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Research Article

Qualitative Behaviors of Functional Differential Equations of Third Order with Multiple Deviating Arguments

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This paper considers nonautonomous functional differential equations of the third order with multiple constant deviating arguments. Using the Lyapunov-Krasovskii functional approach, we find certain sufficient conditions for the solutions to be stable and bounded. We give an example to illustrate the theoretical analysis made in this work and to show the effectiveness of the method utilized here.

1. Introduction

In this paper, we consider nonautonomous differential equation of the third order with constant multiple deviating arguments, $\tau, \tau_i, (i = 1, 2, \dots, n)$ as follows:

$$\begin{aligned} &x'''(t) + a(t)x''(t) + \sum_{i=1}^{n} b_i(t)g_i\big(x'(t-\tau_i)\big) + g\big(x'(t)\big) + h(x(t-\tau)) \\ &\quad = p\big(t, x(t), x(t-\tau_1), \dots, x(t-\tau_n), x', \dots, x'(t-\tau_n) \dots, x''(t), \dots, x''(t-\tau_n)\big). \end{aligned} \tag{1.1}$$

Writing (1.1) as a system of first order equations, we have

$$x' = y, \qquad y' = z,$$

$$z' = -a(t)z - \sum_{i=0}^{n} b_i(t) g_i(y) - h(x) - g(y) + \sum_{i=1}^{n} b_i(t) \int_{t-\tau_i}^{t} g_i'(y(s)) z(s) ds$$
$$+ \int_{t-\tau}^{t} h'(x(s)) y(s) ds + p(t, x, \ldots, x(t-\tau_n), y, \ldots, y(t-\tau_n), z, \ldots, z(t-\tau_n)), \tag{1.2}$$

where τ and τ_i, $(i = 1, 2, \ldots, n)$, are positive constants, that is, fixed delays; the functions a, b_i, g, g_i, h, and p are continuous for their all respective arguments and the primes in (1.1) denote differentiation with respect to t, $t \in \Re^+ = [0, \infty)$. It is also assumed that the derivatives $a'(t) \equiv (d/dt)a(t)$, $b_i'(t) \equiv (d/dt)b_i(t)$, $g_i'(y) \equiv (d/dy)g_i(y)$, and $h'(x) \equiv (d/dx)h(x)$ exist and are continuous; throughout the paper $x(t)$, $y(t)$, and $z(t)$ are abbreviated as x, y, and z, respectively. Finally, the existence and uniqueness of solutions of (1.1) are assumed and all solutions considered are supposed to be real valued.

To the best of our knowledge from the literature, in the last five decades, there has been much attention paid to the discussion of stability and boundedness of solutions of nonlinear differential equations of the third order without a deviating argument. For a comprehensive treatment of the subject, we refer the readers to the book of Reissig et al. [1] as a survey and the papers of Ademola et al. [2], Afuwape et al. [3], Ezeilo [4–13], Ezeilo and Tejumola [14, 15], Mehri and Shadman [16], Ogundare [17], Ogundare and Okecha [18], Omeike [19, 20], Ponzo [21, 22], Swick [23–25], Tejumola [26, 27], Tunç [28–34], Tunç and Ateş [35], Tunç and Ayhan [36], and the references cited in these papers for some works on the topic.

Besides, first, in 1973, Sinha [37] studied the stability of solutions of a third order nonlinear differential equation with a deviating argument. Later, some authors dealt with the stability and boundedness of solutions for various third order nonlinear differential equations with a deviating argument. For some related works, one can refer to the papers of Afuwape and Omeike [38], Omeike [39], Sadek [40, 41], Tejumola and Tchegnani [42], Tunç [43–59], Yao and Meng [60], Zhu [61], and the references thereof.

It should be noted that throughout all the above mentioned papers, Lyapunov's functions or the Lyapunov-Krasovskii functionals have been used as a basic tool to prove the results established there. It is also worth mentioning that the most effective method to study the stability and boundedness of solutions of nonlinear differential equations of higher orders without or with a deviating argument in the literature is still the Lyapunov's direct method, despite its use for a past long period by now.

The motivation for this paper comes from the above mentioned papers and the books. Our results improve and include the results existing in the literature. This work makes also a contribution to the existing studies made in the literature.

2. Main Results

Let $p(\cdot) = 0$.

Our first result is given by the following theorem.

Theorem 2.1. *In addition to the basic assumptions imposed to the functions $a(t)$, $b(t)$, g, g_i, and h appearing in* (1.1), *we assume that there exist positive constants a, α, β, δ, b, b_i, B_i, c, c_1, and L_i such that the following conditions hold:*

(i) $g_i(0) = g(0) = h(0) = 0, a(t) \geq 2\alpha + a$, $\beta_i \leq b_i(t) \leq B_i$,

$$0 < c_1 \leq h'(x) \leq c, \qquad \alpha b - c > \delta, \qquad \frac{g_i(y)}{y} \geq b_i, \quad \frac{g(y)}{y} \geq b, \quad (y \neq 0), \\ \left|g_i'(y)\right| \leq L_i, \tag{2.1}$$

(ii) $\left[\sum_{i=1}^{n}\{\alpha b_i b_i(t)\} - c\right]y^2 \geq 2^{-1}\alpha a'(t)y^2 + \sum_{i=1}^{n} b_i'(t)\int_0^y g_i(\eta)d\eta$.

