

Research Article

Convergence Theorems for Infinite Family of Multivalued Quasi-Nonexpansive Mappings in Uniformly Convex Banach Spaces

Aunyarat Bunyawat^{1,2} and Suthep Suantai^{1,2}

¹ Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand

² Centre of Excellence in Mathematics, CHE, Si Ayutthaya Road, Bangkok 10400, Thailand

Correspondence should be addressed to Suthep Suantai, scmti005@chiangmai.ac.th

Received 13 December 2011; Accepted 6 January 2012

Academic Editor: Simeon Reich

Copyright © 2012 A. Bunyawat and S. Suantai. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We introduce an iterative method for finding a common fixed point of a countable family of multivalued quasi-nonexpansive mapping $\{T_i\}$ in a uniformly convex Banach space. We prove that under certain control conditions, the iterative sequence generated by our method is an approximating fixed point sequence of each T_i . Some strong convergence theorems of the proposed method are also obtained for the following cases: all T_i are continuous and one of T_i is hemicompact, and the domain K is compact.

1. Introduction

Let X be a real Banach space. A subset K of X is called *proximal* if for each $x \in X$, there exists an element $k \in K$ such that

$$d(x, k) = d(x, K), \quad (1.1)$$

where $d(x, K) = \inf\{\|x - y\| : y \in K\}$ is the distance from the point x to the set K . It is clear that every closed convex subset of a uniformly convex Banach space is proximal.

Let X be a uniformly convex real Banach space, K be a nonempty closed convex subset of X , $CB(K)$ be a family of nonempty closed bounded subsets of K , and $P(K)$ be

a nonempty proximal bounded subsets of K . The *Hausdorff metric* on $CB(X)$ is defined by

$$H(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \right\}, \quad (1.2)$$

for all $A, B \in CB(X)$.

An element $p \in K$ is called a *fixed point* of a single valued mapping T if $p = Tp$ and of a multivalued mapping T if $p \in Tp$. The set of fixed points of T is denoted by $F(T)$.

A single valued mapping $T : K \rightarrow K$ is said to be *quasi-nonexpansive* if $\|Tx - p\| \leq \|x - p\|$ for all $x \in K$ and $p \in F(T)$.

A multivalued mapping $T : K \rightarrow CB(K)$ is said to be:

- (i) *quasi-nonexpansive* if $F(T) \neq \emptyset$ and $H(Tx, Tp) \leq \|x - p\|$ for all $x \in K$ and $p \in F(T)$,
- (ii) *nonexpansive* if $H(Tx, Ty) \leq \|x - y\|$ for all $x, y \in K$.

It is well known that every nonexpansive multivalued mapping T with $F(T) \neq \emptyset$ is quasi-nonexpansive. But there exist quasi-nonexpansive mappings that are not nonexpansive. It is clear that if T is a quasi-nonexpansive multivalued mapping, then $F(T)$ is closed.

A map $T : K \rightarrow CB(K)$ is called *hemicompact* if, for any sequence $\{x_n\}$ in K such that $d(x_n, Tx_n) \rightarrow 0$ as $n \rightarrow \infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightarrow p \in K$. We note that if K is compact, then every multivalued mapping T is hemicompact.

In 1969, Nadler [1] proved a fixed point theorem for multivalued contraction mappings and convergence of a sequence. They extended theorems on the stability of fixed points of single-valued mappings and also given a counterexample to a theorem about (ε, λ) -uniformly locally expansive (single-valued) mappings. Later in 1997, Hu et al. [2] obtained common fixed point of two nonexpansive multivalued mappings satisfying certain contractive condition.

In 2005, Sastry and Babu [3] proved the convergence theorem of multivalued maps by defining Ishikawa and Mann iterates and gave example which shows that the limit of the sequence of Ishikawa iterates depends on the choice of the fixed point p and the initial choice of x_0 . In 2007, there is paper which generalized results of Sastry and Babu [3] to uniformly convex Banach spaces by Panyanak [4] and proved a convergence theorem of Mann iterates for a mapping defined on a noncompact domain.

