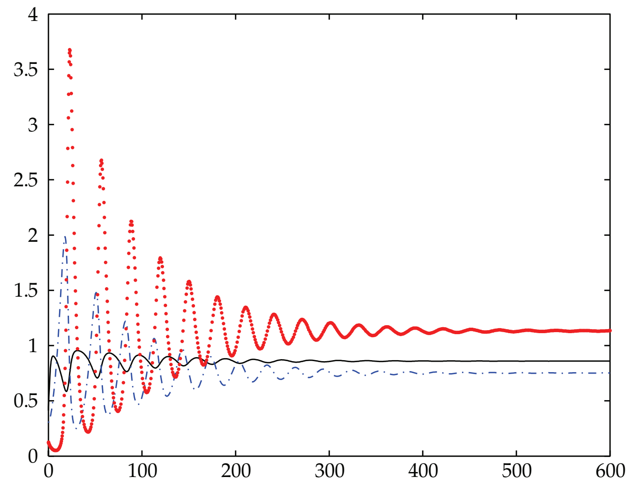
3. Direction and Stability of Hopf Bifurcation

Let $x_1(t) = x(t) - x^*$, $x_2(t) = y(t) - y^*$, $x_3(t) = z(t) - z^*$, $X_i(t) = x_i(\tau t)$ $(i = 1, 2, 3)$, $\tau = \tau_0 + \mu$, $\mu \in \mathbb{R}$, and

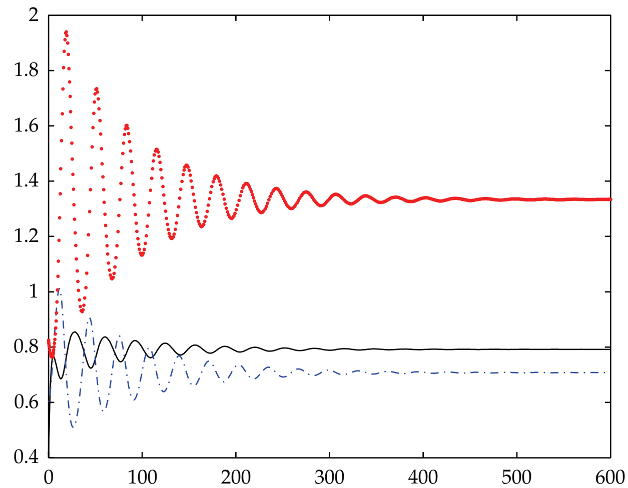
$$B = \begin{pmatrix} m_{11} & m_{12} & 0 \\ m_{21} & 0 & m_{23} \\ 0 & m_{32} & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} n_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.1}$$

The system (1.4) is transformed into a functional differential equation (FDE) in $C = C([-1, 0], \mathbb{R}^3)$, defining

$$L_\mu(\phi) = (\tau_0 + \mu)B\phi(0) + (\tau_0 + \mu)C\phi(-1), \tag{3.2}$$



(a)



(b)

Figure 1: $\overline{E}$ is asymptotically stable for all $\tau > 0$.

where $\phi = (\phi_1, \phi_2, \phi_3)^T \in C([-1,0], \mathbb{R}^3)$, and the nonlinear term is

$$h(\mu, \phi) = (\tau_0 + \mu) \begin{pmatrix} -2\phi_1^2(0) - \dfrac{2ab}{(x^* + b)^2}\phi_2(0)\phi_1(-1) \\ \dfrac{-2bcy^*}{(b + x^*)^3}\phi_1^2(0) + \dfrac{2bc}{(b + x^*)^2}\phi_1(0)\phi_2(0) - 2r\phi_2(0)\phi_3(0) \\ 2dr\phi_2(0)\phi_3(0) \end{pmatrix}. \tag{3.3}$$

So $\mu = 0$ is Hopf bifurcation point.

