

Research Article

A Generalization of Ćirić Quasicontractions

Erdal Karapınar,¹ Kieu Phuong Chi,² and Tran Duc Thanh²

¹ Department of Mathematics, Atilim University, İncek, 06836 Ankara, Turkey

² Department of Mathematics, Vinh University, 182 Le Duan, Vinh City, Vietnam

Correspondence should be addressed to Erdal Karapınar, erdalkarapinar@yahoo.com

Received 26 October 2011; Accepted 4 December 2011

Academic Editor: Simeon Reich

Copyright © 2012 Erdal Karapınar et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We proved a fixed point theorem for a class of maps that satisfy Ćirić's contractive condition dependent on another function. We presented an example to show that our result is a real generalization.

1. Introduction and Preliminaries

The fixed point theorems have extensive applications in many disciplines such as in mathematics, economics, and computer science (see, e.g., [1–5]). The Banach contraction mapping principle [6] is the main motivation of this theory. A self-mapping T on a metric space X is called a contraction if, for each $x, y \in X$, there exists a constant $k \in [0, 1)$ such that $d(Tx, Ty) \leq kd(x, y)$. Banach [6] proved that every contraction on a complete metric space has a unique fixed point.

Many authors have studied on various generalizations of Banach contraction mapping principle. For important and famous examples of such generalizations, we may cite Kannan [7], Reich [8–10], Chatterjea [11], Hardy and Rogers [12], and Ćirić [13, 14]. In this manuscript, we focused on Ćirić type contractions and generalized some of his results [13, 14].

In [14], Ćirić introduced a class of self-maps on a metric space (X, d) which satisfy the following condition:

$$d(Sx, Sy) \leq q \max\{d(x, y), d(x, Sx), d(y, Sy), d(x, Sy), d(y, Sx)\}, \quad (1.1)$$

for every $x, y \in X$ and $0 \leq q < 1$. The maps satisfying Condition (1.1) are said to be *quasicontractions*.

Let T be a self-mapping on a metric space (X, d) . For $Y \subset X$ and for each $x \in X$, we set (cf. [14, 15])

- (1) $\delta(Y) = \sup\{d(x, y) : x, y \in Y\}$,
- (2) $O(x, n) = \{x, Tx, T^2x, \dots, T^n x\}$ for $n \in \mathbb{N}$,
- (3) $O(x, \infty) = \{x, Tx, T^2x, \dots\}$.

A metric space (X, d) is called T -orbitally complete if and only if every Cauchy sequence of $O(x, \infty)$ (for some $x \in X$) converges to a point in X . It is clear that if X is complete, then X is S -orbitally complete for every $T : X \rightarrow X$. We now state the theorem of Ćirić [14].

Theorem 1.1. *Let T be a quasicontraction on T -orbitally complete metric space X . Then*

- (1) T has a unique fixed point u in X ,
- (2) $\lim_{n \rightarrow \infty} T^n x = u$, for $x \in X$.

Other generalizations of Banach's contraction principle and analogous results for metric spaces are abundantly present in the literature (see [7–19] and the references given therein). Rhoades [19] considered many types of contractive definitions and analyzed the relationships among them. Recently, some authors have provided a result on the existence of fixed points for a new class of contractive mappings.

Definition 1.2 (see, e.g., [20, 21]). Let (X, d) be a metric and $T, S : X \rightarrow X$ be two functions. A mapping S is called to be a T contraction if there exists $q \in (0, 1)$ such that

$$d(TSx, TSy) \leq qd(Tx, Ty), \quad \forall x, y \in X. \quad (1.2)$$

In the definition above, if we choose $Tx = x$ for all $x \in X$, then the T contraction becomes a contraction. In [20, 22], some fixed point theorems are proved for T contractions in metric spaces and in 2-metric spaces. In this paper, by the using the same techniques, we shall extend the Ćirić's theorem (Theorem 1.1). For this purpose, we recall the following elementary definition.

Definition 1.3 (see, e.g., [21, 22]). Let (X, d) be a metric space. A mapping $T : X \rightarrow X$ is called *sequentially convergent* if the statement $\{Ty_n\}$ is convergent and implies that $\{y_n\}$ is a convergent sequence for every sequence $\{y_n\}$.