If

$$\tau_0 < \min\left\{\frac{\alpha b}{c + 2\alpha c + \alpha\sum_{i=1}^{n}(B_i L_i)}, \frac{2\alpha}{c + \sum_{i=1}^{n}[(2+\alpha)B_i L_i]}\right\}, \tag{2.2}$$

then the zero solution of (1.1) *is stable.*

Proof. Define the Lyapunov-Krasovskii functional $V(\cdot) = V(t, x_t, y_t, z_t)$ as

$$2V(\cdot) = z^2 + 2\alpha yz + 2\sum_{i=0}^{n} b_i(t)\int_0^y g_i(\eta)d\eta + 2\alpha\int_0^x h(\xi)d\xi + 2\int_0^y g(\eta)d\eta \\ + \alpha a(t)y^2 + 2h(x)y + \lambda\int_{-\tau}^{0}\int_{t+s}^{t} y^2(\theta)d\theta\, ds + \sum_{i=1}^{n}\lambda_i\int_{-\tau_i}^{0}\int_{t+s}^{t} z^2(\theta)d\theta\, ds, \tag{2.3}$$

where λ and λ_i are some positive constants to be chosen later.

Using the assumptions $g_i(y)/y \geq b_i$, $g(y)/y \geq b$, $(y \neq 0)$, $h(0) = 0$, and $0 < c_1 \leq h'(x) \leq c$, we have

$$2b_i(t)\int_0^y g_i(\eta)d\eta = 2b_i(t)\int_0^y \frac{g_i(\eta)}{\eta}\eta\, d\eta \geq (\beta_i b_i)y^2, \\ 2\int_0^y g(\eta)d\eta = 2\int_0^y \frac{g(\eta)}{\eta}\eta\, d\eta \geq by^2, \\ h^2(x) = 2\int_0^x h(\xi)h'(\xi)d\xi \leq 2c\int_0^x h(\xi)d\xi, \tag{2.4}$$

so that

$$
\begin{aligned}
2V(\cdot) &\geq (z+\alpha y)^2 + b\left[y + b^{-1}h(x)\right]^2 + 2\alpha\int_0^x h(\xi)d\xi - \frac{1}{b}h^2(x) + \sum_{i=1}^{n}(\beta_i b_i)y^2 \\
&\quad + \alpha(\alpha+a)y^2 + \lambda\int_{-\tau}^{0}\int_{t+s}^{t} y^2(\theta)d\theta\, ds + \sum_{i=1}^{n}\lambda_i\int_{-\tau_i}^{0}\int_{t+s}^{t} z^2(\theta)d\theta\, ds \\
&\geq (z+\alpha y)^2 + b\left[y + b^{-1}h(x)\right]^2 + 2\alpha\int_0^x h(\xi)d\xi - \frac{2c}{b}\int_0^x h(\xi)d\xi \\
&\quad + \sum_{i=1}^{n}(\beta_i b_i)y^2 + \alpha(\alpha+a)y^2 + \lambda\int_{-\tau}^{0}\int_{t+s}^{t} y^2(\theta)d\theta\, ds + \sum_{i=1}^{n}\lambda_i\int_{-\tau_i}^{0}\int_{t+s}^{t} z^2(\theta)d\theta\, ds.
\end{aligned}
\tag{2.5}
$$

On the other hand, it is obvious that

$$
\begin{aligned}
2\alpha\int_0^x h(\xi)d\xi - \frac{2c}{b}\int_0^x h(\xi)d\xi &= 2b^{-1}(\alpha b - c)\int_0^x h(\xi)d\xi \\
&\geq c_1 b^{-1}(\alpha b - c)x^2,
\end{aligned}
\tag{2.6}
$$

so that

$$
\begin{aligned}
2V(\cdot) \geq{}& (z+\alpha y)^2 + b\left[y + b^{-1}h(x)\right]^2 + c_1 b^{-1}(\alpha b - c)x^2 + \alpha(\alpha+a)y^2 \\
&+ \sum_{i=1}^{n}(\beta_i b_i)y^2 + \lambda\int_{-\tau}^{0}\int_{t+s}^{t} y^2(\theta)d\theta\, ds + \sum_{i=1}^{n}\lambda_i\int_{-\tau_i}^{0}\int_{t+s}^{t} z^2(\theta)d\theta\, ds.
\end{aligned}
\tag{2.7}
$$

Hence, we can obtain some positive constants D_j, $(j = 1,2,3)$, such that

$$
V(\cdot) \geq D_1x^2 + D_2y^2 + D_3z^2 \geq D_4\left(x^2+y^2+z^2\right),
\tag{2.8}
$$

where $D_4 = \min\{D_1, D_2, D_3\}$, since the integrals $\lambda\int_{-\tau}^{0}\int_{t+s}^{t} y^2(\theta)d\theta\, ds$ and $\sum_{i=1}^{n}\lambda_i\int_{-\tau_i}^{0}\int_{t+s}^{t} z^2(\theta)d\theta\, ds$ are nonnegative.