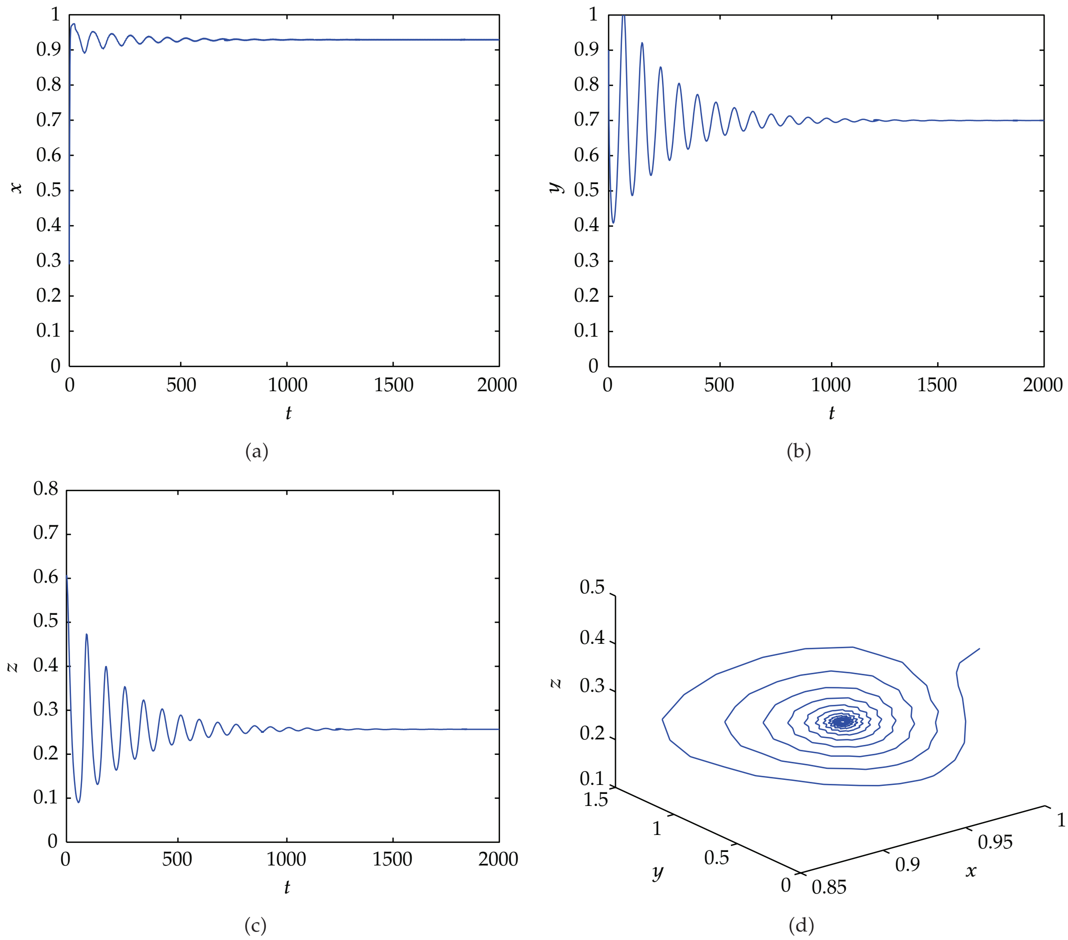


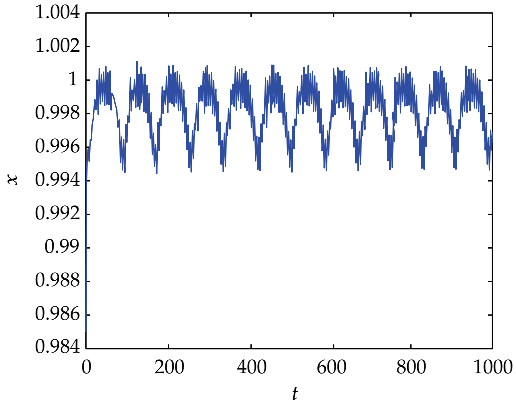
Figure 2: $\overline{E}(0.9977, 0.6954, 0.4432)$ with initial value $(0.985, 0.135, 0.4)$ is asymptotically stable when $\tau = 8.6017 < \tau_0 = 12.6774$.

