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Research Article

Monotonic Positive Solutions of Nonlocal Boundary Value Problems for a Second-Order Functional Differential Equation

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We study the existence of at least one monotonic positive solution for the nonlocal boundary value problem of the second-order functional differential equation $x''(t) = f(t, x(\phi(t)))$, $t \in (0,1)$, with the nonlocal condition $\sum_{k=1}^{m} a_k x(\tau_k) = x_0$, $x'(0) + \sum_{j=1}^{n} b_j x'(\eta_j) = x_1$, where $\tau_k \in (a,d) \subset (0,1)$, $\eta_j \in (c,e) \subset (0,1)$, and $x_0, x_1 > 0$. As an application the integral and the nonlocal conditions $\int_a^d x(t)dt = x_0$, $x'(0) + x(e) - x(c) = x_1$ will be considered.

1. Introduction

The nonlocal boundary value problems of ordinary differential equations arise in a variety of different areas of applied mathematics and physics.

The study of nonlocal boundary value problems was initiated by Il'in and Moiseev [1, 2]. Since then, the non-local boundary value problems have been studied by several authors. The reader is referred to [3–22] and references therein.

In most of all these papers, the authors assume that the function $f : [0,1] \times R^+ \to R^+$ is continuous. They all assume that

$$\begin{aligned} &\lim_{x\to\infty} \frac{f(x)}{x} = 0 \quad \text{or } \infty, \\ &\lim_{x\to 0} \frac{f(x)}{x} = 0 \quad \text{or } \infty. \end{aligned} \tag{1.1}$$

These assumptions are restrictive, and there are many functions that do not satisfy these assumptions.

Here we assume that the function $f : [0,1] \times R^+ \to R^+$ is measurable in $t \in [0,1]$ for all $x \in R^+$ and continuous in $x \in R^+$ for almost all $t \in [0,1]$ is and there exists an integrable function $a \in L_1[0,1]$ and a constant $b > 0$ such that

$$\left|f(t,x)\right| \le |a(t)| + b|x|, \quad \forall (t,x) \in [0,1] \times D. \tag{1.2}$$

Our aim here is to study the existence of at least one monotonic positive solution for the nonlocal problem of the second-order functional differential equation

$$x''(t) = f\left(t, x\left(\phi(t)\right)\right), \quad t \in (0,1), \tag{1.3}$$

with the nonlocal condition

$$\sum_{k=1}^{m} a_k x(\tau_k) = x_0, \qquad x'(0) + \sum_{j=1}^{n} b_j x'\left(\eta_j\right) = x_1, \tag{1.4}$$

where $\tau_k \in (a,d) \subset (0,1), \eta_j \in (c,e) \subset (0,1),$ and $x_0, x_1 > 0$.

As an application, the problem with the integral and nonlocal conditions

$$\int_a^d x(t)dt = x_0, \qquad x'(0) + x(e) - x(c) = x_1, \tag{1.5}$$

is studied.

It must be noticed that the nonlocal conditions

$$\begin{gathered} x(\tau) = x_0, \quad \tau \in (a,d)\ , \qquad x'(0) + x'(\eta) = x_1, \quad \eta \in (c,e), \\ \sum_{k=1}^{m} a_k x(\tau_k) = 0, \quad \tau_k \in (a,d), \qquad x'(0) + \sum_{j=1}^{n} b_j x'\left(\eta_j\right) = 0, \quad \eta_j \in (c,e), \\ \int_a^d x(t)dt = 0, \qquad x'(0) + x(e) = x(c) \end{gathered} \tag{1.6}$$

are special cases of our the nonlocal and integral conditions.

2. Integral Equation Representation

Consider the functional differential equation (1.3) with the nonlocal condition (1.4) with the following assumptions.

(i) $f : [0,1] \times R^+ \to R^+$ is measurable in $t \in [0,1]$ for all $x \in R^+$ and continuous in $x \in R^+$ for almost all $t \in [0,1]$ and there exists an integrable function $a \in L_1[0,1]$, and a constant $b > 0$ such that

$$\left|f(t,x)\right| \le |a(t)| + b|x|, \quad \forall (t,x) \in [0,1] \times D. \tag{2.1}$$

(ii) $\phi : (0,1) \to (0,1)$ is continuous.

(iii) $b < 1/(3-B)$, $B = (\sum_{j=1}^{n} b_j + 1)^{-1}$.

(iv)

$$\sum_{k=1}^{m} a_k > 0, \quad \forall k = 1,2,\ldots,m, \qquad \sum_{j=1}^{n} b_j > 0, \quad \forall j = 1,2,\ldots,n. \tag{2.2}$$

Now, we have the following Lemma.