Let (x, y, z) be a solution of (1.2). Differentiating the Lyapunov-Krasovskii functional $V(\cdot)$ along this solution, we find

$$\begin{aligned}
\frac{d}{dt}V(\cdot) = &-\left[\alpha\sum_{i=1}^{n}\{b_i(t)g_i(y)\}y^{-1} + \alpha g(y)y^{-1} - h'(x) - 2^{-1}\alpha a'(t)\right]y^2 \\
&+ \sum_{i=1}^{n} b_i'(t)\int_0^y g_i(\eta)d\eta - (a(t)-\alpha)z^2 + z\sum_{i=1}^{n} b_i(t)\int_{t-\tau_i}^{t} g_i'(y(s))z(s)ds \\
&+ z\int_{t-\tau}^{t} h'(x(s))y(s)ds \\
&+ \alpha y\sum_{i=1}^{n} b_i(t)\int_{t-\tau_i}^{t} g_i'(y(s))z(s)ds + \alpha y\int_{t-\tau}^{t} h'(x(s))y(s)ds \\
&+ \lambda\tau y^2 - \lambda\int_{t-\tau}^{t} y^2(s)ds + \sum_{i=1}^{n}(\lambda_i\tau_i)z^2 - \sum_{i=1}^{n}\lambda_i\int_{t-\tau_i}^{t} z^2(s)ds.
\end{aligned} \tag{2.9}$$

Using the assumptions of Theorem 2.1 and the estimate $2|mn| \le m^2 + n^2$, we get

$$\begin{gathered}
\begin{aligned}
&\left[\alpha\sum_{i=1}^{n}(b_i(t)g_i(y))y^{-1} + \alpha g(y)y^{-1} - h'(x) - 2^{-1}\alpha a'(t)\right]y^2 - \sum_{i=1}^{n} b_i'(t)\int_0^y g_i(\eta)d\eta \\
&\quad\ge \left[\alpha\sum_{i=1}^{n}(b_i b_i(t)) + \alpha b - c - 2^{-1}\alpha a'(t)\right]y^2 - \sum_{i=1}^{n} b_i'(t)\int_0^y g_i(\eta)d\eta \\
&\quad= \left[\sum_{i=1}^{n}\{\alpha b_i b_i(t)\} - c\right]y^2 - 2^{-1}\alpha a'(t)y^2 - \sum_{i=1}^{n} b_i'(t)\int_0^y g_i(\eta)d\eta + (\alpha b)y^2 \\
&\quad\ge (\alpha b)y^2,
\end{aligned} \\
[a(t) - \alpha]z^2 \ge (\alpha + a)z^2, \\
z\sum_{i=1}^{n} b_i(t)\int_{t-\tau_i}^{t} g_i'(y(s))z(s)ds \le \frac{1}{2}\sum_{i=1}^{n}(B_iL_i\tau_i)z^2 + \frac{1}{2}\sum_{i=1}^{n}(B_iL_i)\int_{t-\tau_i}^{t} z^2(s)ds, \\
\alpha y\sum_{i=1}^{n} b_i(t)\int_{t-\tau_i}^{t} g_i'(y(s))z(s)ds \le \frac{1}{2}\sum_{i=1}^{n}(\alpha B_iL_i\tau_i)y^2 + \frac{1}{2}\sum_{i=1}^{n}(\alpha B_iL_i)\int_{t-\tau_i}^{t} z^2(s)ds, \\
z\int_{t-\tau}^{t} h'(x(s))y(s)ds \le \frac{c}{2}\tau z^2 + \frac{c}{2}\int_{t-\tau}^{t} y^2(s)ds, \\
\alpha y\int_{t-\tau}^{t} h'(x(s))y(s)ds \le \frac{\alpha c}{2}\tau y^2 + \frac{\alpha c}{2}\int_{t-\tau}^{t} y^2(s)ds,
\end{gathered} \tag{2.10}$$