By the Riesz representation theorem, there exists a 3×3 matrix $\eta(\theta, \mu)$ $(-1 \leq \theta \leq 0)$, whose elements are of bounded variation functions such that

$$L_{\mu}(\phi) = \int_{-1}^{0} [d\eta(\theta, \mu)]\phi(\theta), \quad \text{for } \phi \in C\Big([-1, 0], \ \mathbb{R}^3\Big). \tag{3.4}$$

In fact, we can choose

$$\eta(\theta, \mu) = \begin{cases} (\tau_0 + \mu)B, & \theta = 0, \\ 0, & \theta \in (-1, 0), \\ (\tau_0 + \mu)C, & \theta = -1. \end{cases} \tag{3.5}$$



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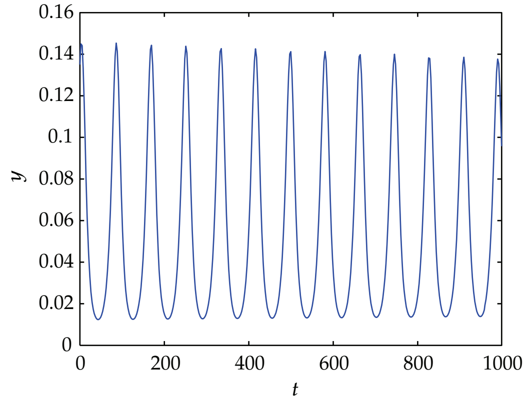

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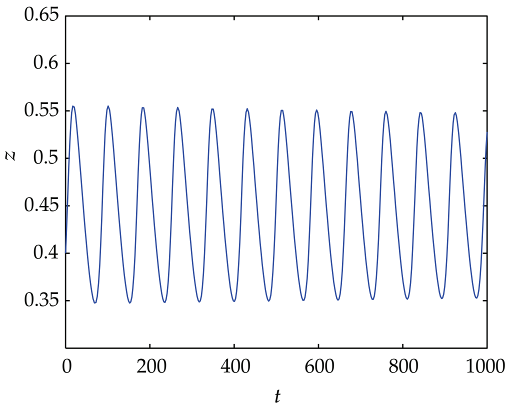
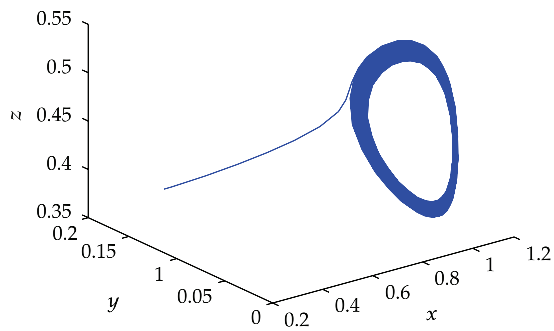


(c)



(d)

Figure 3: A stable periodic orbit of system (2.2) when initial value is (0.985, 0.135, 0.4) and $\tau = 20.7288 > \tau_0 = 12.6774$.

Then, (3.4) is satisfied. For $\phi \in C^1([-1,0], \mathbb{R}^3)$, we define

$$A(\mu)\phi(\theta) = \begin{cases} \dfrac{d\phi(\theta)}{d\theta}, & \theta \in [-1,0), \\ \displaystyle\int_{-1}^{0} [d\eta(\xi,\mu)]\phi(\xi), & \theta = 0, \end{cases}$$
$$R(\mu)\phi(\theta) = \begin{cases} 0, & \theta \in [-1,0), \\ h(\mu,\phi), & \theta = 0. \end{cases} \tag{3.6}$$

So (3.4) is equivalent to the following abstract equation:

$$\dot{x}_t = A(\mu)x_t + R(\mu)x_t, \tag{3.7}$$

where $x = (x_1, x_2, x_3)^T$, $x_t = x(t+\theta)$, for $\theta \in [-1,0]$.