Lemma 2.1. *The solution of the nonlocal problem* (1.3)-(1.4) *can be expressed by the integral equation*

$$\begin{aligned} x(t) &= A\left\{x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s)))ds\right\} \\ &\quad + B\left(t - A\sum_{k=1}^{m} a_k\tau_k\right)\left\{x_1 - \sum_{j=1}^{n} b_j \int_0^{\eta_j} f(s, x(\phi(s)))ds\right\} \\ &\quad + \int_0^t (t-s) f(s, x(\phi(s)))ds, \end{aligned} \tag{2.3}$$

where $A = (\sum_{k=1}^{m} a_k)^{-1}$, $B = (\sum_{j=1}^{n} b_j + 1)^{-1}$.

Proof. Integrating (1.3), we get

$$x'(t) = x'(0) + \int_0^t f(s, x(\phi(s)))ds. \tag{2.4}$$

Integrating (2.4), we obtain

$$x(t) = x(0) + x'(0)t + \int_0^t (t-s) f(s, x(\phi(s)))ds. \tag{2.5}$$

Let $t = \tau_k$, in (2.5), we get

$$\sum_{k=1}^{m} a_k x(\tau_k) = \sum_{k=1}^{n} a_k x(0) + \sum_{k=1}^{n} a_k \tau_k x'(0) + \sum_{k=1}^{m} a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s)))ds, \tag{2.6}$$

and we deduce that

$$x(0) = A\left\{x_0 - \sum_{k=1}^{m} a_k \tau_k x'(0) - \sum_{k=1}^{m} a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s)))ds\right\}, \quad A = \left(\sum_{k=1}^{m} a_k\right)^{-1}. \tag{2.7}$$

Substitute from (2.7) into (2.5), we obtain

$$\begin{aligned} x(t) = A&\left\{x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s)))ds\right\} + x'(0)\left(t - A\sum_{k=1}^{m} a_k \tau_k\right) \\ &+ \int_0^t (t-s) f(s, x(\phi(s)))ds. \end{aligned} \tag{2.8}$$

Let $t = \eta_j$, in (2.4), we obtain

$$\begin{aligned} \sum_{j=1}^{n} b_j x'(\eta_j) &= \sum_{j=1}^{n} b_j x'(0) + \sum_{j=1}^{n} b_j \int_0^{\eta_j} f(s, x(\phi(s)))ds, \\ x_1 - x'(0) &= x'(0)\sum_{j=1}^{n} b_j + \sum_{j=1}^{n} b_j \int_0^{\eta_j} f(s, x(\phi(s)))ds, \end{aligned} \tag{2.9}$$

and we deduce that

$$x'(0) = B\left(x_1 - \sum_{j=1}^{n} b_j \int_0^{\eta_j} f(s, x(\phi(s)))ds\right), \quad B = \left(\sum_{j=1}^{n} b_j + 1\right)^{-1}. \tag{2.10}$$

Substitute from (2.10) into (2.8), we obtain

$$\begin{aligned} x(t) = A&\left\{x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s)))ds\right\} \\ &+ B\left(t - A\sum_{k=1}^{m} a_k \tau_k\right)\left\{x_1 - \sum_{j=1}^{n} b_j \int_0^{\eta_j} f(s, x(\phi(s)))ds\right\}, \\ &+ \int_0^t (t-s) f(s, x(\phi(s)))ds, \end{aligned} \tag{2.11}$$

which proves that the solution of the nonlocal problem (1.3)-(1.4) can be expressed by the integral equation (2.3). □

3. Existence of Solution

We study here the existence of at least one monotonic nondecreasing solution $x \in C[0,1]$ for the integral equation (2.3).

Theorem 3.1. *Assume that (i)–(iv) are satisfied. Then the nonlocal problem* (1.3)-(1.4) *has at least one solution* $x \in C[0,1]$.

Proof. Define the subset $Q_r \subset C(0,1)$ by $Q_r = \{x \in C : |x(t)| \leq r, r = (Ax_0 + Bx_1 + (3-B)\|a\|)/(1-(3-B)b), r > 0\}$. Clear the set Q_r which is nonempty, closed, and convex.