so that

$$\begin{aligned}\frac{d}{dt}V(\cdot) \le\ & -\frac{1}{2}\alpha b y^2 - a z^2 - \frac{1}{2}\left[\alpha b - \left\{\alpha \sum_{i=1}^{n}(B_i L_i \tau_i) + (2\lambda + \alpha c)\tau\right\}\right] y^2 \\ & - \frac{1}{2}\left[2\alpha - \left\{\sum_{i=1}^{n}(B_i L_i \tau_i) + 2\sum_{i=1}^{n}(\lambda_i \tau_i) \ + c\tau\right\}\right] z^2 \\ & + \left[2^{-1}(1+\alpha)c - \lambda\right]\int_{t-\tau}^{t} y^2(s)ds \\ & + \left[2^{-1}(1+\alpha)\sum_{i=1}^{n}(B_i L_i) - \sum_{i=1}^{n}\lambda_i\right]\int_{t-\tau_i}^{t} z^2(s)ds.\end{aligned} \tag{2.11}$$

Let $\lambda = (1/2)(1+\alpha)c$ and $\sum_{i=1}^{n}\lambda_i = (1/2)(1+\alpha)\sum_{i=1}^{n}(B_i L_i)$. Hence,

$$\begin{aligned}\frac{d}{dt}V(\cdot) \le\ & -\frac{1}{2}\alpha b y^2 - a z^2 - \frac{1}{2}\left[\alpha b - \left\{\alpha \sum_{i=1}^{n}(B_i L_i \tau_i) + (2\lambda + \alpha c)\tau\right\}\right] y^2 \\ & - \frac{1}{2}\left[2\alpha - \left\{\sum_{i=1}^{n}(B_i L_i \tau_i) + 2\sum_{i=1}^{n}(\lambda_i \tau_i) + c\tau\right\}\right] z^2.\end{aligned} \tag{2.12}$$

Let $\tau_0 = \max\{\tau, \tau_1, \tau_2, \ldots, \tau_n\}$. Then

$$\begin{aligned}\frac{d}{dt}V(\cdot) \le\ & -\frac{1}{2}\alpha b y^2 - a z^2 - \frac{1}{2}\left[\alpha b - \left\{\alpha \sum_{i=1}^{n}(B_i L_i) \ + (2\lambda + \alpha c)\right\}\tau_0\right] y^2 \\ & - \frac{1}{2}\left[2\alpha - \left\{\sum_{i=1}^{n}(B_i L_i) + 2\sum_{i=1}^{n}\lambda_i + c\right\}\tau_0\right] z^2.\end{aligned} \tag{2.13}$$

Thus, if

$$\tau_0 < \min\left\{\frac{\alpha b}{c + 2\alpha c + \alpha \sum_{i=1}^{n}(B_i L_i)}, \frac{2\alpha}{c + \sum_{i=1}^{n}[(2+\alpha)B_i L_i]}\right\}, \tag{2.14}$$

then

$$\frac{d}{dt}V(\cdot) \le -\frac{1}{2}\alpha b y^2 - a z^2. \tag{2.15}$$

This completes the proof of Theorem 2.1 (Burton [62], Hale [63], and Krasovskiĭ [64]). □

Let $p(\cdot) = 0$.

Our second main result is given by the following theorem.

Theorem 2.2. *In addition to all the assumptions of Theorem 2.1, we assume that the condition*

$$|p(\cdot)| \leq |q(t)| \tag{2.16}$$

holds, where $|q| \in L^1(0,\infty)$. *If*

$$\tau_0 < \min\left\{ \frac{\alpha b}{c + 2\alpha c + \alpha \sum_{i=1}^{n}(B_i L_i)}, \frac{2\alpha}{c + \sum_{i=1}^{n}[(2+\alpha)B_i L_i]} \right\}, \tag{2.17}$$

then, there exists a finite positive constant K *such that the solution* $x(t)$ *of* (1.1) *defined by the initial function*

$$x(t) = \phi(t), \qquad x'(t) = \phi'(t), \qquad x''(t) = \phi''(t) \tag{2.18}$$

satisfies

$$|x(t)| \leq K, \qquad |x'(t)| \leq K, \qquad |x''(t)| \leq K \tag{2.19}$$

for all $t \geq t_0$, *where* $\phi \in C^2([t_0 - r, t_0], \Re)$.

Proof. Under the assumptions of Theorem 2.2, the time derivative of the Lyapunov-Krasovskii functional $V(\cdot)$ satisfies

$$\frac{d}{dt}V(\cdot) \leq -\frac{1}{2}\alpha b y^2 - a z^2 + (\alpha y + z)p(\cdot). \tag{2.20}$$

Using the estimates $|m| < 1 + m^2$ and $D_4(x^2 + y^2 + z^2) \leq V(\cdot)$, it follows that

$$\begin{aligned} \frac{d}{dt}V(\cdot) &\leq (\alpha|y| + |z|)|p(\cdot)| \\ &\leq D_5\left(2 + y^2 + z^2\right)|q(t)| \\ &\leq 2D_5|q(t)| + D_5 D_4^{-1} V(\cdot)|q(t)|, \end{aligned} \tag{2.21}$$

where $D_5 = \max\{1, \alpha\}$.

