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Research Article

Some Delay Integral Inequalities on Time Scales and Their Applications in the Theory of Dynamic Equations

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We establish some delay integral inequalities on time scales, which on one hand provide a handy tool in the study of qualitative as well as quantitative properties of solutions of certain delay dynamic equations on time scales and on the other hand unify some known continuous and discrete results in the literature.

1. Introduction

During the past decades, with the development of the theory of differential and integral equations as well as difference equations, a lot of integral and difference inequalities have been discovered (e.g., see [1–13] and the references therein), which play an important role in the research of boundedness, global existence, stability of solutions of differential and integral equations as well as difference equations. On the other hand, Hilger [14] initiated the theory of time scales as a theory capable to contain both difference and differential calculus in a consistent way. Since then many authors have expounded on various aspects of the theory of dynamic equations on time scales including various inequalities on time scales (e.g., see [15–24], and the references therein). However, delay integral inequalities on time scales have been paid little attention so far. Recent results in this direction include the works of Li [25] and Ma and Pečarić [26] to our best knowledge.

In this paper, we will establish some new delay integral inequalities on time scales, which unify some known continuous and discrete results in the literature. New explicit bounds for unknown functions concerned are obtained due to the presented inequalities. Some applications will be presented for the established results.

Throughout this paper, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_+ = [0, \infty)$, while $\mathbb{Z}$ denotes the set of integers. For two given sets G, H, we denote the set of maps from G to H by (G, H).

2. Some Preliminaries on Time Scales

A time scale is an arbitrary nonempty closed subset of the real numbers. In this paper, $\mathbb{T}$ denotes an arbitrary time scale. On $\mathbb{T}$ we define the forward and backward jump operators $\sigma \in (\mathbb{T}, \mathbb{T})$ and $\rho \in (\mathbb{T}, \mathbb{T})$ such that $\sigma(t) = \inf\{s \in \mathbb{T}, s > t\}$, $\rho(t) = \sup\{s \in \mathbb{T}, s < t\}$.

Definition 2.1. The graininess $\mu \in (\mathbb{T}, \mathbb{R}_+)$ is defined by $\mu(t) = \sigma(t) - t$.

Remark 2.2. Obviously, $\mu(t) = 0$ if $\mathbb{T} = \mathbb{R}$, while $\mu(t) = 1$ if $\mathbb{T} = \mathbb{Z}$.

Definition 2.3. A point $t \in \mathbb{T}$ is said to be left-dense if $\rho(t) = t$ and $t \neq \inf \mathbb{T}$, right-dense if $\sigma(t) = t$ and $t \neq \sup \mathbb{T}$, left-scattered if $\rho(t) < t$, and right-scattered if $\sigma(t) > t$.

Definition 2.4. The set $\mathbb{T}^\kappa$ is defined to be $\mathbb{T}$ if $\mathbb{T}$ does not have a left-scattered maximum, otherwise it is $\mathbb{T}$ without the left-scattered maximum.

Definition 2.5. A function $f \in (\mathbb{T}, \mathbb{R})$ is called rd-continuous if it is continuous in right-dense points and if the left-sided limits exist in left-dense points, while f is called regressive if $1 + \mu(t)f(t) \neq 0$. C_{rd} denotes the set of rd-continuous functions, while $\mathfrak{R}$ denotes the set of all regressive and rd-continuous functions, and $\mathfrak{R}^+ = \{f \mid f \in \mathfrak{R}, 1 + \mu(t)f(t) > 0, \quad \text{for all } t \in \mathbb{T}\}$.

Definition 2.6. For some $t \in \mathbb{T}^\kappa$, and a function $f \in (\mathbb{T}, \mathbb{R})$, the *delta derivative* of f at t is denoted by $f^\Delta(t)$ (provided it exists) with the property such that for every $\varepsilon > 0$ there exists a neighborhood $\mathfrak{U}$ of t satisfying

$$\left| f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s) \right| \leq \varepsilon |\sigma(t) - s| \quad \forall s \in \mathfrak{U}. \tag{2.1}$$

Remark 2.7. If $\mathbb{T} = \mathbb{R}$, then $f^\Delta(t)$ becomes the usual derivative $f'(t)$, while $f^\Delta(t) = f(t+1) - f(t)$ if $\mathbb{T} = \mathbb{Z}$, which represents the forward difference.

Definition 2.8. For a, $b \in \mathbb{T}$ and a function $f \in (\mathbb{T}, \mathbb{R})$, the *Cauchy integral* of f is defined by

$$\int_a^b f(t)\Delta t = F(b) - F(a), \tag{2.2}$$

where $F^\Delta(t) = f(t)$, $t \in \mathbb{T}^\kappa$.

The following two theorems include some important properties for *delta derivative* and the Cauchy integral on time scales.

Theorem 2.9 (see [27]). *If $f, g \in (\mathbb{T}, \mathbb{R})$, and $t \in \mathbb{T}^\kappa$, then*

(i)

$$f^{\Delta}(t) = \begin{cases} \dfrac{f(\sigma(t)) - f(t)}{\mu(t)} & \text{if } \mu(t) \neq 0, \\ \lim\limits_{s \to t} \dfrac{f(t) - f(s)}{t - s} & \text{if } \mu(t) = 0. \end{cases} \tag{2.3}$$

(ii) *If f, g are delta differentials at t, then fg is also delta differential at t, and*

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t). \tag{2.4}$$

Theorem 2.10 (see [27]). *If $a, b, c \in \mathbb{T}$, $\alpha \in \mathbb{R}$, and f, $g \in C_{\mathrm{rd}}$, then*

(i) $\int_a^b [f(t) + g(t)]\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t$,

(ii) $\int_a^b (\alpha f)(t)\Delta t = \alpha \int_a^b f(t)\Delta t$,

(iii) $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t$,

(iv) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t$,

(v) $\int_a^a f(t)\Delta t = 0$,

(vi) *if $f(t) \geq 0$ for all $a \leq t \leq b$, then $\int_a^b f(t)\Delta t \geq 0$.*

Definition 2.11. The cylinder transformation $\xi_h : \mathbb{C}_h \to \mathbb{Z}_h$ is defined by

$$\xi_h(z) = \begin{cases} \dfrac{\mathrm{Log}(1 + hz)}{h}, & \text{if } h \neq 0 \left(\text{for } z \neq -\dfrac{1}{h}\right), \\ z, & \text{if } h = 0, \end{cases} \tag{2.5}$$

where Log is the principal logarithm function.

Definition 2.12. For $p \in \mathfrak{R}$ and $s, t \in \mathbb{T}$, the *exponential function* is defined by

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right). \tag{2.6}$$

Remark 2.13. If $\mathbb{T} = \mathbb{R}$, then for s, $t \in \mathbb{R}$, $e_p(t, s) = \exp(\int_s^t p(\tau)d\tau)$. If $\mathbb{T} = \mathbb{Z}$, then for s, $t \in \mathbb{Z}$ and $s < t$, $e_p(t, s) = \prod_{\tau=s}^{t-1}[1 + p(\tau)]$.

The following two theorems include some known properties on the *exponential function*.

Theorem 2.14 (see [28]). *If $p \in \mathfrak{R}$, then the following conclusions hold:*

(i) $e_p(t, t) \equiv 1$, *and* $e_0(t, s) \equiv 1$,

(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$,

(iii) $e_p(t, s) = 1/e_p(s, t) = e_{\ominus p}(s, t)$,

where $\ominus p = -p/(1 + \mu p)$.

Theorem 2.15 (see [28]). *If* $p \in \Re$*, and fix* $t_0 \in \mathbb{T}$*, then the exponential function* $e_p\ (t, t_0)$ *is the unique solution of the following initial value problem:*

$$\begin{aligned} y^{\Delta}(t) &= p(t)y(t), \\ y(t_0) &= 1. \end{aligned} \tag{2.7}$$

For more details about the calculus of time scales, we advise to refer to [29].