Later in 2008, Song and Wang [5] shown that strong convergence for Mann and Ishikawa iterates of multivalued nonexpansive mapping T under some appropriate conditions. In 2009, Shahzad and Zegeye [6] proved strong convergence theorems of quasi-nonexpansive multivalued mapping for the Ishikawa iteration. They also constructed an iteration scheme which removes the restriction of T with $Tp = \{p\}$ for any $p \in F(T)$ which relaxed compactness of the domain of T .

Recently, Abbas et al. [7] established weak- and strong-convergence theorems of two multivalued nonexpansive mappings in a real uniformly convex Banach space by one-step iterative process to approximate common fixed points under some basic boundary conditions.

A fixed points and common fixed points theorem of multivalued maps in uniformly convex Banach space or in complete metric spaces or in convex metric spaces have been intensively studied by many authors; for instance, see [8–23].

In this paper, we generalize and modify the iteration of Abbas et al. [7] from two mapping to the infinite family mappings $\{T_i : i \in \mathbb{N}\}$ of multivalued quasi-nonexpansive mapping in a uniformly convex Banach space.

Let $\{T_i\}$ be a countable family of multivalued quasi-nonexpansive mapping from a bounded and closed convex subset K of a Banach space into $P(K)$ with $F := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and let $p \in F$. For $x_1 \in K$, we define

$$x_{n+1} = \alpha_{n,0}x_n + \sum_{i=1}^{\infty} \alpha_{n,i}x_{n,i}, \tag{1.3}$$

where the sequences $\{\alpha_{n,i}\} \subset [0, 1)$ satisfying $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$ and $x_{n,i} \in T_i x_n$ such that $d(p, x_{n,i}) = d(p, T_i x_n)$ for $i \in \mathbb{N}$. The main purpose of this paper is to prove strong convergence of the iterative scheme (1.3) to a common fixed point of T_i .

2. Preliminaries

Before to say the main theorem, we need the following lemmas.

Lemma 2.1 (see [24]). *Suppose that X is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all positive integers n . Also suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of X such that $\limsup_{n \rightarrow \infty} \|x_n\| \leq r$, $\limsup_{n \rightarrow \infty} \|y_n\| \leq r$, and $\lim_{n \rightarrow \infty} \|t_n x_n + (1 - t_n)y_n\| = r$ hold for some $r \geq 0$. Then $\limsup_{n \rightarrow \infty} \|x_n - y_n\| = 0$.*

Lemma 2.2 (see [25]). *Let E be a uniformly convex Banach space. For arbitrary $r > 0$, let $B_r(0) := \{x \in E : \|x\| \leq r\}$. Then, for any given sequence $\{x_n\}_{n=1}^{\infty} \subset B_r(0)$ and for any given sequence $\{\lambda_n\}_{n=1}^{\infty}$ of positive numbers such that $\sum_{i=1}^{\infty} \lambda_i = 1$, there exists a continuous strictly increasing convex function*

$$g : [0, 2r] \longrightarrow \mathbb{R}, \quad g(0) = 0, \tag{2.1}$$

such that for any positive integers i, j with $i < j$, the following inequality holds:

$$\left\| \sum_{n=1}^{\infty} \lambda_n x_n \right\|^2 \leq \sum_{n=1}^{\infty} \lambda_n \|x_n\|^2 - \lambda_i \lambda_j g(\|x_i - x_j\|). \tag{2.2}$$

3. Main Results

We first prove that the sequence $\{x_n\}$ generated by (1.3) is an approximating fixed point sequence of each T_i ($i \in \mathbb{N}$).

Lemma 3.1. *Let K be a nonempty bounded and closed convex subset of a uniformly convex Banach space X . For $i \in \mathbb{N}$, let $\{T_i\}$ be a sequence of multivalued quasi-nonexpansive mappings from K into $P(K)$ with $F := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and $p \in F$. Let $\{x_n\}$ be a sequence defined by (1.3). Then*

- (i) $\|x_{n+1} - p\| \leq \|x_n - p\|$,
- (ii) $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists.