2. Main Results

The aim of this section is to prove the following result.

Theorem 2.1. *Let (X, d) be a metric space. Let $T : X \rightarrow X$ be a one-to-one, continuous, and sequentially convergent mapping and S be a self-mapping of X . Suppose that there exists $q \in [0, 1)$,*

$$d(TSx, TSy) \leq q \max\{d(Tx, Ty), d(Tx, TSx), d(Ty, TSy), d(Tx, TSy), d(Ty, TSx)\}, \quad (2.1)$$

for every $x, y \in X$ and every Cauchy sequence of the form $\{TS^n x\}$ for $x \in X$ converges in X . Then

- (i) $d(TS^i, TS^j) \leq q\delta[O(Tx, n)]$, for all $i, j \in \{1, 2, \dots, n\}$, for all $x \in X$ and $n \in \mathbb{N}$,
- (ii) $\delta[O(Tx, \infty)] \leq 1/(1 - q)d(Tx, TSx)$, for all $x \in X$,
- (iii) S has a unique fixed point $b \in X$,
- (iv) $\lim TS^n x = Tb$.

Proof. We will mainly follow the arguments in the proof of the Ćirić's theorem. Let T, S be defined as in Theorem 2.1. We start with the proof of (i). Let $x \in X$ be arbitrary element. Let $n \geq 1$ be integer and $i, j \in \{1, 2, \dots, n\}$. Then

$$TS^{i-1}x, TS^i, TS^{j-1}x, TS^jx \in O(Tx, n). \quad (2.2)$$

Note that we use S^0 as the identity self-mapping. Due to (2.1), we have

$$\begin{aligned} d(TS^i x, TS^j x) &= d(TS(S^{i-1}x), TS(S^{j-1}x)) \\ &\leq q \max\{d(TS^{i-1}x, TS^{j-1}x); d(TS^{i-1}x, TS^i x), \\ &\quad d(TS^{j-1}x, TS^j x), d(TS^{i-1}x, TS^j x), d(TS^i x, TS^{j-1}x)\} \\ &\leq q\delta[O(Tx, n)]. \end{aligned} \quad (2.3)$$

Thus, (i) is proved. Let us show (ii). Observe that

$$\delta[O(Tx, 1)] \leq \delta[O(Tx, 2)] \leq \dots \leq \delta[O(Tx, n)] \leq \delta[O(Tx, n+1)] \leq \dots, \quad (2.4)$$

which implies that

$$\delta[O(Tx, \infty)] = \sup\{\delta[O(Tx, n)] : n \in \mathbb{N}\}. \quad (2.5)$$

So we only need to show that $\delta[O(Tx, n)] \leq 1/(1 - q)d(Tx, TSx)$ for all $n \in \mathbb{N}$. Indeed, since $O(Tx, n) = \{Tx, TSx, \dots, TS^n x\}$ is a finite set for each n and

$$(TS^i, TS^j) \leq q\delta[O(Tx, n)], \quad \forall i, j \in \{1, 2, \dots, n\} \quad (2.6)$$

with $0 \leq q < 1$, we infer that there exists $k \in \{1, 2, \dots, n\}$ such that

$$d(Tx, TS^k x) = \delta[O(Tx, n)]. \quad (2.7)$$

Applying the triangle inequality and by (i), we get that

$$\begin{aligned} d(Tx, TS^k x) &\leq d(Tx, TSx) + d(TSx, TS^k x) \\ &\leq d(Tx, TSx) + q\delta[O(Tx, n)] \\ &= d(Tx, TSx) + qd(Tx, TS^k x). \end{aligned} \quad (2.8)$$

Hence, $\delta[O(Tx, n)] = d(Tx, TS^k x) \leq 1/(1 - q)d(Tx, TSx)$. Thus, (ii) is proved. \square

To prove (iii), choose $x_0 \in X$ and define the iterative sequence $\{x_n\}$ and $\{y_n\}$ as follows:

$$x_{n+1} = Sx_n = S^{n+1}x_0, \quad y_n = Tx_n = TS^n x_0, \quad n = 0, 1, 2, \dots \quad (2.9)$$