3. Main Results

In the rest of this paper, for the sake of convenience, we denote $\mathbb{T}_0 = [t_0, \infty) \cap \mathbb{T}$, where $t_0 \in \mathbb{T}$, and always assume $\mathbb{T}_0 \subset \mathbb{T}^{\kappa}$.

Lemma 3.1 (see [30, Theorem 2.2]). *Let* $t_0 \in \mathbb{T}^{\kappa}$ *and* $\omega : \mathbb{T} \times \mathbb{T}^{\kappa} \to \mathbb{R}$ *be continuous at* (t, t)*, where* $t \geq t_0$*,* $t \in \mathbb{T}^{\kappa}$ *with* $t > t_0$*. Assume that* $\omega^{\Delta}(t, \cdot)$ *is rd-continuous on* $[t_0, \sigma(t)]$*. If for any* $\varepsilon > 0$*, there exists a neighborhood* U *of* t*, independent of* $\tau \in [t_0, \sigma(t)]$*, such that*

$$\left|\omega(\sigma(t), \tau) - \omega(s, \tau) - \omega^{\Delta}(t, \tau)(\sigma(t) - s)\right| \leq \varepsilon|\sigma(t) - s| \quad \forall s \in U, \tag{3.1}$$

where ω^{Δ} *denotes the derivative of* ω *with respect to the first variable, then*

$$g(t) := \int_{t_0}^{t} \omega(t, \tau)\Delta\tau \tag{3.2}$$

implies

$$g^{\Delta}(t) = \int_{t_0}^{t} \omega^{\Delta}(t, \tau)\Delta\tau + \omega(\sigma(t), t). \tag{3.3}$$

Theorem 3.2. *Suppose* $u \in C_{\mathrm{rd}}(\mathbb{T}_0, \mathbb{R}_+)$*,* $\omega \in (\mathbb{R}_+, \mathbb{R}_+)$ *with* $\omega(u) > 0$ *for* $u > 0$*, and* ω *is nondecreasing.* $\tau_i \in (\mathbb{T}_0, \mathbb{T})$ *with* $\tau_i(t) \leq t$*,* $i = 1, 2$*, and* $-\infty < \alpha = \inf\{\min\{\tau_i(t),\ i = 1, 2\},\ t \in \mathbb{T}_0\} \leq t_0$*.* $f, g \in C_{\mathrm{rd}}([\alpha, \infty) \cap \mathbb{T}, \mathbb{R}_+)$*.* p, q, C *are constants, and* $p > q > 0$*,* $C \geq 0$*. If for* $t \in \mathbb{T}_0$*,* $u(t)$ *satisfies the following inequality:*

$$u^{p}(t) \leq C^{p/(p-q)} + \frac{p}{p-q}\int_{t_0}^{t}\left[f(\tau_1(s))u^{q}(\tau_1(s))\omega(u(\tau_1(s))) + g(\tau_2(s))u^{q}(\tau_2(s))\right]\Delta s \tag{3.4}$$

with the initial condition

$$\begin{aligned} &u(t) = \phi(t), \quad t \in [\alpha, t_0] \cap \mathbb{T}, \\ &\phi(\tau_i(t)) \le C^{1/(p-q)}, \quad \forall t \in \mathbb{T}_0,\ \tau_i(t) \le t_0,\ i = 1, 2, \end{aligned} \tag{3.5}$$

where $\phi \in C_{\mathrm{rd}}([\alpha, t_0] \cap \mathbb{T}, \mathbb{R}_+)$, *then,*

$$u(t) \le \left\{ G^{-1}\left[G\left(C + \int_{t_0}^{t} g(\tau_2(s))\Delta s \right) + \int_{t_0}^{t} f(\tau_1(s))\Delta s \right] \right\}^{1/(p-q)}, \quad t \in \mathbb{T}_0, \tag{3.6}$$

where $G(x) = \int_1^x 1/\omega(r^{1/(p-q)})dr,\ x > 0$ *with* $G(\infty) = \infty$, *and* G^{-1} *is the inverse of* G.

Proof. Assume $C > 0$. Denote the right side of (3.4) by $z(t)$. Then

$$u(t) \le z^{1/p}(t), \quad t \in \mathbb{T}_0. \tag{3.7}$$

If $\tau_i(t) \ge t_0$, for $t \in \mathbb{T}_0$, since $\tau_i(t) \le t$, then $\tau_i(t) \in \mathbb{T}_0$, and from (3.7) we have

$$u(\tau_i(t)) \le z^{1/p}(\tau_i(t)) \le z^{1/p}(t), \quad i = 1, 2. \tag{3.8}$$

If $\tau_i(t) \le t_0$, from (3.5) we obtain

$$u(\tau_i(t)) = \phi(\tau_i(t)) \le C^{1/(p-q)} \le z^{1/p}(t), \quad i = 1, 2. \tag{3.9}$$

So from (3.8) and (3.9) we always have

$$u(\tau_i(t)) \le z^{1/p}(t), \quad i = 1, 2,\ t \in \mathbb{T}_0. \tag{3.10}$$

Furthermore,

$$\begin{aligned} z^{\Delta}(t) &= \frac{p}{p-q}\left[f(\tau_1(t))u^q(\tau_1(t))\omega(u(\tau_1(t))) + g(\tau_2(t))u^q(\tau_2(t)) \right] \\ &\le \frac{p}{p-q}\left[f(\tau_1(t))z^{q/p}(t)\omega\left(z^{1/p}(t)\right) + g(\tau_2(t))z^{q/p}(t) \right], \end{aligned} \tag{3.11}$$

that is,

$$\frac{z^{\Delta}(t)}{z^{q/p}(t)} \le \frac{p}{p-q}\left[f(\tau_1(t))\omega\left(z^{1/p}(t)\right) + g(\tau_2(t)) \right], \tag{3.12}$$

According to [29, Theorem 1.90], considering $z^{\Delta}(t) \geq 0$, we have

$$\begin{aligned}\left(\frac{p}{p-q} z^{(p-q)/p}(t)\right)^{\Delta} &= z^{\Delta}(t) \int_0^1 \left[z(t) + h\mu(t) z^{\Delta}(t)\right]^{-q/p} dh \\ &= \frac{z^{\Delta}(t)}{z^{q/p}(t)} \int_0^1 \left[1 + h\mu(t) \frac{z^{\Delta}(t)}{z(t)}\right]^{-q/p} dh \\ &\leq \frac{z^{\Delta}(t)}{z^{q/p}(t)}.\end{aligned} \tag{3.13}$$

Combining (3.12) and (3.13), we obtain

$$\left(z^{(p-q)/p}(t)\right)^{\Delta} \leq f(\tau_1(t))\omega\left(z^{1/p}(t)\right) + g(\tau_2(t)). \tag{3.14}$$

Setting $t = s$ in (3.14), an integration with respect to s from t_0 to t yields

$$z^{(p-q)/p}(t) - z^{(p-q)/p}(t_0) \leq \int_{t_0}^{t} \left[f(\tau_1(s))\omega\left(z^{1/p}(s)\right) + g(\tau_2(s))\right]\Delta s. \tag{3.15}$$

Since $z(t_0) = C^{p/(p-q)}$, then (3.15) implies

$$z(t) \leq \left\{C + \int_{t_0}^{t} \left[f(\tau_1(s))\omega\left(z^{1/p}(s)\right) + g(\tau_2(s))\right]\Delta s\right\}^{p/(p-q)}. \tag{3.16}$$

Fix $T \in \mathbb{T}_0$, and let $t \in [t_0, T] \cap \mathbb{T}$. Then,

$$z(t) \leq \left\{C + \int_{t_0}^{T} g(\tau_2(s))\Delta s + \int_{t_0}^{t} f(\tau_1(s))\omega\left(z^{1/p}(s)\right)\Delta s\right\}^{p/(p-q)}. \tag{3.17}$$