Let H be an operator defined by

$$
\begin{aligned}
(Hx)(t) = {} & A\left\{x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s)))ds\right\} \\
& + B\left(t - A\sum_{k=1}^{m} a_k \tau_k\right)\left\{x_1 - \sum_{j=1}^{n} b_j \int_0^{\eta_j} f(s, x(\phi(s)))ds\right\} \\
& + \int_0^t (t-s) f(s, x(\phi(s)))ds.
\end{aligned}
\tag{3.1}
$$

Let $x \in Q_r$, then

$$
\begin{aligned}
|(Hx)(t)| \leq {} & A\left\{x_0 + \sum_{k=1}^{m} a_k \int_0^{\tau_k} (\tau_k - s)\left|f(s, x(\phi(s)))\right|ds\right\} \\
& + B\left(t - A\sum_{k=1}^{m} a_k \tau_k\right)\left\{x_1 + \sum_{j=1}^{n} b_j \int_0^{\eta_j} \left|f(s, x(\phi(s)))\right|ds\right\} \\
& + \int_0^t (t-s)\left|f(s, x(\phi(s)))\right|ds \\
\leq {} & A\left\{x_0 + \sum_{k=1}^{m} a_k \int_0^{1} \left[|a(s)| + b\left|x(\phi(s))\right|\right]ds\right\} \\
& + B\left\{x_1 + \sum_{j=1}^{n} b_j \int_0^{1} \left[|a(s)| + b\left|x(\phi(s))\right|\right]ds\right\} \\
& + \int_0^1 \left[|a(s)| + b\left|x(\phi(s))\right|\right]ds \\
\leq {} & Ax_0 + \|a\| + b\sup_{t\in I}\left|x(\phi(t))\right| + Bx_1 + B\sum_{j=1}^{n} b_j \|a\| \\
& + bB\sum_{j=1}^{n} b_j \sup_{t\in I}\left|x(\phi(t))\right| + \|a\| + b\sup_{t\in I}\left|x(\phi(t))\right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq Ax_0 + Bx_1 + 2\|a\| + 2b\|x\| + (1-B)\|a\| + b(1-B)\|x\| \\
&\leq Ax_0 + Bx_1 + (3-B)\|a\| + (3-B)br \leq r,
\end{aligned}
\tag{3.2}
$$

then $H : Q_r \to Q_r$ and $\{Hx(t)\}$ is uniformly bounded in Q_r.

Also for $t_1, t_2 \in [0,1]$ such that $t_1 < t_2$, we have

$$
\begin{aligned}
(Hx)(t_2) - (Hx)(t_1) = {}& B\left(t_2 - A\sum_{k=1}^{m} a_k\tau_k\right)\left\{x_1 - \sum_{j=1}^{n} b_j\int_0^{\eta_j} f(s, x(\phi(t)))ds\right\} \\
&+ \int_0^{t_2}(t_2 - s) f(s, x(\phi(t)))ds \\
&- B\left(t_1 - A\sum_{k=1}^{m} a_k\tau_k\right)\left\{x_1 - \sum_{j=1}^{n} b_j\int_0^{\eta_j} f(s, x(\phi(t)))ds\right\} \\
&- \int_0^{t_1}(t_1 - s) f(s, x(\phi(t)))ds \\
= {}& B(t_2 - t_1)\left\{x_1 - \sum_{j=1}^{n} b_j\int_0^{\eta_j} f(\mathrm{s}, x(\phi(t)))ds\right\} \\
&+ \int_0^{t_1}(t_2 - t_1) f(s, x(\phi(t)))ds \\
&+ \int_{t_1}^{t_2}(t_2 - s) f(s, x(\phi(t)))ds.
\end{aligned}
\tag{3.3}
$$

Then

$$
\begin{aligned}
|(Hx)(t_2) - (Hx)(t_1)| \leq {}& B|t_2 - t_1|\left\{x_1 + \sum_{j=1}^{n} b_j\int_0^{\eta_j}\left[|a(s)| + b\left|x(\phi(s))\right|\right]ds\right\} \\
&+ |t_2 - t_1|\int_0^{t_1}\left[|a(s)| + b\left|x(\phi(s))\right|\right]ds \\
&+ \int_{t_1}^{t_2}(t_2 - s)\left[|a(s)| + b\left|x(\phi(s))\right|\right]ds
\end{aligned}
$$

$$\leq B|t_2 - t_1|x_1 + \sum_{j=1}^{n} b_j [\|a\| + br] + |t_2 - t_1|[\|a\| + br] + \int_{t_1}^{t_2} \|a\| ds + br[t_2 - t_1]. \tag{3.4}$$

The above inequality shows that

$$|(Hx)(t_2) - (Hx)(t_1)| \longrightarrow 0 \quad \text{as } t_2 \longrightarrow t_1. \tag{3.5}$$

Therefore $\{Hx(t)\}$ is equicontinuous. By the Arzelà-Ascoli theorem, $\{Hx(t)\}$ is relatively compact.