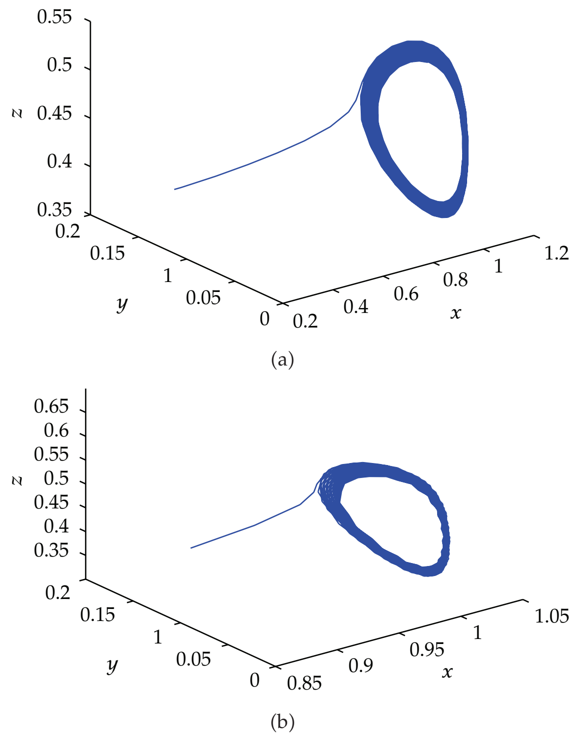


Figure 4: The stable periodic orbits of system (2.2) when $\tau = 200.7288$ and $\tau = 400.8768$ with parameters given by (3).

For $\psi \in C^1([0,1], \mathbb{R}^{3^*})$, we define

$$A^*\psi(s) = \begin{cases} -\dfrac{d\psi(s)}{ds}, & s \in (0,1], \\ \displaystyle\int_{-1}^{0} \psi(-\xi)\, d\eta(\xi, 0), & s = 0, \end{cases} \tag{3.8}$$

and a bilinear form

$$\langle \psi(s), \phi(\theta) \rangle = \overline{\psi}(0)\phi(0) - \int_{-1}^{0} \int_{\xi=0}^{\theta} \overline{\psi}(\xi - \theta)\, d\eta(\theta)\phi(\xi)\, d\xi, \tag{3.9}$$

where $\eta(\theta) = \eta(\theta, 0)$. Then, $A(0)$ and A^* are adjoint operators. We know that $\pm i\omega_0\tau_0$ are eigenvalues of $A(0)$, and, therefore, they are also eigenvalues of $A^*(0)$. The vector $q(\theta) = (q_1, 1, q_3)^T e^{i\omega_0\tau_0\theta}(\theta \in [-1, 0])$ and $q^*(s) = D(q_1^*, 1, q_3^*)e^{i\omega_0\tau_0 s}(s \in [0,1])$ are the eigenvectors of $A(0)$ and A^* corresponding to the eigenvalue $i\omega_0\tau_0$ and $-i\omega_0\tau_0$, respectively, satisfying $\langle q^*(s), q(\theta)\rangle = 1$, $\langle q^*(s), \overline{q}(\theta)\rangle = 0$ with $q_1 = m_{12}/(i\omega_0 - n_{11}e^{-i\omega_0} - m_{11})$, $q_3 = m_{32}/i\omega_0$, $q_1^* = m_{21}/(-i\omega_0 - n_{11}e^{i\omega_0} - m_{11})$, $q_3^* = m_{23}/i\omega_0$, and $D = 1/(q_1^*\overline{q_1} + 1 + q_3^*\overline{q_3} + \overline{q_1}q_1^* n_{11}e^{-i\omega_0\tau_0})$. Following the same algorithms as Hassard et al. [26], we can obtain the coefficients which will be used to determine the important quantities:

$$
\begin{aligned}
g_{02} &= 2\overline{D}\tau_0\left[-\overline{q_1}^2\overline{q_1^*} - \frac{ab}{(b+x^*)^2}\overline{q_1}\overline{q_1^*}e^{i\omega_0\tau_0} - \frac{bcy^*}{(b+x^*)^2}\overline{q_1} - r\overline{q_3} + d\overline{q_3}\overline{q_3^*}\right],\\
g_{20} &= 2\overline{D}\tau_0\left[-q_1^2\overline{q_1^*} - \frac{abq_1\overline{q_1^*}}{(b+x^*)^2}e^{-i\omega_0\tau_0} - \frac{bcy^*}{(b+x^*)^3}q_1^2 + \frac{bc}{(b+x^*)^2}q_1 - rq_3 + dq_3\overline{q_3^*}\right],\\
g_{11} &= 2\overline{D}\tau_0\left[-2q_1\overline{q_1}\overline{q_1^*} - \frac{ab}{(b+x^*)^2}\left(2\operatorname{Re}\left\{q_1e^{-i\omega_0\tau_0}\right\}\right) - \frac{2bcy^*}{(b+x^*)^3}q_1\overline{q_1} + \frac{2bc}{(b+x^*)^2}\operatorname{Re}\{q_1\}\right.\\
&\qquad\left. -2r\operatorname{Re}\{q_3\} + 2d\operatorname{Re}\left\{q_3\overline{q_3^*}\right\}\right],\\
g_{21} &= 2\overline{D}\tau_0\left[\left(-4q_1\overline{q_1^*} - \frac{4bc}{(b+x^*)^3}q_1 + \frac{2bc}{(b+x^*)^2}\right)W_{11}^1(0) + \left(\frac{2bc}{(b+x^*)^2}q_1 - \frac{2ab}{(b+x^*)^2}q_1\overline{q_1^*}e^{-i\omega_0\tau_0}\right.\right.\\
&\qquad\left. +2dq_3\overline{q_3^*} - 2rq_3\right)W_{11}^2(0) + \left(2d\overline{q_3^*} - 2r\right)W_{11}^3(0) - \frac{2ab\overline{q_1^*}}{(b+x^*)^2}W_{11}^1(-1)\\
&\qquad + \left(-4\overline{q_1^*}\overline{q_1} - \frac{2bc}{(b+x^*)^3}\overline{q_1} + \frac{bc}{(b+x^*)^2}\right)W_{20}^1(0) + \left(-\frac{ab\overline{q_1^*}}{(b+x^*)^2}\overline{q_1}e^{i\omega_0\tau_0} + \frac{bc}{(b+x^*)^2}\overline{q_1}\right.\\
&\qquad\left.\left. +d\overline{q_3^*}\overline{q_3} - r\overline{q_3}\right)W_{20}^2(0) + \left(d\overline{q_3^*} - r\right)W_{20}^3(0) - \frac{ab\overline{q_1^*}}{(b+x^*)^2}W_{20}^1(-1)\right].
\end{aligned}
\tag{3.10}
$$

Since there are $W_{20}(\theta)$ and $W_{11}(\theta)$ in g_{21}, we still need to compute them. From [25], we have

$$
W_{20}(\theta) = \frac{ig_{20}}{\omega_0\tau_0}q(\theta) + \frac{i\overline{g}_{02}}{3\omega_0\tau_0}\overline{q}(\theta) + E_1e^{2i\omega_0\tau_0\theta}. \tag{3.11}
$$

According to

$$
\left[2i\omega_0\tau_0 I - \int_{-1}^{0} d\eta(\theta)e^{2i\omega_0\theta}\right]E_1 = h_{z^2}, \tag{3.12}
$$

where

$$
h_{z^2} = \begin{pmatrix} \left(-2 + \dfrac{2aby^*}{(b+x^*)^3}\right)q_1^2 - \dfrac{2ab}{(b+x^*)^2}q_1 \\ -2rq_3 + \dfrac{bc}{(b+x^*)^2}q_1e^{-i\omega_0\tau_0} - \dfrac{4bcy^*}{(b+x^*)^3}q_1^2e^{-2i\omega_0\tau_0} \\ 2dq_3e^{-i\omega_0\tau_0} \end{pmatrix}, \tag{3.13}
$$

we have $E_1 = (E_1^{(1)}, E_1^{(2)},\ E_1^{(3)})^T$, where $E_1^{(i)} = 2\Delta_1^{(i)}/\Delta_1\ (i = 1, 2, 3)$, with