Denote $v(t)$ by $C + \int_{t_0}^{T} g(\tau_2(s))\Delta s + \int_{t_0}^{t} f(\tau_1(s))\omega(z^{1/p}(s))\Delta s$. Then,

$$z(t) \leq v^{p/(p-q)}(t), \quad t \in [t_0, T] \cap \mathbb{T}, \tag{3.18}$$

$$v^{\Delta}(t) = f(\tau_1(t))\omega\left(z^{1/p}(t)\right) \leq f(\tau_1(t))\omega\left(v^{1/(p-q)}(t)\right), \tag{3.19}$$

that is,

$$\frac{v^{\Delta}(t)}{\omega\left(v^{1/(p-q)}(t)\right)} \leq f(\tau_1(t)). \tag{3.20}$$

On the other hand, for $t \in [t_0, T] \cap \mathbb{T}$, if $\sigma(t) > t$, then

$$\begin{aligned} [G(v(t))]^{\Delta} &= \frac{G(v(\sigma(t))) - G(v(t))}{\sigma(t) - t} = \frac{1}{\sigma(t) - t} \int_{v(t)}^{v(\sigma(t))} \frac{1}{\omega\left(r^{1/(p-q)}\right)} dr \\ &\leq \frac{v(\sigma(t)) - v(t)}{\sigma(t) - t} \frac{1}{\omega\left(v^{1/(p-q)}(t)\right)} = \frac{v^{\Delta}(t)}{\omega\left(v^{1/(p-q)}(t)\right)}. \end{aligned} \tag{3.21}$$

If $\sigma(t) = t$, then

$$\begin{aligned} [G(v(t))]^{\Delta} &= \lim_{s \to t} \frac{G(v(t)) - G(v(s))}{t - s} = \lim_{s \to t} \frac{1}{t - s} \int_{v(s)}^{v(t)} \frac{1}{\omega\left(r^{1/(p-q)}\right)} dr \\ &= \lim_{s \to t} \frac{v(t) - v(s)}{t - s} \frac{1}{\omega\left(\xi^{1/(p-q)}\right)} = \frac{v^{\Delta}(t)}{\omega\left(v^{1/(p-q)}(t)\right)}, \end{aligned} \tag{3.22}$$

where ξ lies between $v(s)$ and $v(t)$. So from (3.21) and (3.22) we always have

$$[G(v(t))]^{\Delta} \leq \frac{v^{\Delta}(t)}{\omega\left(v^{1/(p-q)}(t)\right)}. \tag{3.23}$$

Combining (3.20) and (3.23), we deduce

$$[G(v(t))]^{\Delta} \leq f(\tau_1(t)). \tag{3.24}$$

Setting $t = s$ in (3.24), an integration with respect to s from t_0 to t yields

$$G(v(t)) - G(v(t_0)) \leq \int_{t_0}^{t} f(\tau_1(s)) \Delta s. \tag{3.25}$$

Considering $v(t_0) = C + \int_{t_0}^{T} g(\tau_2(s)) \Delta s$, and G is increasing, then we obtain

$$v(t) \leq G^{-1}\left[G\left(C + \int_{t_0}^{T} g(\tau_2(s)) \Delta s\right) + \int_{t_0}^{t} f(\tau_1(s)) \Delta s\right], \quad t \in [t_0, T] \cap \mathbb{T}. \tag{3.26}$$

Combining (3.7), (3.18), and (3.26), we have

$$u(t) \leq \left\{G^{-1}\left[G\left(C + \int_{t_0}^{T} g(\tau_2(s)) \Delta s\right) + \int_{t_0}^{t} f(\tau_1(s)) \Delta s\right]\right\}^{1/(p-q)}, \quad t \in [t_0, T] \cap \mathbb{T}. \tag{3.27}$$

Setting $t = T$ in (3.27), considering that $T \in \mathbb{T}_0$ is selected arbitrarily, after substituting T with t, we get the desired result.

If $C = 0$, then we carry out the process above with C replaced by ε, where $\varepsilon > 0$, and after letting $\varepsilon \to 0$, we also get the desired result. So the proof is complete. □

Remark 3.3. If we take $\mathbb{T} = \mathbb{R}$, $t_0 = 0$, then Theorem 3.2 reduces to [31, Theorem 2.1]. If $\mathbb{T} = \mathbb{R}$, $t_0 = 0$, $p = 2$, $q = 1$, then Theorem 3.2 reduces to [32, Theorem 1]. If $\mathbb{T} = \mathbb{R}$, $t_0 = 0$, $p = 2$, $q = 1$, $\tau_1(t) = \tau_2(t) = t$, then Theorem 3.2 reduces to [33, Theorem 3(a6)]. If we take $\mathbb{T} = \mathbb{Z}$, $t_0 = 0$, $p = 2$, $q = 1$, $\tau_1(t) = \tau_2(t) = t$, then Theorem 3.2 reduces to [33, Theorem 6(b6)].

Theorem 3.4. *Suppose u, $a \in C_{\mathrm{rd}}(\mathbb{T}_0, \mathbb{R}_+)$, and a is nondecreasing. $\alpha, \phi, \tau_i, i = 1,2$ are defined as in Theorem 3.2. $f, h, f_t^{\Delta}, h_t^{\Delta} \in C_{\mathrm{rd}}(\mathbb{T}_0 \times ([\alpha, \infty) \cap \mathbb{T}), \mathbb{R}_+)$, $g, d, g_t^{\Delta}, d_t^{\Delta} \in C_{\mathrm{rd}}(\mathbb{T}_0^2, \mathbb{R}_+)$, where $f_t^{\Delta}, h_t^{\Delta}, g_t^{\Delta}, d_t^{\Delta}$ denote the delta derivative of f, h, g, d with respect to the first variable. If for $t \in \mathbb{T}_0$, $u(t)$ satisfies the following inequality:*

$$
\begin{aligned}
u(t) \le a(t) &+ \int_{t_0}^{t} \left[f(t, \tau_1(s)) u(\tau_1(s)) + g(t,s) u(s) \right] \Delta s \\
&+ \int_{t_0}^{t} h(t, \tau_2(s)) u(\tau_2(s)) \Delta s \int_{t_0}^{t} d(t,s) u(s) \Delta s
\end{aligned}
\tag{3.28}
$$

with the initial condition

$$
\begin{gathered}
u(t) = \phi(t), \quad t \in [\alpha, t_0] \cap \mathbb{T}, \\
\phi(\tau_i(t)) \le a(t), \quad \forall t \in \mathbb{T}_0,\ \tau_i(t) \le t_0,\ i = 1,2,
\end{gathered}
\tag{3.29}
$$

then

$$
u(t) \le \frac{a(t) e_{\ominus(-F_1)}(t, t_0)}{1 + a(t) \int_{t_0}^{t} e_{\ominus(-F_1)}(\sigma(s), t_0) F_2(s) \Delta s}, \quad t \in \mathbb{T}_0
\tag{3.30}
$$

provided that $1 + a(t) \int_{t_0}^{t} e_{\ominus(-F_1)}(\sigma(s), t_0) F_2(s) \Delta s > 0$ and $1 - \mu(t) F_1(t) > 0$ for $\forall t \in \mathbb{T}_0$, where

$$
F_1(t) = \left\{ \int_{t_0}^{t} \left[f(t, \tau_1(s)) + g(t,s) \right] \Delta s \right\}^{\Delta}, \qquad F_2(t) = \left\{ \int_{t_0}^{t} h(t, \tau_2(s)) \Delta s \int_{t_0}^{t} d(t,s) \Delta s \right\}^{\Delta}.
\tag{3.31}
$$