We assert that $\{y_n\}$ is a Cauchy sequence. Without loss of generality, we may assume that $n < m$, where $n, m \in \mathbb{N}$. By (i), we obtain

$$\begin{aligned} d(y_n, y_m) &= d(TS^n x_0, TS^m x_0) \\ &= d(TSS^{n-1} x_0, TS^{m-n+1} S^{n-1} x_0) \\ &\leq q\delta[O(TS^{n-1} x_0, m - n + 1)]. \end{aligned} \quad (2.10)$$

According to the assumption $n < m$, there exists $l \in \{1, m - n + 1\}$ such that

$$\delta[O(TS^{n-1} x_0, m - n + 1)] = d(TS^{n-1} x_0, TS^l S^{n-1} x_0). \quad (2.11)$$

Again, by the mentioned assumption, we have

$$\begin{aligned} d(TS^{n-1} x_0, TS^l S^{n-1} x_0) &= d(TSS^{n-2} x_0, TS^{l+1} S^{n-2} x_0) \\ &\leq q\delta[O(TS^{n-2} x_0, l + 1)]. \end{aligned} \quad (2.12)$$

So we can get the following system of inequalities:

$$d(y_n, y_m) = d(TS^n x_0, TS^m x_0) \leq q\delta[O(TS^{n-1} x_0, m - n + 1)] \leq q^2\delta[O(TS^{n-2} x_0, m - n + 2)]. \quad (2.13)$$

By the same argument, we obtain

$$d(y_n, y_m) = d(TS^n x_0, TS^m x_0) \leq \dots \leq q^n \delta[O(TSx_0, m)]. \quad (2.14)$$

Since $\delta[O(TSx_0, m)] \leq \delta[O(TSx_0, \infty)] \leq 1/(1-q)d(Tx_0, TSx_0)$, we have

$$d(y_n, y_m) = d(TS^n x_0, TS^m x_0) \leq \frac{q^n}{1-q} d(Tx_0, TSx_0). \quad (2.15)$$

This implies that $\{y_n\} = \{TS^n x_0\}$ is a Cauchy sequence. Since every Cauchy sequence of the form $\{TS^n x_0\}$ converges in X , we deduce that

$$\lim TS^n x_0 = a \in X. \quad (2.16)$$

Since T is sequentially convergent, it follows that $\lim S^n x_0 = b \in X$. By the hypothesis that T is continuous, we get

$$\lim TS^n x_0 = Tb. \quad (2.17)$$

Therefore, $Tb = a$. We shall show that $Sb = b$. To achieve this, we consider the following inequalities:

$$\begin{aligned} d(TSb, Tb) &\leq d(Tb, TS^{n+1}x_0) + d(TS^{n+1}x_0, TSb) \\ &\leq d(Tb, TS^{n+1}x_0) + q \max\{d(Tb, TS^n x_0), d(Tb, TSb), \\ &\quad d(TS^n x_0, TS^{n+1}x_0), d(Tb, TS^{n+1}x_0), d(TS^n x_0, TSb)\} \\ &\leq d(Tb, TS^{n+1}x_0) + q \max\{d(Tb, TS^n x_0), d(Tb, TSb), d(TS^n x_0, TS^{n+1}x_0), \\ &\quad d(Tb, TS^{n+1}x_0), d(TS^n x_0, Tb) + d(Tb, TSb)\} \\ &\leq d(Tb, TS^{n+1}x_0) + q [d(Tb, TS^{n+1}x_0) + d(TS^n x_0, TS^{n+1}x_0) \\ &\quad + d(TS^n x_0, Tb) + d(Tb, TSb)]. \end{aligned} \quad (2.18)$$

Hence,

$$d(Tb, TSb) \leq \frac{1}{1-q} [(1+q)d(Tb, TS^{n+1}x_0) + qd(Tb, TS^n x_0) + qd(TS^n x_0, TS^{n+1}x_0)]. \quad (2.19)$$