Proof. By (1.3), we have

$$\begin{aligned}
\|x_{n+1} - p\| &\leq \alpha_{n,0}\|x_n - p\| + \sum_{i=1}^{\infty} \alpha_{n,i}\|x_{n,i} - p\| \\
&= \alpha_{n,0}\|x_n - p\| + \sum_{i=1}^{\infty} \alpha_{n,i}d(T_i x_n, p) \\
&\leq \alpha_{n,0}\|x_n - p\| + \sum_{i=1}^{\infty} \alpha_{n,i}H(T_i x_n, T_i p) \\
&\leq \alpha_{n,0}\|x_n - p\| + \sum_{i=1}^{\infty} \alpha_{n,i}\|x_n - p\| \\
&= \|x_n - p\|.
\end{aligned} \tag{3.1}$$

So (i) is obtained. (ii) follows from (i). \square

Theorem 3.2. Let K be a nonempty bounded and closed convex subset of a uniformly convex Banach space X . For $i \in \mathbb{N}$, let $\{T_i\}$ be a sequence of multivalued quasi-nonexpansive mappings from K into $P(K)$ with $F := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and $p \in F$. Let $\{x_n\}$ be a sequence defined by (1.3) with $\liminf_{n \rightarrow \infty} \alpha_{n,0}\alpha_{n,j} > 0$ for all $j \in \mathbb{N}$. Then $\lim_{n \rightarrow \infty} d(x_n, T_i x_n) = 0$ for all $i \in \mathbb{N}$.

Proof. For $j \in \mathbb{N}$, by Lemma 2.2, we get

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \left\| \alpha_{n,0}(x_n - p) + \sum_{i=1}^{\infty} \alpha_{n,i}(x_{n,i} - p) \right\|^2 \\
&\leq \alpha_{n,0}\|x_n - p\|^2 + \sum_{i=1}^{\infty} \alpha_{n,i}\|x_{n,i} - p\|^2 - \alpha_{n,0}\alpha_{n,j}g(\|x_n - x_{n,j}\|) \\
&= \alpha_{n,0}\|x_n - p\|^2 + \sum_{i=1}^{\infty} \alpha_{n,i}(d(T_i x_n, p))^2 - \alpha_{n,0}\alpha_{n,j}g(\|x_n - x_{n,j}\|) \\
&\leq \alpha_{n,0}\|x_n - p\|^2 + \sum_{i=1}^{\infty} \alpha_{n,i}(H(T_i x_n, T_i p))^2 - \alpha_{n,0}\alpha_{n,j}g(\|x_n - x_{n,j}\|) \\
&\leq \alpha_{n,0}\|x_n - p\|^2 + \sum_{i=1}^{\infty} \alpha_{n,i}\|x_n - p\|^2 - \alpha_{n,0}\alpha_{n,j}g(\|x_n - x_{n,j}\|) \\
&= \|x_n - p\|^2 - \alpha_{n,0}\alpha_{n,j}g(\|x_n - x_{n,j}\|).
\end{aligned} \tag{3.2}$$

Thus, $0 < \alpha_{n,0}\alpha_{n,j}g(\|x_n - x_{n,j}\|) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2$. It follows that $\lim_{n \rightarrow \infty} g(\|x_n - x_{n,j}\|) = 0$. By property of g , we have $\lim_{n \rightarrow \infty} \|x_n - x_{n,j}\| = 0$. Thus $\lim_{n \rightarrow \infty} d(x_n, T_j x_n) = 0$ for $i \in \mathbb{N}$. \square

Theorem 3.3. Let X be a uniformly convex real Banach space and K be a bounded and closed convex subset of X . For $i \in \mathbb{N}$, let $\{T_i\}$ be a sequence of multivalued quasi-nonexpansive and continuous mappings from K into $P(K)$ with $F := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and $p \in F$. Let $\{x_n\}$ be a sequence defined by

(1.3) with $\liminf_{n \rightarrow \infty} \alpha_{n,0} \alpha_{n,j} > 0$ for all $j \in \mathbb{N}$. Assume that one of T_i is hemicompact. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}$.