Proof. Assume $a(0) > 0$. Fix $T \in \mathbb{T}_0$, and let $t \in [t_0, T] \cap \mathbb{T}$. If the right side of (3.28) is $a(t) + z(t)$, then

$$
u(t) \le a(t) + z(t),
\tag{3.32}
$$

and similar to the process of (3.8)–(3.10), we have

$$
u(\tau_i(t)) \le a(t) + z(t), \quad i = 1,2.
\tag{3.33}
$$

Furthermore, by Lemma 3.1 and Theorem 2.9(ii)

$$
\begin{aligned}
z^{\Delta}(t) = {} & \int_{t_0}^{t}\left[f_t^{\Delta}(t,\tau_1(s))u(\tau_1(s)) + g_t^{\Delta}(t,s)u(s)\right]\Delta s + f(\sigma(t),\tau_1(t))u(\tau_1(t)) + g(\sigma(t),t)u(t) \\
& + \left[\int_{t_0}^{t} h_t^{\Delta}(t,\tau_2(s))u(\tau_2(s))\Delta s + h(\sigma(t),\tau_2(t))u(\tau_2(t))\right]\int_{t_0}^{\sigma(t)} d(\sigma(t),s)u(s)\Delta s \\
& + \int_{t_0}^{t} h(t,\tau_2(s))u(\tau_2(s))\Delta s\left[\int_{t_0}^{t} d_t^{\Delta}(t,s)u(s)\Delta s + d(\sigma(t),t)\right] \\
\le {} & [a(t)+z(t)]\left\{\int_{t_0}^{t}\left[f_t^{\Delta}(t,\tau_1(s)) + g_t^{\Delta}(t,s)\right]\Delta s + f(\sigma(t),\tau_1(t)) + g(\sigma(t),t)\right\} \\
& + [a(t)+z(t)]^2\left\{\left[\int_{t_0}^{t} h_t^{\Delta}(t,\tau_2(s))\Delta s + h(\sigma(t),\tau_2(t))\right]\int_{t_0}^{\sigma(t)} d(\sigma(t),s)\Delta s\right. \\
& \qquad\qquad \left. + \int_{t_0}^{t} h(t,\tau_2(s))\Delta s\left[\int_{t_0}^{t} d_t^{\Delta}(t,s)\Delta s + d(\sigma(t),t)\right]\right\} \\
\le {} & [a(T)+z(t)]\left\{\int_{t_0}^{t}[f(t,\tau_1(s)) + g(t,s)]\Delta s\right\}^{\Delta} \\
& + [a(T)\ + z(t)]^2\left\{\int_{t_0}^{t} h(t,\tau_2(s))\Delta s\int_{t_0}^{t} d(t,s)\Delta s\right\}^{\Delta},
\end{aligned}
\tag{3.34}
$$

that is,

$$
\begin{aligned}
\frac{z^{\Delta}(t)}{[a(T)+z(t)]^2} \le {} & \frac{1}{a(T)+z(t)}\left\{\int_{t_0}^{t}[f(t,\tau_1(s)) + g(t,s)]\Delta s\right\}^{\Delta} \\
& + \left\{\int_{t_0}^{t} h(t,\tau_2(s))\Delta s\int_{t_0}^{t} d(t,s)\Delta s\right\}^{\Delta} \\
\le {} & \frac{1}{a(T)+z(t)}F_1(t) + F_2(t),
\end{aligned}
\tag{3.35}
$$

where F_1, F_2 are defined in (3.31).

Considering $z^{\Delta}(t) \ge 0$, from (3.35), we deduce

$$
\frac{z^{\Delta}(t)}{[a(T)+z(t)][a(T)+z(\sigma(t))]} \le \frac{z^{\Delta}(t)}{[a(T)+z(t)]^2} \le \frac{1}{a(T)+z(t)}F_1(t) + F_2(t). \tag{3.36}
$$

Let $v(t) = 1/(a(T) + z(t))$. Then, by Theorem 2.9(ii), $v^{\Delta}(t) = -z^{\Delta}(t)/[a(T) + z(t)][a(T) + z(\sigma(t))]$, and (3.36) implies

$$v^{\Delta}(t) + v(t)F_1(t) \geq F_2(t). \tag{3.37}$$

On the other hand, since $1 - \mu(t)F_1(t) > 0$, then $1 + \mu(t)[\ominus(-F_1)(t)] = 1/(1 - \mu(t)F_1(t)) > 0$. So $\ominus(-F_1) \in \Re_+$, and $e_{\ominus(-F_1)}(t, t_0) > 0$, $\forall t \in \mathbb{T}_0$. By Theorem 2.14(i), we have $e_{\ominus(-F_1)}(t_0, t_0) = 1$. Furthermore, by a combination of Theorem 2.9(ii), Theorems 2.15, and 2.14, we obtain

$$\begin{aligned}
\left[v(t)e_{\ominus(-F_1)}(t,t_0)\right]^{\Delta} &= \left[e_{\ominus(-F_1)}(t,t_0)\right]^{\Delta}v(t) + e_{\ominus(-F_1)}(\sigma(t),t_0)v^{\Delta}(t) \\
&= (\ominus(-F_1))(t)e_{\ominus(-F_1)}(t,t_0)v(t) + e_{\ominus(-F_1)}(\sigma(t),t_0)v^{\Delta}(t) \\
&= e_{\ominus(-F_1)}(\sigma(t),t_0)\left[\frac{(\ominus(-F_1))(t)}{1+\mu(t)(\ominus(-F_1))(t)}v(t) + v^{\Delta}(t)\right] \\
&= e_{\ominus(-F_1)}(\sigma(t),t_0)\left[v^{\Delta}(t) + F_1(t)v(t)\right].
\end{aligned} \tag{3.38}$$

Combining (3.37) and (3.38), we deduce

$$\left[v(t)e_{\ominus(-F_1)}(t,t_0)\right]^{\Delta} \geq e_{\ominus(-F_1)}(\sigma(t),t_0)F_2(t). \tag{3.39}$$

Setting $t = s$ in (3.39), an integration with respect to s from t_0 to t yields

$$v(t)e_{\ominus(-F_1)}(t,t_0) - v(t_0)e_{\ominus(-F_1)}(t_0,t_0) \geq \int_{t_0}^{t} e_{\ominus(-F_1)}(\sigma(s),t_0)F_2(s)\Delta s. \tag{3.40}$$

Considering $v(t_0) = 1/a(T)$, it is then followed by

$$v(t) \geq \frac{1 + a(T)\int_{t_0}^{t} e_{\ominus(-F_1)}(\sigma(s),t_0)F_2(s)\Delta s}{a(T)e_{\ominus(-F_1)}(t,t_0)}, \tag{3.41}$$

and furthermore

$$a(T) + z(t) \leq \frac{a(T)e_{\ominus(-F_1)}(t,t_0)}{1 + a(T)\int_{t_0}^{t} e_{\ominus(-F_1)}(\sigma(s),t_0)F_2(s)\Delta s}. \tag{3.42}$$

Combining (3.32) and (3.42), we obtain

$$u(t) \leq \frac{a(T)e_{\ominus(-F_1)}(t,t_0)}{1 + a(T)\int_{t_0}^{t} e_{\ominus(-F_1)}(\sigma(s),t_0)F_2(s)\Delta s}, \quad t \in [t_0, T] \cap \mathbb{T}. \tag{3.43}$$

Setting $t = T$ in (3.43), since $T \in \mathbb{T}_0$ is selected arbitrarily, after substituting T with t, we get the desired result.

If $a(0) = 0$, then we carry out the process above with $a(t)$ replaced by $a(t)+\varepsilon$, and after letting $\varepsilon \to 0$, we also get the desired result. So the proof is complete. □

Remark 3.5. If we take $\mathbb{T} = \mathbb{R}$, $t_0 = 0$, then Theorem 3.4 reduces to [34, Corollary 2.5]. If $\mathbb{T} = \mathbb{R}$, $t_0 = 0$, $\tau_1(t) = \tau_2(t) = t$, $a(t) \equiv C$, where C is a nonnegative constant, and $f(t,s)$, $g(t,s)$, $h(t,s)$ are replaced by $f(s)$, $g(s)$, $h(s)$, then Theorem 3.4 reduces to [35, Theorem 1]. If we take $\mathbb{T} = \mathbb{Z}$, $t_0 = 0$, $\tau_1(t) = \tau_2(t) = t$, $(t) \equiv C$, where C is a nonnegative constant, and $f(t,s)$, (t,s), $h(t,s)$ are replaced by $f(s)$, $g(s)$, $h(s)$, then Theorem 3.4 reduces to [35, Theorem 5].