$$\begin{aligned}
\Delta_1 &= \left(-4\omega_0^2\tau_0^2 - m_{23}m_{32} - m_{12}m_{21}\right)(2i\omega_0\tau_0) + \left(4\omega_0^2\tau_0^2 + m_{23}m_{32}\right)\left(n_{11}e^{-2i\omega_0\tau_0} + m_{11}\right),\\
\Delta_1^{(1)} &= \left(4\omega_0^2\tau_0^2 + m_{23}m_{32}\right)\left(2q_1^2 + \frac{2ab}{(b+x^*)^2}q_1e^{-i\omega_0\tau_0}\right)\\
&\quad + 2m_{12}i\omega_0\tau_0\left[-\frac{2bcy^*}{(b+x^*)^3}q_1^2 + \frac{2bc}{(b+x^*)^2}q_1 - 2rq_3\right] + 2m_{12}m_{23}dq_3,\\
\Delta_1^{(2)} &= 2m_{21}i\omega_0\tau_0\left[-2q_1^2 - \frac{2ab}{(b+x^*)^2}q_1e^{-i\omega_0\tau_0}\right]\\
&\quad + 2i\omega_0\tau_0\left(2i\omega_0\tau_0 - n_{11}e^{-i\omega_0\tau_0} - m_{11}\right)\left[-\frac{2bcy^*}{(b+x^*)^3}q_1^2 + \frac{2bc}{(b+x^*)^2}q_1 - 2rq_3\right]\\
&\quad + 2m_{23}\left(2i\omega_0\tau_0 - n_{11}e^{-2i\omega_0\tau_0} - m_{11}\right)dq_3,\\
\Delta_1^{(3)} &= m_{21}m_{32}\left[-2q_1^2 - \frac{2ab}{(b+x^*)^2}q_1e^{-i\omega_0\tau_0}\right]\\
&\quad + m_{32}\left(2i\omega_0\tau_0 - n_{11}e^{-i\omega_0\tau_0} - m_{11}\right)\left[-\frac{2bcy^*}{(b+x^*)^3}q_1^2 + \frac{2bc}{(b+x^*)^2}q_1 - 2rq_3\right]\\
&\quad + 2\left[2i\omega_0\tau_0\left(2i\omega_0\tau_0 - n_{11}e^{-i\omega_0\tau_0} - m_{11}\right) - m_{12}m_{21}\right]dq_3.
\end{aligned} \tag{3.14}$$

And, similarly,

$$W_{11}(\theta) = -\frac{ig_{11}}{\omega_0\tau_0}q(\theta) + \frac{i\overline{g}_{11}}{\omega_0\tau_0}\overline{q}(\theta) + E_2. \tag{3.15}$$

According to

$$\left(\int_{-1}^{0} d\eta(\theta)\right)E_2 = -h_{z\overline{z}}, \tag{3.16}$$

where

$$h_{z\overline{z}} = \begin{pmatrix} \left(-2 + \dfrac{2aby^*}{(b+x^*)^3}\right)q_1\overline{q_1} - \dfrac{2ab}{(b+x^*)^2}\left(q_1 + \overline{q_1}\right) \\ -r\left(q_3 + \overline{q_3}\right) + \dfrac{bc}{(b+x^*)^2}\left(\overline{q_1}e^{i\omega_0\tau_0} + q_1e^{-i\omega_0\tau_0}\right) - \dfrac{4bcy^*}{(b+x^*)^3}q_1\overline{q_1} \\ d\left(q_3\overline{q_3^*}e^{i\omega_0\tau_0} + q_3\overline{q_3^*}e^{-i\omega_0\tau_0}\right) \end{pmatrix}, \tag{3.17}$$

we have $E_2 = (E_2^{(1)}, E_2^{(2)}, E_2^{(3)})^T$, where $E_2^{(i)} = 2\Delta_2^{(i)}/\Delta_2\ (i = 1,2,3)$, with

$$\begin{aligned}
\Delta_2 &= m_{23}m_{32}(m_{11}+n_{11}),\\
\Delta_2^{(1)} &= -m_{23}m_{32}\left(-4q_1\overline{q_1} - \frac{2ab}{(b+x^*)^2}\left(2\mathrm{Re}\left\{q_1 e^{-i\omega_0\tau_0}\right\}\right)\right) + 2m_{12}m_{23}d\left(2\mathrm{Re}\{q_3\}\right),\\
\Delta_2^{(2)} &= -2m_{23}(m_{11}+n_{11})d\left(2\mathrm{Re}\{q_3\}\right),\\
\Delta_2^{(3)} &= m_{21}m_{32}\left(-4q_1\overline{q_1} - \frac{2ab}{(b+x^*)^2}\left(2\mathrm{Re}\left\{q_1 e^{-i\omega_0\tau_0}\right\}\right)\right)\\
&\quad - m_{32}(m_{11}+n_{11})\left[\frac{-4bcy^*}{(b+x^*)^3}q_1\overline{q_1} + \frac{2bc}{(b+x^*)^2}\ \left(2\mathrm{Re}\{q_1\}\right)\right] - 2m_{12}m_{21}d\left(2\mathrm{Re}\{q_3\}\right).
\end{aligned} \tag{3.18}$$