Integrating the above estimate from 0 to t, using the assumption $|q| \in L^1(0,\infty)$ and the Gronwall-Bellman inequality (see Gronwall [65] and Mitrinović [66]), we can conclude that all solutions of (1.1) are bounded. □

Example 2.3. Consider the nonlinear differential equation of the third order with two deviating arguments as follows:

$$\begin{aligned} & x'''(t) + \left(11 + \frac{1}{1+t^2}\right)x''(t) + 2\left(1+e^{-t}\right)x'(t-\tau_1) + 2\left(2+e^{-t}\right)x'(t-\tau_2) \\ & \quad + 4x'(t) + x(t-\tau) + \operatorname{arc} tg x(t-\tau) = \frac{4}{1+t^2+x^2(t)+x^2(t-\tau_1)+x'^2(t-\tau_2)}. \end{aligned} \tag{2.22}$$

Writing (2.22) as a system of first order equations, we obtain

$$\begin{aligned} & x' = y, \qquad y' = z, \\ z' = & -\left(11 + \frac{1}{1+t^2}\right)z - 2\left(5+2e^{-t}\right)y - x - \operatorname{arc} tg x \\ & + 2\left(1+e^{-t}\right)\int_{t-\tau_1}^{t} z(s)ds + 2\left(2+e^{-t}\right)\int_{t-\tau_2}^{t} z(s)ds + \int_{t-\tau}^{t} y(s)ds \\ & + \int_{t-\tau}^{t} \frac{y(s)}{1+x^2(s)}ds + \frac{4}{1+t^2+\cdots+y^2(t-\tau_2)}. \end{aligned} \tag{2.23}$$

It follows that (2.22) is special case of (1.1), and when we compare (2.22) with (1.1) we obtain the following estimates:

$$\begin{aligned} & a(t) = 11 + \frac{1}{1+t^2} \geq 11 = 2 \times 5 + 1, \quad \alpha = 5, \ a = 1, \\ & b_1(t) = 1 + \frac{1}{e^t}, \quad 1 \leq 1 + \frac{1}{e^t} \leq 2, \quad \beta_1 = 1, \quad B_1 = 2, \\ & g_1(y) = 2y, \quad g_1(0) = 0, \quad \frac{g_1(y)}{y} = 2 = b_1, \quad (y \neq 0), \\ & g_1'(y) = 2 = L_1, \qquad \int_0^y g_1(\eta)d\eta = \int_0^y 2\eta\, d\eta = y^2, \\ & b_2(t) = 2 + \frac{1}{e^t}, \quad 2 \leq 2 + \frac{1}{e^t} \leq 3, \quad \beta_2 = 2, \quad B_2 = 3, \\ & g_2(y) = 2y, \quad g_2(0) = 0, \quad \frac{g_2(y)}{y} = 2 = b_2, \quad (y \neq 0), \end{aligned}$$

$$g_2'(y) = 2 = L_2, \qquad \int_0^y g_2(\eta)d\eta = \int_0^y 2\eta\, d\eta = y^2,$$
$$g(y) = 4y, \quad g(0) = 0, \quad \frac{g(y)}{y} = 4 = b, \quad (y \neq 0),$$
$$h(x) = x + \operatorname{arc} tgx, \qquad h(0) = 0, \qquad h'(x) = 1 + \frac{1}{1+x^2},$$
$$0 < 2^{-1} < h'(x) \leq 2, \qquad c_1 = 2^{-1}, \qquad c = 2,$$
$$a'(t) = -\frac{2t}{(1+t^2)^2}, \quad b_1'(t) = -\frac{1}{e^t}, \quad b_2'(t) = -\frac{1}{e^t}, \quad (t \geq 0),$$
$$p(t, x, \dots, z) = \frac{4}{1 + t^2 + \cdots + y^2(t-\tau_2)} \leq \frac{4}{1+t^2} = q(t). \tag{2.24}$$

In view of the above discussion, it follows that

$$\alpha b - c = 4 > 0,$$
$$\left[\sum_{i=1}^{2}\{\alpha b_i b_i(t)\} - c\right]y^2 \geq 2^{-1}\alpha a'(t)y^2 + \sum_{i=1}^{2} b_i'(t)\int_0^y g_i(\eta)d\eta$$
$$= [\alpha b_1 b_1(t) + \alpha b_2 b_2(t) - c]y^2 = \left[5(1+e^{-t}) + 10(2+e^{-t}) - 4\right]y^2$$
$$= (21 + 15e^{-t})y^2, \quad (t \geq 0),$$
$$\frac{1}{2}\alpha a'(t)y^2 + b_1'(t)\int_0^y g_1(\eta)d\eta + b_1'(t)\int_0^y g_1(\eta)d\eta \tag{2.25}$$
$$= -\left[\frac{5t}{(1+t^2)^2}\right]y^2 - 2e^{-t}y^2, \quad (t \geq 0),$$
$$(21 + 15e^{-t})y^2 \geq -\left[\frac{5t}{(1+t^2)^2}\right] - 2e^{-t}y^2,$$
$$\int_0^\infty |q(s)|ds = \int_0^\infty \frac{4}{1+s^2}ds = 2\pi < \infty,$$