Next we will study the delay integral inequality on time scales with the following form

$$\begin{aligned} u^p(t) \le\ & C^{p/(p-q)} + \frac{p}{p-q}\int_{t_0}^{t} f(\tau_1(s))u^p(\tau_1(s))\Delta s \\ & + \frac{p}{p-q}\int_{t_0}^{t}\left[f(\tau_1(s))u^q(\tau_1(s))\int_{t_0}^{s} g(\tau_1(\xi))u^{p-q}(\tau_1(\xi))\Delta\xi + h(\tau_2(s))u^q(\tau_2(s))\right]\Delta s, \end{aligned} \tag{3.44}$$

where $u, f, g, p, q, C, \alpha, \phi, \tau_i, i = 1,2$ are defined as in Theorem 3.2, and $h \in C_{\mathrm{rd}}([\alpha,\infty)\cap\mathbb{T},\mathbb{R}_+)$.

Lemma 3.6. *Suppose u, f, g, τ_1 are defined as in Theorem 3.2, and $(f+g)(\tau_1(\cdot)) \in \Re_+$, then for $t \in \mathbb{T}_0$,*

$$u(t) \le 1 + \int_{t_0}^{t} f(\tau_1(s))u(s)\Delta s + \int_{t_0}^{t}\left[f(\tau_1(s))\int_{t_0}^{s} g(\tau_1(\xi))u(\xi)\Delta\xi\right]\Delta s \tag{3.45}$$

implies

$$u(t) \le 1 + \int_{t_0}^{t} f(\tau_1(s))e_{(f+g)(\tau_1(\cdot))}(s,t_0)\Delta s. \tag{3.46}$$

Proof. Denote the right side of (3.46) by $z(t)$. Then, $u(t) \le z(t)$, $t \in \mathbb{T}_0$, and

$$\begin{aligned} z^{\Delta}(t) &= f(\tau_1(t))u(t) + f(\tau_1(t))\int_{t_0}^{t} g(\tau_1(\xi))u(\xi)\Delta\xi \\ &\le f(\tau_1(t))z(t) + f(\tau_1(t))\int_{t_0}^{t} g(\tau_1(\xi))z(\xi)\Delta\xi. \end{aligned} \tag{3.47}$$

Let $m(t) = z(t) + \int_{t_0}^{t} g(\tau_1(\xi))z(\xi)\Delta\xi$. Then, $z(t) \le m(t)$, and $z^{\Delta}(t) \le f(\tau_1(t))m(t)$. Furthermore,

$$m^{\Delta}(t) = z^{\Delta}(t) + g(\tau_1(t))z(t) \le f(\tau_1(t))m(t) + g(\tau_1(t))m(t). \tag{3.48}$$

Since $(f+g)(\tau_1(\cdot)) \in \Re_+$, by [28, Theorem 5.4], we have $m(t) \le e_{(f+g)(\tau_1(\cdot))}(t,t_0)$. So,

$$z^{\Delta}(t) \le f(\tau_1(t))e_{(f+g)(\tau_1(\cdot))}(t,t_0). \tag{3.49}$$

Using $z(t_0) = 1$, an integration for (3.49) from t_0 to t yields

$$z(t) \leq 1 + \int_{t_0}^{t} f(\tau_1(s)) e_{(f+g)(\tau_1(\cdot))}(s, t_0)\, \Delta s, \tag{3.50}$$

which confirms the desired inequality. So the proof is complete. □

Theorem 3.7. *If for $t \in \mathbb{T}_0$, $u(t)$ satisfies the inequality* (3.44) *with the initial condition* (3.5)*, then*

$$u(t) \leq \left\{ J(t) \left[1 + \int_{t_0}^{t} f(\tau_1(s)) e_{(f+g)(\tau_1(\cdot))}(s, t_0) \Delta s \right] \right\}^{1/(p-q)}, \quad t \in \mathbb{T}_0 \tag{3.51}$$

provided $(f + g)(\tau_1(\cdot)) \in \Re_+$, where

$$J(t) = C + \int_{t_0}^{t} h(\tau_2(s)) \Delta s. \tag{3.52}$$

Proof. Let

$$\begin{aligned} z(t) &= (C + \varepsilon)^{p/(p-q)} \\ &\quad + \frac{p}{p-q} \int_{t_0}^{t} \left[f(\tau_1(s)) u^p(\tau_1(s)) \Delta s + \frac{p}{p-q} \int_{t_0}^{t} f(\tau_1(s)) u^q(\tau_1(s)) \right. \\ &\quad \left. \times \int_{t_0}^{s} g(\tau_1(\xi)) u^{p-q}(\tau_1(\xi)) \Delta\xi + h(\tau_2(s)) u^q(\tau_2(s)) \right] \Delta s, \end{aligned} \tag{3.53}$$

where $\varepsilon > 0$ is an arbitrary small constant. Then,

$$u(t) \leq z^{1/p}(t), \quad t \in \mathbb{T}_0, \tag{3.54}$$

and similar to (3.8)–(3.10)

$$u(\tau_i(t)) \leq z^{1/p}(t), \quad i = 1, 2, \ t \in \mathbb{T}_0. \tag{3.55}$$

Furthermore,

$$\begin{aligned}
z^{\Delta}(t) &= \frac{p}{p-q} f(\tau_1(t)) u^p(\tau_1(t)) \\
&\quad + \frac{p}{p-q}\left[f(\tau_1(t)) u^q(\tau_1(t)) \int_{t_0}^{t} g(\tau_1(\xi)) u^{p-q}(\tau_1(\xi))\Delta\xi + h(\tau_2(t)) u^q(\tau_2(t))\right] \\
&\le \frac{p}{p-q} f(\tau_1(t)) z(t) \\
&\quad + \frac{p}{p-q}\left[f(\tau_1(t)) z^{q/p}(t) \int_{t_0}^{t} g(\tau_1(\xi)) z^{(p-q)/p}(\xi)\Delta\xi + h(\tau_2(t)) z^{q/p}(t)\right].
\end{aligned} \tag{3.56}$$

Using (3.13), we obtain $((p/(p-q))z^{(p-q)/p}(t))^{\Delta} \le (z^{\Delta}(t)/z^{q/p}(t))$. So (3.56) implies

$$\left(z^{(p-q)/p}(t)\right)^{\Delta} \le f(\tau_1(t)) z^{(p-q)/p}(t) + f(\tau_1(t)) \int_{t_0}^{t} g(\tau_1(\xi)) z^{(p-q)/p}(\xi)\Delta\xi + h(\tau_2(t)). \tag{3.57}$$

Considering $z(t_0) = (C+\varepsilon)^{p/(p-q)}$, an integration for (3.57) from t_0 to t yields

$$\begin{aligned}
z^{(p-q)/p}(t) &\le C + \varepsilon + \int_{t_0}^{t} f(\tau_1(s)) z^{(p-q)/p}(s)\Delta s \\
&\quad + \int_{t_0}^{t}\left[f(\tau_1(s)) \int_{t_0}^{s} g(\tau_1(\xi)) z^{(p-q)/p}(\xi)\Delta\xi + h(\tau_2(s))\right]\Delta s \\
&= J_{\varepsilon}(t) + \int_{t_0}^{t} f(\tau_1(s)) z^{(p-q)/p}(s)\Delta s \\
&\quad + \int_{t_0}^{t}\left[f(\tau_1(s)) \int_{t_0}^{s} g(\tau_1(\xi)) z^{(p-q)/p}(\xi)\Delta\xi\right]\Delta s,
\end{aligned} \tag{3.58}$$

where $J_{\varepsilon}(t) = J(t) + \varepsilon$, and $J(t)$ is defined in (3.52). Then,

$$\frac{z^{(p-q)/p}(t)}{J_{\varepsilon}(t)} \le 1 + \int_{t_0}^{t} f(\tau_1(s)) \frac{z^{(p-q)/p}(s)}{J_{\varepsilon}(s)}\Delta s + \int_{t_0}^{t}\left[f(\tau_1(s)) \int_{t_0}^{s} g(\tau_1(\xi)) \frac{z^{(p-q)/p}(\xi)}{J_{\varepsilon}(\xi)}\Delta\xi\right]\Delta s. \tag{3.59}$$