Since all conditions of the Schauder theorem hold, then H has a fixed point in Q_r which proves the existence of at least one solution $x \in C[0,1]$ of the integral equation (2.3), where

$$\begin{aligned} \lim_{t \to 0^+} x(t) &= A\left\{x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s))) ds\right\} \\ &\quad - BA \sum_{k=1}^{m} a_k \tau_k \left\{x_1 - \sum_{j=1}^{n} b_j \int_0^{\eta_j} f(s, x(\phi(s))) ds\right\} = x(0), \\ \lim_{t \to 1^-} x(t) &= A\left\{x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s))) ds\right\} \\ &\quad + B\left(1 - A\sum_{k=1}^{m} a_k \tau_k\right)\left\{x_1 - \sum_{j=1}^{n} b_j \int_0^{\eta_j} f(s, x(\phi(s))) ds\right\} \\ &\quad + \int_0^1 (1 - s) f(s, x(\phi(s))) ds = x(1). \end{aligned} \tag{3.6}$$

To complete the proof, we prove that the integral equation (2.3) satisfies nonlocal problem (1.3)-(1.4). Differentiating (2.3), we get

$$x'(t) = B\left\{x_1 - \sum_{j=1}^{n} b_j \int_0^{\eta_j} f(s, x(\phi(s))) ds\right\} + \int_0^t f(s, x(\phi(s))) ds, \tag{3.7}$$

$$x''(t) = f(t, x(\phi(t))). \tag{3.8}$$

Let $t = \tau_k$ in (2.3), we obtain

$$x(\tau_k) = A\left\{x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s)))ds\right\} + \int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s)))ds, \tag{3.9}$$

which proves

$$\sum_{k=1}^{m} a_k x(\tau_k) = x_0. \tag{3.10}$$

Also let $t = \eta_j$ in (3.7), we obtain

$$x'(\eta_j) = B\left\{x_1 - \sum_{j=1}^{n} b_j \int_0^{\eta_j} f(s, x(\phi(s)))ds\right\} + \int_0^{\eta_j} f(s, x(\phi(s)))ds, \tag{3.11}$$

then

$$\sum_{j=1}^{n} b_j x'(\eta_j) = B\sum_{j=1}^{n} b_j \left\{x_1 - \sum_{j=1}^{n} b_j \int_0^{\eta_j} f(s, x(\phi(s)))ds\right\} + \sum_{j=1}^{n} b_j \int_0^{\eta_j} f(s, x(\phi(s)))ds. \tag{3.12}$$

Let $t = 0$ in (3.7), we obtain

$$x'(0) = B\left\{x_1 - \sum_{j=1}^{n} b_j \int_0^{\eta_j} f(s, x(\phi(s)))ds\right\}. \tag{3.13}$$

Adding (3.12) and (3.13), we obtain

$$x'(0) + \sum_{j=1}^{n} b_j x'(\eta_j) = x_1. \tag{3.14}$$

This implies that there exists at least one solution $x \in C[0,1]$ of the nonlocal problem (1.3) and (1.4). This completes the proof. □

Corollary 3.2. *The solution of the problem* (1.3)-(1.4) *is monotonic nondecreasing.*

Proof. Let $t_1 < t_2$, we deduce from (2.3) that

$$\begin{aligned}
x(t_1) = A\left\{x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s)))ds\right\} \\
+ B\left(t_1 - A\sum_{k=1}^{m} a_k\tau_k\right)\left\{x_1 - \sum_{j=1}^{n} b_j \int_0^{\eta_j} f(s, x(\phi(s)))ds\right\} \\
+ \int_0^{t_1} (t_1 - s) f(s, x(\phi(s)))ds \\
< A\left\{x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s)))ds\right\} \\
+ B\left(t_2 - A\sum_{k=1}^{m} a_k\tau_k\right)\left\{x_1 - \sum_{j=1}^{n} b_j \int_0^{\eta_j} f(s, x(\phi(s)))ds\right\} \\
+ \int_0^{t_2} (t_2 - s) f(s, x(\phi(s)))ds = x(t_2),
\end{aligned} \tag{3.15}$$

which proves that the solution x of the problem (1.3)-(1.4) is monotonic nondecreasing. □

3.1. Positive Solution

Let $b_j = 0, j = 1, 2, \ldots n$ and $x_1 = 0$, then the nonlocal problem condition (1.4) will be

$$\sum_{k=1}^{m} a_k x(\tau_k) = x_0, \qquad x'(0) = 0. \tag{3.16}$$

Theorem 3.3. *Let the assumptions (i)–(iv) of Theorem 3.1 be satisfied. Then the solution of the nonlocal problem* (1.3)–(3.16) *is positive* $t \in [d, 1]$.