Letting $n \rightarrow \infty$ and combining with the fact that $\lim TS^n x_0 = Tb$, we can conclude that

$$d(TSb, Tb) = 0. \quad (2.20)$$

Therefore, $TSb = Tb$. Since T is one to one, we obtain that $Sb = b$. Now we will show that b is the unique fixed point of S . Suppose that b and b' are fixed points of S . Then $Sb = b$, $Sb' = b'$, and applying (2.1),

$$\begin{aligned} d(Tb, Tb') &= d(TSb, TSb') \\ &\leq q \max\{d(Tb, TSb), d(Tb', TSb'), d(Tb, TSb'), d(Tb', TSb), d(Tb, Tb')\} \\ &= qd(Tb, Tb'). \end{aligned} \quad (2.21)$$

Since $0 \leq q < 1$, we infer that $d(Tb, Tb') = 0$. Hence, $Tb = Tb'$. Since T is one-to-one, we may conclude that $b = b'$. This proves (iii) of Theorem 2.1. Proof of (iv) is directly obtained from (2.17). The theorem is proved.

Remark 2.2. Theorem 1.1 is obtained from Theorem 2.1 if we choose $Tx = x$ for every $x \in X$.

The following example shows that Theorem 2.1 is indeed an extension of Theorem 1.1.

Example 2.3. Let $X = [0, +\infty)$ with metric induced by $d(x, y) = |x - y|$. Then X is a complete metric space. Consider the function $Sx = x^2/(x + 1)$ for all $x \in X$. It is easy to compute that

$$\begin{aligned} d(Sx, S2x) &= \frac{4x^2}{2x+1} - \frac{x^2}{x+1} = \frac{x^2(2x+3)}{(2x+1)(x+1)}, \\ d(x, Sx) &= x - \frac{x^2}{x+1} = \frac{x}{x+1}, \\ d(2x, S2x) &= 2x - \frac{4x^2}{2x+1} = \frac{2x}{2x+1}, \\ d(x, S2x) &= \left| \frac{4x^2}{2x+1} - x \right| = \left| \frac{2x^2 - x}{2x+1} \right|, \\ d(2x, Sx) &= 2x - \frac{x^2}{x+1} = \frac{x^2 + 2x}{x+1}, \\ d(x, 2x) &= x. \end{aligned} \quad (2.22)$$

Therefore, if x is large enough, we have

$$\max\{d(x, 2x), d(x, Sx), d(2x, S2x), d(x, S2x), d(2x, Sx)\} = \frac{x^2 + 2x}{x+1}. \quad (2.23)$$

It follows that the quasicontractive condition

$$d(Sx, S2x) \leq q \max\{d(x, 2x), d(x, Sx), d(2x, S2x), d(x, S2x), d(2x, Sx)\} \quad (2.24)$$

is equivalent to

$$\frac{x(2x+3)}{(2x+1)(x+2)} \leq q. \quad (2.25)$$

Letting $x \rightarrow \infty$, we get that $q \geq 1$ which contradicts the fact that $q \in (0, 1)$. Hence S is not a quasicontraction. Therefore, we can not apply Theorem 1.1 for S . However, it is easy to see that 0 is the unique fixedpoint of S .

Now we shall show that S satisfies Theorem 2.1 with $Tx = e^x - 1$ and $q = 1/2$. Indeed, T is one-to-one, continuous and sequentially convergent on $[0, +\infty)$. We have

$$\begin{aligned} d(TSx, TSy) &= \left| e^{x^2/(x+1)} - e^{y^2/(y+1)} \right|, \\ d(Tx, Ty) &= |e^x - e^y|. \end{aligned} \tag{2.26}$$

Now, we will show that

$$d(TSx, TSy) \leq \frac{1}{2}d(Tx, Ty). \tag{2.27}$$

If $x = y$, then the inequality above holds. So we can assume that $x > y$. Then (2.27) is equivalent to

$$\left| e^{x^2/(x+1)} - e^{y^2/(y+1)} \right| \leq \frac{1}{2}|e^x - e^y|, \quad \forall x > y. \tag{2.28}$$