Denote $z^{(p-q)/p}(t)/J_{\varepsilon}(t) = v(t)$. Then,

$$v(t) \le 1 + \int_{t_0}^{t} f(\tau_1(s)) v(s)\Delta s + \int_{t_0}^{t}\left[f(\tau_1(s)) \int_{t_0}^{s} g(\tau_1(\xi)) v(\xi)\Delta\xi\right]\Delta s. \tag{3.60}$$

A suitable application of Lemma 3.6 to (3.60) yields

$$v(t) \leq 1 + \int_{t_0}^{t} f(\tau_1(s)) e_{(f+g)(\tau_1(\cdot))}(s, t_0) \Delta s. \tag{3.61}$$

Combining (3.54) and (3.61), we obtain

$$u(t) \leq \left\{ J_\varepsilon(t) \left[1 + \int_{t_0}^{t} f(\tau_1(s)) e_{(f+g)(\tau_1(\cdot))}(s, t_0) \Delta s \right] \right\}^{1/(p-q)}. \tag{3.62}$$

After letting $\varepsilon \to 0$, we get the desired result. So the proof is complete. □

Remark 3.8. If we take $\mathbb{T} = \mathbb{R}$, $t_0 = 0$, $p = 2$, $q = 1$, $\tau_1(t) = \tau_2(t) = t$, then Theorem 3.7 reduces to [33, Theorem 1(a2)]. If we take $\mathbb{T} = \mathbb{Z}$, $t_0 = 0$, $p = 2$, $q = 1$, $\tau_1(t) = \tau_2(t) = t$, then Theorem 3.7 reduces to [33, Theorem 4(b2)].

Now we present a more general inequality than in Theorem 3.7.

Theorem 3.9. *Suppose $u, f, g, \omega, p, q, C, \alpha, \phi, \tau_i, i = 1, 2$ are defined as in Theorem 3.2, and $h \in C_{\mathrm{rd}}([\alpha, \infty) \cap \mathbb{T}, \mathbb{R}_+)$. If for $t \in \mathbb{T}_0$, $u(t)$ satisfies the following inequality:*

$$\begin{aligned} u^p(t) \leq C^{p/(p-q)} &+ \frac{p}{p-q} \int_{t_0}^{t} f(\tau_1(s)) u^p(\tau_1(s)) \Delta s + \frac{p}{p-q} \int_{t_0}^{t} \Big[f(\tau_1(s)) u^q(\tau_1(s)) \\ &\times \int_{t_0}^{s} g(\tau_1(\xi)) \, \omega(u(\tau_1(\xi))) \Delta \xi + h(\tau_2(s)) u^q(\tau_2(s)) \Big] \Delta s \end{aligned} \tag{3.63}$$

with the initial condition (3.5), *then for $t \in \mathbb{T}_0$,*

$$u(t) \leq \left\{ J(t) + \int_{t_0}^{t} f(\tau_1(s)) \widetilde{G}^{-1} \left\{ \widetilde{G}(J(s)) + \int_{t_0}^{s} \left[f(\tau_1(\xi)) + g(\tau_1(\xi)) \right] \Delta \xi \right\} \Delta s \right\}^{1/(p-q)}, \tag{3.64}$$

where $J(t)$ is defined as in Theorem 3.7, and

$$\widetilde{G}(x) = \int_{1}^{x} \frac{1}{r + \omega\left(r^{1/(p-q)}\right)} dr, \quad x > 0 \tag{3.65}$$

with $\widetilde{G}(\infty) = \infty$, and $\widetilde{G}^{-1}$ is the inverse of $\widetilde{G}$.

Proof. Let the right side of (3.63) is $z(t)$, $\varepsilon > 0$ is an arbitrary small constant, and $J_\varepsilon(t)$ is defined as in Theorem 3.7. Then, $u(t) \le z^{1/p}(t),\ t \in \mathbb{T}_0$, and similar to the process of (3.53)–(3.58), we obtain

$$\begin{aligned} z^{(p-q)/p}(t) \le J_\varepsilon(t) &+ \int_{t_0}^{t} f(\tau_1(s)) z^{(p-q)/p}(s)\,\Delta s \\ &+ \int_{t_0}^{t}\left[f(\tau_1(s)) \int_{t_0}^{s} g(\tau_1(\xi))\omega\left(z^{1/p}(\xi)\right)\Delta\xi\right]\Delta s. \end{aligned} \tag{3.66}$$

Fix $T \in \mathbb{T}_0$, and let $t \in [t_0, T] \cap \mathbb{T}$. Considering $J_\varepsilon(t)$ is nondecreasing, we deduce that

$$\begin{aligned} z^{(p-q)/p}(t) \le J_\varepsilon(T) &+ \int_{t_0}^{t} f(\tau_1(s)) z^{(p-q)/p}(s)\Delta s \\ &+ \int_{t_0}^{t}\left[f(\tau_1(s)) \int_{t_0}^{s} g(\tau_1(\xi))\omega\left(z^{1/p}(\xi)\right)\Delta\xi\right]\Delta s. \end{aligned} \tag{3.67}$$

Denote the right side of (3.66) by $v(t)$. Then, $z^{(p-q)/p}(t) \le v(t),\ t \in [t_0, T] \cap \mathbb{T}$, and

$$\begin{aligned} v^{\Delta}(t) &= f(\tau_1(t)) z^{(p-q)/p}(t) + f(\tau_1(t)) \int_{t_0}^{t} g(\tau_1(\xi))\omega\left(z^{1/p}(\xi)\right)\Delta\xi \\ &\le f(\tau_1(t)) v(t) + f(\tau_1(t)) \int_{t_0}^{t} g(\tau_1(\xi))\omega\left(v^{1/(p-q)}(\xi)\right)\Delta\xi. \end{aligned} \tag{3.68}$$

Denote $m(t) = v(t) + \int_{t_0}^{t} g(\tau_1(\xi))\omega(v^{1/(p-q)}(\xi))\Delta\xi$. Then, $v(t) \le m(t)$, and

$$v^{\Delta}(t) \le f(\tau_1(t)) m(t). \tag{3.69}$$

Furthermore,

$$\begin{aligned} m^{\Delta}(t) &= v^{\Delta}(t) + g(\tau_1(t))\omega\left(v^{1/(p-q)}(t)\right) \le f(\tau_1(t)) m(t) + g(\tau_1(t))\omega\left(m^{1/(p-q)}(t)\right) \\ &\le \left[f(\tau_1(t)) + g(\tau_1(t))\right]\left[m(t) + \omega\left(m^{1/(p-q)}(t)\right)\right], \end{aligned} \tag{3.70}$$

that is,

$$\frac{m^{\Delta}(t)}{m(t) + \omega(m^{1/(p-q)}(t))} \le f(\tau_1(t)) + g(\tau_1(t)). \tag{3.71}$$

On the other hand, similar to (3.21)–(3.23), we have

$$\left[\widetilde{G}(m(t))\right]^{\Delta} \le \frac{m^{\Delta}(t)}{m(t) + \omega(m^{1/(p-q)}(t))}. \tag{3.72}$$