that is, $|q| \in L^1(0,\infty)$, and

$$\tau_0 < \min\left\{\frac{\alpha b}{c + 2\alpha c + \alpha\sum_{i=1}^{2}(B_i L_i)}, \frac{2\alpha}{c + \sum_{i=1}^{2}[(2+\alpha)B_i L_i]}\right\} = \min\left\{\frac{5}{18}, \frac{5}{36}\right\} = \frac{5}{36}. \tag{2.26}$$

Thus, all the assumptions of Theorems 2.1 and 2.2 hold. This shows that the zero solution of (2.22) is stable and all solutions of the same equation are bounded, when $p(\cdot) = 0$ and $\neq 0$, respectively.

References

[1] R. Reissig, G. Sansone, and R. Conti, *Non-Linear Differential Equations of Higher Order*, Noordhoff International Publishing, Leyden, Mass, USA, 1974, Translated from the German.

[2] A. T. Ademola, R. Kehinde, and O. M. Ogunlaran, "A boundedness theorem for a certain third order nonlinear differential equation," *Journal of Mathematics and Statistics*, vol. 4, no. 2, pp. 88–93, 2008.

[3] A. U. Afuwape, O. A. Adesina, and E. P. Ebiendele, "On the stability and the periodicity properties of solutions of a certain third order nonlinear differential equation," *Acta Mathematica*, vol. 23, no. 2, pp. 149–159, 2007.

[4] J. O. C. Ezeilo, "On the boundedness of solutions of a certain differential equation of the third order," *Proceedings of the London Mathematical Society*, vol. 9, no. 3, pp. 74–114, 1959.

[5] J. O. C. Ezeilo, "On the stability of solutions of certain differential equations of the third order," *The Quarterly Journal of Mathematics*, vol. 11, no. 2, pp. 64–69, 1960.

[6] J. O. C. Ezeilo, "A note on a boundedness theorem for some third order differential equations," *Journal of the London Mathematical Society*, vol. 36, pp. 439–444, 1961.

[7] J. O. C. Ezeilo, "A boundedness theorem for some non-linear differential equations of the third order," *Journal of the London Mathematical Society*, vol. 37, pp. 469–474, 1962.

[8] J. O. C. Ezeilo, "A stability result for the solutions of certain third order differential equations," *Journal of the London Mathematical Society*, vol. 37, pp. 405–409, 1962.

[9] J. O. C. Ezeilo, "A boundedness theorem for a certain third-order differential equation," *Proceedings of the London Mathematical Society*, vol. 13, no. 3, pp. 99–124, 1963.

[10] J. O. C. Ezeilo, "An elementary proof of a boundedness theorem for a certain third order differential equation," *Journal of the London Mathematical Society*, vol. 38, pp. 11–16, 1963.

[11] J. O. C. Ezeilo, "A stability result for a certain third order differential equation," *Annali di Matematica Pura ed Applicata*, vol. 72, no. 4, pp. 1–9, 1966.

[12] J. O. C. Ezeilo, "A generalization of a boundedness theorem for a certain third-order differential equation," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 63, pp. 735–742, 1967.

[13] J. O. C. Ezeilo, "On the stability of the solutions of some third order differential equations," *Journal of the London Mathematical Society*, vol. 43, pp. 161–167, 1968.

[14] J. O. C. Ezeilo and H. O. Tejumola, "Boundedness and periodicity of solutions of a certain system of third-order non-linear differential equations," *Annali di Matematica Pura ed Applicata*, vol. 74, no. 4, pp. 283–316, 1966.

[15] J. O. C. Ezeilo and H. O. Tejumola, "Boundedness theorems for certain third order differential equations," *Atti della Accademia Nazionale dei Lincei*, vol. 55, pp. 194–201, 1974.

[16] B. Mehri and D. Shadman, "Boundedness of solutions of certain third order differential equation," *Mathematical Inequalities & Applications*, vol. 2, no. 4, pp. 545–549, 1999.

[17] B. S. Ogundare, "On the boundedness and the stability results for the solutions of certain third order non-linear differential equations," *Kragujevac Journal of Mathematics*, vol. 29, pp. 37–48, 2006.

[18] B. S. Ogundare and G. E. Okecha, "On the boundedness and the stability of solution to third order non-linear differential equations," *Annals of Differential Equations*, vol. 24, no. 1, pp. 1–8, 2008.