It follows from the fact that ϕ and ψ are increasing, (2.28) reduces to

$$e^{x^2/(x+1)} - \frac{e^x}{2} \leq e^{y^2/(y+1)} - \frac{e^y}{2}, \quad \forall x > y. \tag{2.29}$$

Consider the function $\varphi(t) = e^{t^2/(t+1)} - e^t/2$, ($t \geq 0$). It is easy to check that φ is decreasing on $[0, +\infty)$. So we can deduce that

$$\begin{aligned} d(TSx, TSy) &\leq \frac{1}{2}d(Tx, Ty) \\ &\leq \frac{1}{2} \max\{d(Tx, Ty); d(Tx, TSx); \\ &\quad d(Ty, TSy), d(Tx, TSy), d(Ty, TSx)\} \end{aligned} \tag{2.30}$$

for every $x, y \in X$. Therefore, we can apply Theorem 2.1 for S .

As an immediate consequence of Theorem 2.1, we have the following corollary.

Corollary 2.4. *Let (X, d) be a metric space. Let $T : X \rightarrow X$ be a one-to-one, continuous, and sequentially convergent mapping, and let S be a self-mapping of X . Suppose that there exists $q \in [0, 1)$,*

$$d(TSx, TSy) \leq q \max \left\{ d(Tx, Ty), d(Tx, TSx), d(Ty, TSy), \frac{d(Tx, TSy) + d(Ty, TSx)}{2} \right\} \tag{2.31}$$

for every $x, y \in X$ and every Cauchy sequence of the form $\{TS^n x\}$ for $x \in X$ converges in X . Then,

- (a) S has a unique fixed point $b \in X$,
- (b) $\lim TS^n x = Tb$ for $x \in X$.

This corollary is also a consequence of the main result of [21], which is established by a contractive condition of integral type.

We also get the following corollary.

Corollary 2.5. Let (X, d) be a metric space. Let $T : X \rightarrow X$ be a one-to-one, continuous, and sequentially convergent mapping and S be a self-mapping of X . Suppose that there exists $q \in [0, 1)$,

$$d(TS^k x, TS^k y) \leq q \max \left\{ d(Tx, Ty), d(Tx, TS^k x), \right. \\ \left. d(Ty, TS^k y), d(Tx, TS^k y), d(Ty, TS^k x) \right\} \quad (2.32)$$

for every $x, y \in X$, for some positive integer k and every Cauchy sequence of the form $\{TS^n x\}$ for $x \in X$ converges in X . Then, S has a unique fixed point $b \in X$.

Proof. Applying Theorem 2.1 for the map S^k , we can deduce that S^k has a unique fixed point $b \in X$. This yields that $S^k b = b$ and $S^k(Sb) = S^{k+1}b = S(S^k b) = Sb$. It follows that Sb is a fixed point of S^k . By the uniqueness of fixed point of S^k , we infer that $Sb = b$. Therefore, b is a fixed point of S . Now, if a is a fixed point of S , then $S^k a = S^{k-1}a = \dots = Sa = a$. Hence, a is a fixed point of S^k , so that $a = b$. This proves the uniqueness of fixed point of S . \square

The following corollary is an extension of the main result of [12].

Corollary 2.6. Let (X, d) be a metric space. Let $T : X \rightarrow X$ be a one-to-one, continuous, and sequentially convergent mapping, and let S be a self-mapping of X . Suppose that there exists $a_i \geq 0$, $i = 1, 2, \dots, 5$ satisfying that $\sum_{i=1}^n a_i < 1$ such that

$$d(TSx, TSy) \leq a_1 d(Tx, Ty) + a_2 d(Tx, TSx) + a_3 d(Ty, TSy) + a_4 d(Tx, TSy) + a_5 d(Ty, TSx) \quad (2.33)$$

for every $x, y \in X$ and every Cauchy sequence of the form $\{TS^n x\}$ for $x \in X$ converges in X . Then

- (a) S has a unique fixed point $b \in X$.
- (b) $\lim TS^n x = Tb$.

Proof. It is easy to see that the condition (2.1) is a consequence of the condition (2.33). This is enough to prove the corollary. \square

Remark 2.7. In the same way in Corollary 2.6, we can get the extensions of Kannan's contractions (see [7]), and Reich's contractions (see [8–10]).