Proof. Suppose that T_i is hemicompact for some $i_0 \in \mathbb{N}$. By Theorem 3.2, we have $\lim_{n \rightarrow \infty} d(x_n, T_i x_n) = 0$ for all $i \in \mathbb{N}$. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k \rightarrow \infty} x_{n_k} = q \in K$. From continuity of T_i , we get $d(x_{n_k}, T_i x_{n_k}) \rightarrow d(q, T_i q)$. This implies that $d(q, T_i q) = 0$ and $q \in F$. Since $\lim_{n \rightarrow \infty} \|x_n - q\|$ exists, it follows that $\{x_n\}$ converges strongly to q . \square

Theorem 3.4. Let X be a uniformly convex real Banach space and K be a compact convex subset of X . For $i \in \mathbb{N}$, let $\{T_i\}$ be a sequence of multivalued quasi-nonexpansive mappings from K into $P(K)$ with $F := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and $p \in F$. Let $\{x_n\}$ be a sequence defined by (1.3) with $\liminf_{n \rightarrow \infty} \alpha_{n,0} \alpha_{n,j} > 0$ for all $j \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}$.

Proof. From the compactness of K , there exists a subsequence $\{x_{n_k}\}_{n=k}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $\lim_{k \rightarrow \infty} \|x_{n_k} - q\| = 0$ for some $q \in K$. Thus, it follows by Theorem 3.2 that,

$$\begin{aligned} d(q, T_i q) &\leq d(q, x_{n_k}) + d(x_{n_k}, T_i x_{n_k}) + H(T_i x_{n_k}, T_i q) \\ &\leq 2\|x_{n_k} - q\| + d(x_{n_k}, T_i x_{n_k}) \rightarrow 0 \quad \text{as } k \rightarrow \infty. \end{aligned} \quad (3.3)$$

Hence $q \in F$. By Lemma 3.1(ii), $\lim_{n \rightarrow \infty} \|x_n - q\|$ exists. Hence $\lim_{n \rightarrow \infty} x_n = q$. The proof is complete. \square

Acknowledgments

This research is supported by the Centre of Excellence in Mathematics, the Commission on Higher Education, Thailand. The first author is supported by the Graduate School, Chiang Mai University, Thailand.

References

- [1] S. B. Nadler Jr., "Multi-valued contraction mappings," *Pacific Journal of Mathematics*, vol. 30, pp. 475–488, 1969.
- [2] T. Hu, J.-C. Huang, and B. E. Rhoades, "A general principle for Ishikawa iterations for multi-valued mappings," *Indian Journal of Pure and Applied Mathematics*, vol. 28, no. 8, pp. 1091–1098, 1997.
- [3] K. P. R. Sastry and G. V. R. Babu, "Convergence of Ishikawa iterates for a multi-valued mapping with a fixed point," *Czechoslovak Mathematical Journal*, vol. 55(130), no. 4, pp. 817–826, 2005.
- [4] B. Panyanak, "Mann and Ishikawa iterative processes for multivalued mappings in Banach spaces," *Computers & Mathematics with Applications*, vol. 54, no. 6, pp. 872–877, 2007.
- [5] Y. Song and H. Wang, "Erratum to: "Mann and Ishikawa iterative processes for multivalued mappings in Banach spaces" [*Comput. Math. Appl.* vol. 54, no. 6, 872–877, 2007] by B. Panyanak," *Computers & Mathematics with Applications*, vol. 55, no. 12, pp. 2999–3002, 2008.
- [6] N. Shahzad and H. Zegeye, "On Mann and Ishikawa iteration schemes for multi-valued maps in Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 3-4, pp. 838–844, 2009.
- [7] M. Abbas, S. H. Khan, A. R. Khan, and R. P. Agarwal, "Common fixed points of two multivalued nonexpansive mappings by one-step iterative scheme," *Applied Mathematics Letters*, vol. 24, no. 2, pp. 97–102, 2011.
- [8] H.-K. Xu, "Multivalued nonexpansive mappings in Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 43, no. 6, pp. 693–706, 2001.