[19] M. O. Omeike, "New result in the ultimate boundedness of solutions of a third-order nonlinear ordinary differential equation," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 9, no. 1, article 15, p. 8, 2008.

[20] M. O. Omeike, "Further results on global stability of solutions of certain third-order nonlinear differential equations," *Acta Universitatis Palackianae Olomucensis*, vol. 47, pp. 121–127, 2008.

[21] P. J. Ponzo, "On the stability of certain nonlinear differential equations," *IEEE Transactions on Automatic Control*, vol. AC-10, pp. 470–472, 1965.

[22] P. J. Ponzo, "Some stability conditions for linear differential equations," *IEEE Transactions on Automatic Control*, vol. AC-13, no. 6, pp. 721–722, 1968.

[23] K. E. Swick, "A boundedness result for the solutions of certain third order differential equations," *Annali di Matematica Pura ed Applicata*, vol. 86, no. 4, pp. 169–180, 1970.

[24] K. E. Swick, "Asymptotic behavior of the solutions of certain third order differential equations," *SIAM Journal on Applied Mathematics*, vol. 19, pp. 96–102, 1970.

[25] K. E. Swick, "Boundedness and stability for a nonlinear third order differential equation," *Atti della Accademia Nazionale dei Lincei*, vol. 56, no. 6, pp. 859–865, 1974.

[26] H. O. Tejumola, "On the boundedness and periodicity of solutions of certain third-order non-linear differential equations," *Annali di Matematica Pura ed Applicata*, vol. 83, no. 4, pp. 195–212, 1969.

[27] H. O. Tejumola, "A note on the boundedness and the stability of solutions of certain third-order differential equations," *Annali di Matematica Pura ed Applicata*, vol. 92, no. 4, pp. 65–75, 1972.
[28] C. Tunç, "Global stability of solutions of certain third-order nonlinear differential equations," *Panamerican Mathematical Journal*, vol. 14, no. 4, pp. 31–35, 2004.
[29] C. Tunç, "Uniform ultimate boundedness of the solutions of third-order nonlinear differential equations," *Kuwait Journal of Science & Engineering*, vol. 32, no. 1, pp. 39–48, 2005.
[30] C. Tunç, "Boundedness of solutions of a third-order nonlinear differential equation," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 6, no. 1, article 3, p. 6, 2005.
[31] C. Tunç, "On the asymptotic behavior of solutions of certain third-order nonlinear differential equations," *Journal of Applied Mathematics and Stochastic Analysis*, no. 1, pp. 29–35, 2005.
[32] C. Tunç, "On the boundedness of solutions of certain nonlinear vector differential equations of third order," *Bulletin Mathématique de la Société des Sciences Mathématiques de Roumanie*, vol. 49(97), no. 3, pp. 291–300, 2006.
[33] C. Tunç, "On the stability and boundedness of solutions of nonlinear vector differential equations of third order," *Nonlinear Analysis A*, vol. 70, no. 6, pp. 2232–2236, 2009.
[34] C. Tunç, "Some new results on the boundedness of solutions of a certain nonlinear differential equation of third order," *International Journal of Nonlinear Science*, vol. 7, no. 2, pp. 246–256, 2009.
[35] C. Tunç and M. Ateş, "Stability and boundedness results for solutions of certain third order nonlinear vector differential equations," *Nonlinear Dynamics*, vol. 45, no. 3-4, pp. 273–281, 2006.
[36] C. Tunç and T. Ayhan, "New boundedness results for a kind of nonlinear differential equations of third order," *Journal of Computational Analysis and Applications*, vol. 13, no. 3, pp. 477–484, 2011.
[37] A. S. C. Sinha, "On stability of solutions of some third and fourth order delay-differential equations," *Information and Computation*, vol. 23, pp. 165–172, 1973.
[38] A. U. Afuwape and M. O. Omeike, "On the stability and boundedness of solutions of a kind of third order delay differential equations," *Applied Mathematics and Computation*, vol. 200, no. 1, pp. 444–451, 2008.
[39] M. Omeike, "Stability and boundedness of solutions of some non-autonomous delay differential equation of the third order," *Analele Ştiinţifice ale Universităţii*, vol. 55, supplement 1, pp. 49–58, 2009.
[40] A. I. Sadek, "Stability and boundedness of a kind of third-order delay differential system," *Applied Mathematics Letters*, vol. 16, no. 5, pp. 657–662, 2003.
[41] A. I. Sadek, "On the stability of solutions of some non-autonomous delay differential equations of the third order," *Asymptotic Analysis*, vol. 43, no. 1-2, pp. 1–7, 2005.
[42] H. O. Tejumola and B. Tchegnani, "Stability, boundedness and existence of periodic solutions of some third and fourth order nonlinear delay differential equations," *Journal of the Nigerian Mathematical Society*, vol. 19, pp. 9–19, 2000.
[43] C. Tunç, "New results about stability and boundedness of solutions of certain non-linear third-order delay differential equations," *The Arabian Journal for Science and Engineering A*, vol. 31, no. 2, pp. 185–196, 2006.
[44] C. Tunç, "On asymptotic stability of solutions to third order nonlinear differential equations with retarded argument," *Communications in Applied Analysis*, vol. 11, no. 3-4, pp. 515–527, 2007.
[45] C. Tunç, "Stability and boundedness of solutions of nonlinear differential equations of third-order with delay," *Differential Equations and Control Processes*, no. 3, pp. 1–13, 2007.
[46] C. Tunç, "On the boundedness of solutions of third-order differential equations with delay," *Differentsiale nye Uravneniya*, vol. 44, no. 4, pp. 446–454, 574, 2008.
[47] C. Tunç, "On the boundedness of solutions of delay differential equations of third order," *The Arabian Journal for Science and Engineering A*, vol. 34, no. 1, pp. 227–237, 2009.
[48] C. Tunç, "Stability criteria for certain third order nonlinear delay differential equations," *Portugaliae Mathematica*, vol. 66, no. 1, pp. 71–80, 2009.
[49] C. Tunç, "Boundedness criteria for certain third order nonlinear delay differential equations," *Journal of Concrete and Applicable Mathematics*, vol. 7, no. 2, pp. 126–138, 2009.
[50] C. Tunç, "A new boundedness result to nonlinear differential equations of third order with finite lag," *Communications in Applied Analysis*, vol. 13, no. 1, pp. 1–10, 2009.
[51] C. Tunç, "On the stability and boundedness of solutions to third order nonlinear differential equations with retarded argument," *Nonlinear Dynamics*, vol. 57, no. 1-2, pp. 97–106, 2009.
[52] C. Tunç, "A new result on the stability of solutions of a nonlinear differential equation of third-order with finite lag," *Southeast Asian Bulletin of Mathematics*, vol. 33, no. 5, pp. 947–958, 2009.
[53] C. Tunç, "Bounded solutions to nonlinear delay differential equations of third order," *Topological Methods in Nonlinear Analysis*, vol. 34, no. 1, pp. 131–139, 2009.