So combining (3.71) and (3.72), we deduce

$$\left[\widetilde{G}(m(t))\right]^{\Delta} \leq f(\tau_1(t)) + g(\tau_1(t)). \tag{3.73}$$

Considering $m(t_0) = z(t_0) = J_\varepsilon(T)$, an integration for (3.73) from t_0 to t yields

$$\widetilde{G}(m(t)) \leq \widetilde{G}(J_\varepsilon(T)) + \int_{t_0}^{t} \left[f(\tau_1(\xi)) + g(\tau_1(\xi))\right] \Delta\xi. \tag{3.74}$$

Since $\widetilde{G}$ is increasing, then

$$m(t) \leq \widetilde{G}^{-1}\left\{\widetilde{G}(J_\varepsilon(T)) + \int_{t_0}^{t} \left[f(\tau_1(\xi)) + g(\tau_1(\xi))\right] \Delta\xi\right\}. \tag{3.75}$$

Combining (3.69) and (3.75), we obtain

$$v^{\Delta}(t) \leq f(\tau_1(t))\widetilde{G}^{-1}\left\{\widetilde{G}(J_\varepsilon(T)) + \int_{t_0}^{t} \left[f(\tau_1(\xi)) + g(\tau_1(\xi))\right] \Delta\xi\right\}. \tag{3.76}$$

Setting $t = T$ in (3.76), since T is selected from $\mathbb{T}_0$ arbitrarily, then in fact (3.76) holds for all $t \in \mathbb{T}_0$, that is,

$$v^{\Delta}(t) \leq f(\tau_1(t))\widetilde{G}^{-1}\left\{\widetilde{G}(J_\varepsilon(t)) + \int_{t_0}^{t} \left[f(\tau_1(\xi)) + g(\tau_1(\xi))\right] \Delta\xi\right\}, \quad \forall t \in \mathbb{T}_0. \tag{3.77}$$

Considering $v(t_0) = J_\varepsilon(T)$, an integration for (3.77) from t_0 to t yields

$$v(t) \leq J_\varepsilon(T) + \int_{t_0}^{t} f(\tau_1(s))\widetilde{G}^{-1}\left\{\widetilde{G}(J_\varepsilon(s)) + \int_{t_0}^{s} \left[f(\tau_1(\xi)) + g(\tau_1(\xi))\right] \Delta\xi\right\} \Delta s, \tag{3.78}$$

which implies

$$\begin{aligned} u(t) \leq \Bigg\{ & J_\varepsilon(T) + \int_{t_0}^{t} f(\tau_1(s))\widetilde{G}^{-1} \\ & \times\left\{\widetilde{G}(J_\varepsilon(s)) + \int_{t_0}^{s} \left[f(\tau_1(\xi)) + g(\tau_1(\xi))\right] \Delta\xi\right\} \Delta s\Bigg\}^{1/(p-q)}, \quad t \in [t_0, T] \cap \mathbb{T}. \end{aligned} \tag{3.79}$$

Setting $t = T$ in the above inequality, considering T is selected from $\mathbb{T}_0$ arbitrarily, after replacing T with t and letting $\varepsilon \to 0$, we get the desired result. So the proof is complete. □

Remark 3.10. If we take $\mathbb{T} = \mathbb{R}$, $t_0 = 0$, $p = 2$, $q = 1$, $\tau_1(t) = \tau_2(t) = t$, then Theorem 3.9 reduces to [33, Theorem 3(a7)]. If we take $\mathbb{T} = \mathbb{Z}$, $t_0 = 0$, $p = 2$, $q = 1, \tau_1(t) = \tau_2(t) = t$, then Theorem 3.9 reduces to [33, Theorem 6(b7)].

Following in a similar manner as the proof of Theorem 3.7 and 3.9, then we present two more theorems as follows, the special cases of which ($p = 2$, $q = 1$, $t_0 = 0$, $\tau_1(t) = \tau_2(t) = t$) unify the continuous [33, Theorems 1(a3), 3(a8)] and discrete inequalities [33, Theorems 4(b3), 6(b8)].

Theorem 3.11. *Suppose $u, f, g, h, p, q, C, \alpha, \phi, \tau_i, i = 1, 2$ are defined as in Theorem 3.9. If for $t \in \mathbb{T}_0$, $u(t)$ satisfies the following inequality:*

$$\begin{aligned} u^p(t) \le C^{p/(p-q)} + \frac{p}{p-q} \int_{t_0}^{t} \Bigg[& f(\tau_1(s)) u^q(\tau_1(s)) \\ & \times \int_{t_0}^{s} g(\tau_1(\xi)) u^{p-q}(\tau_1(\xi)\Delta\xi + h(\tau_2(s)) u^q(\tau_2(s)) \Bigg] \Delta s \end{aligned} \tag{3.80}$$

with the initial condition (3.5), *then*

$$u(t) \le [J(t) e_H(t, t_0)]^{1/(p-q)}, \quad t \in \mathbb{T}_0 \tag{3.81}$$

provided $H \in \Re_+$, where $H(t) = f(\tau_1(t)) \int_{t_0}^{t} g(\tau_1(\xi))\Delta\xi$, and $J(t)$ is defined as in Theorem 3.7.

Theorem 3.12. *Suppose $u, f, g, h, \omega, p, q, C, \alpha, \phi, \tau_i, i = 1, 2$ are defined as in Theorem 3.9. If for $t \in \mathbb{T}_0$, $u(t)$ satisfies the following inequality:*

$$\begin{aligned} u^p(t) \le C^{p/(p-q)} + \frac{p}{p-q} \int_{t_0}^{t} \Bigg[& f(\tau_1(s)) u^q(\tau_1(s)) \\ & \times \int_{t_0}^{s} g(\tau_1(\xi)) \omega(u(\tau_1(\xi))\Delta\xi + h(\tau_2(s)) u^q(\tau_2(s)) \Bigg] \Delta s \end{aligned} \tag{3.82}$$

with the initial condition (3.5), *then*

$$u(t) \le \left\{ G^{-1} \left[G(J(t)) + \int_{t_0}^{t} f(\tau_1(s)) \int_{t_0}^{s} g(\tau_1(\xi))\Delta\xi\Delta s \right] \right\}^{1/(p-q)}, \quad t \in \mathbb{T}_0, \ t \in \mathbb{T}_0, \tag{3.83}$$

where G is defined as in Theorem 3.2, and $J(t)$ is defined as in Theorem 3.7.

4. Some Applications

In this section, we will present some applications for the results which we have established above and apply them to qualitative and quantitative analysis of solutions of certain delay dynamic equations on time scales.

Example 4.1. Consider the following delay dynamic integral equation on time scales:

$$u^p(t) = C + \int_{t_0}^{t} F(\tau(s), u(\tau(s)))\Delta s, \quad t \in \mathbb{T}_0 \tag{4.1}$$

with the initial condition

$$\begin{aligned} u(t) &= \phi(t), \quad t \in [\alpha, t_0] \cap \mathbb{T}, \\ |\phi(\tau(t))| &\le |C|^{1/p}, \quad \forall t \in \mathbb{T}_0, \ \tau(t) \le t_0, \end{aligned} \tag{4.2}$$

where $u \in C_{\mathrm{rd}}(\mathbb{T}_0, \mathbb{R})$, $\tau \in (\mathbb{T}_0, \mathbb{T})$ with $\tau(t) \le t$, $-\infty < \alpha = \inf\{\tau(t), t \in \mathbb{T}_0\} \le t_0$, $\phi \in C_{\mathrm{rd}}([\alpha, t_0] \cap \mathbb{T}, \mathbb{R})$, $F \in C_{\mathrm{rd}}(([\alpha, \infty) \cap \mathbb{T}) \times \mathbb{R}, \mathbb{R})$, and p is a constant with $p > 1$.