Proof. Let $b_j = 0, j = 1, 2, \ldots n$ and $x_1 = 0$ in the integral equation (2.3) and the nonlocal condition (1.4), then the solution of the nonlocal problem (1.3)–(3.16) will be given by the integral equation

$$x(t) = A\left\{x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s)))ds\right\} + \int_0^{t} (t - s)\ f(s, x(\phi(s)))ds, \tag{3.17}$$

where $A = \left(\sum_{k=1}^{m} a_k\right)^{-1}$.

Let $t \in [d, 1]$, then

$$\begin{aligned} &\int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s))) ds \le \int_0^t (t - s) f(s, x(\phi(s))) ds, \quad \tau_k \le t, \\ &\sum_{k=1}^m a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s))) ds \le \sum_{k=1}^m a_k \int_0^t (t - s) f(s, x(\phi(s))) ds. \end{aligned} \tag{3.18}$$

Multiplying by $A = (\sum_{k=1}^m a_k)^{-1}$, we obtain

$$\begin{aligned} A \sum_{k=1}^m a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s))) ds &\le A \sum_{k=1}^m a_k \int_0^t (t - s) f(s, x(\phi(s))) ds \\ &= \int_0^t (t - s) f(s, x(\phi(s))) ds, \end{aligned} \tag{3.19}$$

and the solution x of the nonlocal problem (1.3) and (3.16), given by the integral equation (3.17), is positive for $t \in [d, 1]$. This complete the proof. □

Example 3.4. Consider the nonlocal problem of the second-order functional differential equation (1.3) with two-point boundary condition

$$x'(0) = 0, \qquad x(\eta) = x_0, \quad \eta \in (a, d) \subset (0, 1). \tag{3.20}$$

Applying our results here, we deduce that the two-point boundary value problem (1.3)–(3.20) has at least one monotonic nondecreasing solution $x \in C[0, 1]$ represented by the integral equation

$$x(t) = x_0 - \int_0^\eta (\eta - s) f(s, x(\phi(s))) ds + \int_0^t (t - s) f(s, x(\phi(s))) ds. \tag{3.21}$$

This the solution is positive with $t > \eta$.

4. Nonlocal Integral Condition

Let $x \in C[0, 1]$ be the solution of the nonlocal problem (1.3) and (1.4).

Let $a_k = t_k - t_{k-1}, \tau_k \in (t_{k-1}, t_k) \subset (a, d) \subset (0, 1)$ and let $b_j = \xi_j - \xi_{j-1}, \eta_j \in (\xi_{j-1}, \xi_j) \subset (c, e) \subset (0, 1)$, then

$$\sum_{k=1}^m (t_k - t_{k-1}) x(\tau_k) = x_0, \qquad x'(0) + \sum_{j=1}^n (\xi_j - \xi_{j-1}) x'(\eta_j) = x_1. \tag{4.1}$$

From the continuity of the solution x of the nonlocal problem (1.3) and (1.4), we obtain

$$\lim_{m\to\infty}\sum_{k=1}^{m}(t_k - t_{k-1})x(\tau_k) = \int_a^d x(s)ds,$$
$$x'(0) + \lim_{n\to\infty}\sum_{j=1}^{n}(\xi_j - \xi_{j-1})x'(\eta_j) = x'(0) + \int_c^e x'(s)ds, \tag{4.2}$$

and the nonlocal condition (1.4) transformed to the integral condition

$$\int_a^d x(s)ds = x_0, \qquad x'(0) + x(e) - x(c) = x_1, \tag{4.3}$$

and the solution of the integral equation (2.3) will be

$$x(t) = (d-a)^{-1}\left\{x_0 - \int_a^d\int_0^t (t-s)f(s,x(\phi(s)))ds\,dt\right\}$$
$$+ ((b-c)+1)^{-1}(t-1)\left\{x_1 - \int_c^e\int_0^t f(s,x(\phi(s)))ds\,dt\right\} \tag{4.4}$$
$$+ \int_0^t f(s,x(\phi(s)))ds.$$

Now, we have the following theorem.

Theorem 4.1. *Let the assumptions (i)–(iv) of Theorem 3.1 be satisfied. Then the nonlocal problem*

$$x''(t) = f(t,x(\phi(t))), \quad t \in (0,1),$$
$$\int_a^d x(s)ds = x_0, \qquad x'(0) + x(e) - x(c) = x_1 \tag{4.5}$$

has at least one monotonic nondecreasing solution $x \in C[0,1]$ *represented by* (4.4).

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