References

- [1] K. C. Border, *Fixed Point Theorems with Applications to Economics and Game Theory*, Cambridge University Press, New York, NY, USA, 1985.

- [2] E. A. Ok, *Real Analysis with Economic Applications*, Princeton University Press, Princeton, NJ, USA, 2007.
- [3] J. W. de Bakker and J. I. Zucker, "Processes and the denotational semantics of concurrency," *Information and Control*, vol. 54, no. 1-2, pp. 70–120, 1982.
- [4] R. Heckmann, "Approximation of metric spaces by partial metric spaces," *Applied Categorical Structures*, vol. 7, no. 1-2, pp. 71–83, 1999.
- [5] S. G. Matthews, "Partial metric topology," Research Report 212, Department of Computer Science, University of Warwick, 1992.
- [6] S. Banach, "Sur les operations dans les ensembles abstraits et leur application aux equations itegrales," *Fundamenta Mathematicae*, vol. 3, pp. 133–181, 1922.
- [7] R. Kannan, "Some results on fixed points," *Bulletin of the Calcutta Mathematical Society*, vol. 60, pp. 71–76, 1968.
- [8] S. Reich, "Kannan's fixed point theorem," *Bollettino della Unione Matematica Italiana IV*, vol. 4, no. 4, pp. 1–11, 1971.
- [9] S. Reich, "Some remarks concerning contraction mappings," *Canadian Mathematical Bulletin*, vol. 14, pp. 121–124, 1971.
- [10] S. Reich, "Fixed points of contractive functions," *Bollettino della Unione Matematica Italiana IV*, vol. 5, pp. 26–42, 1972.
- [11] S. K. Chatterjea, "Fixed-point theorems," *Comptes Rendus de l'Académie Bulgare des Sciences*, vol. 25, pp. 727–730, 1972.
- [12] G. E. Hardy and T. D. Rogers, "A generalization of a fixed point theorem of Reich," *Canadian Mathematical Bulletin*, vol. 16, pp. 201–206, 1973.
- [13] L. B. Ćirić, "Generalized contraction and fixed point theorems," *Publications de l'Institut Mathématique*, vol. 12, no. 26, pp. 19–26, 1971.
- [14] L. B. Ćirić, "A generalization of Banach's contraction principle," *Proceedings of the American Mathematical Society*, vol. 45, pp. 267–273, 1974.
- [15] L. Ćirić, "On contraction type mappings," *Mathematica Balkanica*, vol. 1, pp. 52–57, 1971.
- [16] M. Edelstein, "On fixed and periodic points under contractive mappings," *Journal of the London Mathematical Society II*, vol. 37, pp. 74–79, 1962.
- [17] E. Karapınar, "Fixed point theory for cyclic weak ϕ -contraction," *Applied Mathematics Letters. An International Journal of Rapid Publication*, vol. 24, no. 6, pp. 822–825, 2011.
- [18] E. Karapınar, "Fixed point theorems in cone Banach spaces," *Fixed Point Theory and Applications*, vol. 2009, Article ID 609281, 9 pages, 2009.
- [19] B. E. Rhoades, "A comparison of various definitions of contractive mappings," *Transactions of the American Mathematical Society*, vol. 226, pp. 257–290, 1977.
- [20] K. P. Chi and H. T. Thuy, "A fixed point theorem in 2-metric spaces for a class of maps that satisfy a contractive condition dependent on an another function," *Lobachevskii Journal of Mathematics*, vol. 31, no. 4, pp. 338–346, 2010.
- [21] S. Moradi and M. Omid, "A fixed-point theorem for integral type inequality depending on another function," *International Journal of Mathematical Analysis*, vol. 4, no. 29–32, pp. 1491–1499, 2010.
- [22] K. P. Chi, "On a fixed point theorem for certain class of maps satisfying a contractive condition depended on an another function," *Lobachevskii Journal of Mathematics*, vol. 30, no. 4, pp. 289–291, 2009.



Hindawi

Submit your manuscripts at
<http://www.hindawi.com>