- [9] Lj. B. Ćirić and J. S. Ume, "Multi-valued non-self-mappings on convex metric spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 60, no. 6, pp. 1053–1063, 2005.
- [10] Y. Liu, J. Wu, and Z. Li, "Common fixed points of single-valued and multivalued maps," *International Journal of Mathematics and Mathematical Sciences*, no. 19, pp. 3045–3055, 2005.
- [11] J. S. Jung, "Strong convergence theorems for multivalued nonexpansive nonself-mappings in Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 66, no. 11, pp. 2345–2354, 2007.
- [12] L. Ćirić, "Fixed point theorems for multi-valued contractions in complete metric spaces," *Journal of Mathematical Analysis and Applications*, vol. 348, no. 1, pp. 499–507, 2008.
- [13] M. Abbas and B. E. Rhoades, "Fixed point theorems for two new classes of multivalued mappings," *Applied Mathematics Letters*, vol. 22, no. 9, pp. 1364–1368, 2009.
- [14] S. Hong, D. Guan, and L. Wang, "Hybrid fixed points of multivalued operators in metric spaces with applications," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 11, pp. 4106–4117, 2009.
- [15] N. Shahzad and H. Zegeye, "On Mann and Ishikawa iteration schemes for multi-valued maps in Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 3-4, pp. 838–844, 2009.
- [16] Y. Song and H. Wang, "Convergence of iterative algorithms for multivalued mappings in Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 4, pp. 1547–1556, 2009.
- [17] W. Chulamjiak and S. Suantai, "A hybrid method for a countable family of multivalued maps, equilibrium problems, and variational inequality problems," *Discrete Dynamics in Nature and Society*, vol. 2010, Article ID 349158, 14 pages, 2010.
- [18] S. Hong, "Fixed points of multivalued operators in ordered metric spaces with applications," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 72, no. 11, pp. 3929–3942, 2010.
- [19] Q. Kiran and T. Kamran, "Fixed point theorems for generalized contractive multi-valued maps," *Computers & Mathematics with Applications*, vol. 59, no. 12, pp. 3813–3823, 2010.
- [20] W. Chulamjiak and S. Suantai, "A common fixed point of Ishikawa iteration with errors for two quasi-nonexpansive multi-valued maps in Banach spaces," *Bulletin of Mathematical Analysis and Applications*, vol. 3, no. 2, pp. 110–117, 2011.
- [21] W. Chulamjiak and S. Suantai, "Approximation of common fixed points of two quasi-nonexpansive multi-valued maps in Banach spaces," *Computers & Mathematics with Applications*, vol. 61, no. 4, pp. 941–949, 2011.
- [22] S. H. Khan, I. Yildirim, and B. E. Rhoades, "A one-step iterative process for two multivalued nonexpansive mappings in Banach spaces," *Computers & Mathematics with Applications*, vol. 61, no. 10, pp. 3172–3178, 2011.
- [23] A. Latif and A. A. N. Abdou, "Multivalued generalized nonlinear contractive maps and fixed points," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 74, no. 4, pp. 1436–1444, 2011.
- [24] J. Schu, "Weak and strong convergence to fixed points of asymptotically nonexpansive mappings," *Bulletin of the Australian Mathematical Society*, vol. 43, no. 1, pp. 153–159, 1991.
- [25] S.-s. Chang, J. K. Kim, and X. R. Wang, "Modified block iterative algorithm for solving convex feasibility problems in Banach spaces," *Journal of Inequalities and Applications*, vol. 2010, Article ID 869684, 14 pages, 2010.



Hindawi

Submit your manuscripts at
<http://www.hindawi.com>