[54] C. Tunç, "Bound of solutions to third-order nonlinear differential equations with bounded delay," *Journal of the Franklin Institute*, vol. 347, no. 2, pp. 415–425, 2010.
[55] C. Tunç, "Some stability and boundedness conditions for non-autonomous differential equations with deviating arguments," *Electronic Journal of Qualitative Theory of Differential Equations*, no. 1, pp. 1–12, 2010.
[56] C. Tunç, "On some qualitative behaviors of solutions to a kind of third order nonlinear delay differential equations," *Electronic Journal of Qualitative Theory of Differential Equations*, no. 12, pp. 1–19, 2010.
[57] C. Tunç, "On the stability and boundedness of solutions of nonlinear third order differential equations with delay," *Filomat*, vol. 24, no. 3, pp. 1–10, 2010.
[58] C. Tunç, "Stability and bounded of solutions to non-autonomous delay differential equations of third order," *Nonlinear Dynamics*, vol. 62, no. 4, pp. 945–953, 2010.
[59] C. Tunç, "Asymptotic stable and bounded solutions of a kind of nonlinear differential equations with variable delay," *Functional Differential Equations*, vol. 17, no. 3-4, pp. 345–354, 2010.
[60] H. Yao and W. Meng, "On the stability of solutions of certain non-linear third-order delay differential equations," *International Journal of Nonlinear Science*, vol. 6, no. 3, pp. 230–237, 2008.
[61] Y. F. Zhu, "On stability, boundedness and existence of periodic solution of a kind of third order nonlinear delay differential system," *Annals of Differential Equations*, vol. 8, no. 2, pp. 249–259, 1992.
[62] T. A. Burton, *Stability and Periodic Solutions of Ordinary and Functional-Differential Equations*, vol. 178 of *Mathematics in Science and Engineering*, Academic Press, Orlando, Fla, USA, 1985.
[63] J. K. Hale, "Sufficient conditions for stability and instability of autonomous functional-differential equations," *Journal of Differential Equations*, vol. 1, pp. 452–482, 1965.
[64] N. N. Krasovskiĭ, *Stability of Motion. Applications of Lyapunov's Second Method to Differential Systems and Equations with Delay*, Translated by J. L. Brenner, Stanford University Press, Stanford, Calif, USA, 1963.
[65] T. H. Gronwall, "Note on the derivatives with respect to a parameter of the solutions of a system of differential equations," *Annals of Mathematics*, vol. 20, no. 4, pp. 292–296, 1919.
[66] D. S. Mitrinović, *Analytic Inequalities*, Springer, New York, NY, USA, 1970, In cooperation with P. M. Vasić. Die Grundlehren der mathematischen Wissenschaften, Band 165.