Theorem 4.2. *Suppose $u(t)$ is a solution of* (4.1)-(4.2)*, and $|F(t,u)| \le f(t)|u|^p + g(t)|u|^{p-1}$, where $f, \ g \in C_{rd}([\alpha, \infty) \cap \mathbb{T}, \mathbb{R}_+)$, then*

$$u(t) \le \left[C + \int_{t_0}^{t} g(\tau(s))\Delta s\right] \exp\left[\int_{t_0}^{t} f(\tau(s))\Delta s\right], \quad t \in \mathbb{T}_0. \tag{4.3}$$

Proof. In fact, from (4.1) we have

$$\begin{aligned} |u(t)|^p &\le |C| + \int_{t_0}^{t} |F(\tau(s), u(\tau(s)))|\Delta s \\ &\le |C| + \int_{t_0}^{t} \left[f(\tau(s))|u(\tau(s))|^p + g(\tau(s))|u(\tau(s))|^{p-1}\right]\Delta s \\ &= |C| + \int_{t_0}^{t} \left[f(\tau(s))|u(\tau(s))|^{p-1}\omega(|u(\tau(s))|) + g(\tau(s))|u(\tau(s))|^{p-1}\right]\Delta s, \end{aligned} \tag{4.4}$$

where $\omega(r) = r$. Then a suitable application of Theorem 3.2 (with $\tau_1 = \tau_2 = \tau$, $q = p - 1$) to (4.4) yields

$$u(t) \le G^{-1}\left[G\left(C + \int_{t_0}^{t} g(\tau(s))\Delta s\right) + \int_{t_0}^{t} f(\tau(s))\Delta s\right], \quad t \in \mathbb{T}_0, \tag{4.5}$$

where $G(x) = \int_1^x (1/r)\,dr = lnx$, for all $x > 0$. Using the expression of G in (4.5), we obtain the desired result, and the proof is complete. □

Example 4.3. Consider the following delay dynamic differential equation on time scales

$$\left(u^3(t)\right)^{\Delta} = F\left[\tau(t), u(\tau(t)), \int_{t_0}^{t} M(\tau(\xi), u(\tau(\xi)))\Delta\xi\right], \quad t \in \mathbb{T}_0 \tag{4.6}$$

with the initial condition

$$\begin{gathered} u(t_0) = u_0, \\ u(t) = \phi(t), \quad t \in [\alpha, t_0] \cap \mathbb{T}, \\ \left|\phi(\tau(t))\right| \le |u_0|, \quad \forall t \in \mathbb{T}_0,\ \tau(t) \le t_0, \end{gathered} \tag{4.7}$$

where u, τ, α, ϕ are defined as in Example 4.1, $F \in C_{\mathrm{rd}}(([\alpha,\infty) \cap \mathbb{T}) \times \mathbb{R}^2, \mathbb{R})$, $M \in C_{\mathrm{rd}}(([\alpha,\infty) \cap \mathbb{T}) \times \mathbb{R}, \mathbb{R})$.

Theorem 4.4. *Suppose $u(t)$ is a solution of* (4.6)-(4.7), *and* $|F(t,u,v)| \le f(t)|u|^3 + f(t)|u||v|$, $|M(t,u)| \le g(t)|u|^2$, *where $f, g \in C_{\mathrm{rd}}([\alpha,\infty) \cap \mathbb{T}, \mathbb{R}_+)$, then*

$$u(t) \le |u_0| \sqrt{\left[1 + \int_{t_0}^{t} f(\tau(s)) e_{(f+g)(\tau(\cdot))}(s, t_0) \Delta s\right]}, \quad t \in \mathbb{T}_0 \tag{4.8}$$

provided $(f+g)(\tau(\cdot)) \in \mathfrak{R}_+$.

Proof. The equivalent integral form of (4.6)-(4.7) is denoted by

$$u^3(t) = u_0^3 + \int_{t_0}^{t} F\left[\tau(s), u(\tau(s)), \int_{t_0}^{s} M(\tau(\xi), u(\tau(\xi)))\Delta\xi\right]\Delta s. \tag{4.9}$$

So,

$$\begin{aligned} |u(t)|^3 &\le |u_0|^3 + \int_{t_0}^{t} \left| F\left[\tau(s), u(\tau(s)), \int_{t_0}^{s} M(\tau(\xi), u(\tau(\xi)))\Delta\xi\right]\right| \Delta s \\ &\le |u_0|^3 + \int_{t_0}^{t} \left[f(\tau(s))|u(\tau(s))|^3 + f(\tau(s))|u(\tau(s))| \left| \int_{t_0}^{s} M(\tau(\xi), u(\tau(\xi)))\Delta\xi \right| \right] \Delta s \\ &\le \left|u_0^2\right|^{3/2} + \int_{t_0}^{t} \left[f(\tau(s))|u(\tau(s))|^3 + f(\tau(s))|u(\tau(s))| \int_{t_0}^{s} g(\tau(\xi))|u(\tau(\xi))|^2 \Delta\xi \right] \Delta s. \end{aligned} \tag{4.10}$$

Then a suitable application of Theorem 3.7 (with $p = 3$, $q = 1$, $\tau_1 = \tau_2 = \tau$, $h(t) \equiv 0$) to (4.10) yields the desired inequality (4.8), and the proof is complete. □

Theorem 4.5. *If under the conditions of Theorem 4.4, for $t \in \mathbb{T}_0$, $\int_{t_0}^{t} f(\tau_1(s)) e_{(f+g)(\tau_1(.))}(s, t_0)\Delta s \le L$, where $L > 0$ is a constant, then the trivial solution of* (4.6)-(4.7) *is uniformly stable.*

Theorem 4.6. *Assume* $|F(t,u_1,v_1) - F(t,u_2,v_2)| \le f(t)|u_1^3 - u_2^3| + f(t)\sqrt{|u_1^3 - u_2^3|}|v_1 - v_2|$, $|M(t,u_1) - M(t,u_2)| \le g(t)|\sqrt{|u_1^3 - u_2^3|}$, *where f, g are defined as in Theorem 4.4, then* (4.6)-(4.7), *have at most one solution.*

Proof. Suppose $u_1(t),\ u_2(t)$ are two solutions of (4.6)-(4.7). By (4.9) we have

$$\begin{aligned}
\left|u_1^3(t)-u_2^3(t)\right| \le \int_{t_0}^{t}\Bigg| & F\left[\tau(s),u_1(\tau(s)),\int_{t_0}^{s}M(\tau(\xi),u_1(\tau(\xi)))\Delta\xi\right]\\
& -F\left[\tau(s),u_2(\tau(s)),\int_{t_0}^{s}M(\tau(\xi),u_2(\tau(\xi)))\Delta\xi\right]\Bigg|\Delta s\\
\le & \int_{t_0}^{t}\Bigg[f(\tau(s))\left|u_1^3(\tau(s))-u_2^3(\tau(s))\right|+f(\tau(s))\sqrt{\left|u_1^3(\tau(s))-u_2^3(\tau(s))\right|}\\
& \times\int_{t_0}^{s}[M(\tau(\xi),u_1(\tau(\xi)))-M(\tau(\xi),u_2(\tau(\xi)))]\Delta\xi\Bigg]\Delta s\\
\le & \int_{t_0}^{t}\Bigg[f(\tau(s))\left|u_1^3(\tau(s))-u_2^3(\tau(s))\right|+f(\tau(s))\sqrt{\left|u_1^3(\tau(s))-u_2^3(\tau(s))\right|}\\
& \times\int_{t_0}^{s}g(\tau(\xi))\sqrt{\left|u_1^3(\tau(\xi))-u_2^3(\tau(\xi))\right|}\Delta\xi\Bigg]\Delta s.
\end{aligned} \tag{4.11}$$

Treat $|u_1^3(t)-u_2^3(t)|$ as one variable, and applying Theorem 3.7 to (4.11) (with $p=1$, $q=1/2$, $\tau_1=\tau_2=\tau$, $h(t)\equiv 0$) yields $|u_1^3-u_2^3|\le 0$, which implies $u_1(t)\equiv u_2(t)$, and the proof is complete. □

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