Consequently, g_{ij} can be expressed explicitly by the parameters and delay. Thus, we can compute the following values:

$$\begin{gathered}
c_1(0) = \frac{i}{2\omega_0\tau_0}\left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2},\\
\mu_2 = -\frac{Re(c_1(0))}{Re(\lambda'(\tau_0))},\\
T_2 = -\frac{Imc_1(0) + \mu_2 Im\lambda'(\tau_0)}{\omega_0\tau_0},\\
\beta_2 = 2Re(c_1(0)),
\end{gathered} \tag{3.19}$$

which determine the properties of bifurcating periodic solutions at the critical value τ_0. That is, μ_2 determines the direction of Hopf bifurcation: if $\mu_2 > 0$ ($\mu_2 < 0$), then Hopf bifurcation at τ_0 is forward (or backward); β_2 determines the stability of bifurcating periodic solutions: the bifurcating periodic solutions on the center manifold are stable (unstable) if $\beta_2 < 0$ ($\beta_2 > 0$); T_2 determines the period of the bifurcating periodic solutions: the period increases (decreases) if $T_2 > 0$ ($T_2 < 0$).

4. Numerical Simulations and Discussions

In this part, we perform some numerical simulations. The results not only support our previous parts but also predict the existence of global Hopf bifurcations periodic solutions. However, to prove this observation theoretically is of great challenge.

We choose the parameters as follows:

(1) $a = 0.321, b = 0.860, c = 0.532, r = 0.12, s = 0.312, d = 0.415, l = 0.13$,

(2) $a = 0.432, b = 0.677, c = 0.321, r = 0.12, s = 0.235, d = 0.332, l = 0.013$.

Thus, all the conditions in Theorem 2.4 are satisfied. $\overline{E}(0.855, 0.7519, 1.1272)$ with initial value (0.812, 0.702, 0.925) for (1) and $\overline{E}(0.7915, 0.7078, 1.3339)$ with initial value (0.607, 0.602, 0.895) for (2) are asymptotically stable for all $\tau > 0$ (see Figure 1). From a biological sense, the prey, predator, and top predator will have a short-term fluctuation in the initial stage as the effect of τ. But the population would tend to a steady level after a long period of time.

We choose another set of parameters which satisfy the assumptions in Theorem 2.6:

(3) $\tau = 8.6017$, $a = 0.036$, $s = 0.0160$, $d = 0.296$, $b = 0.1347$, $r = 0.9200$, $l = 0.1500$, $c = 0.6330$.

Through (2.14), we have $\tau_0 = 12.6774$, $\omega_0 = 14.7193$, and $c_1(0) = -0.4617 + i0.6024$. Utilizing Theorem 2.6, we know that the equilibrium of the delay model (2.2) is asymptotically stable when $\tau < \tau_0$ (see Figure 2).

Hopf bifurcation occurs when $\tau = \tau_0$, and the bifurcating periodic solution is orbitally asymptotically for $\tau > \tau_0$ (see Figure 3).

In addition, the periodic solution of system (2.2) still exists when τ is large and its amplitude is larger compared with the solution in Figure 4. The numerical results of Figure 4 show the global existence of periodic solutions generated by the Hopf bifurcation. How to explain the phenomenon theoretically needs further researches.

5. Conclusion

In this paper, we analyze the dynamics of the equilibria coexistence for a class of three-dimensional Gause-type predator-prey model. We obtain the stability of this equilibrium and also claim that the introduced delay changes its stability while a Hopf bifurcation occurs. The existence of the bifurcation periodic solutions for sufficiently large delay has been shown by numerical simulations.

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