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*Review Article*

# On Some Recent Developments in Ulam's Type Stability

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We present a survey of some selected recent developments (results and methods) in the theory of Ulam's type stability. In particular we provide some information on hyperstability and the fixed point methods.

## 1. Introduction

The theory of Ulam's type stability (also quite often connected, e.g., with the names of Bourgin, Găvruţa, Ger, Hyers, and Rassias) is a very popular subject of investigations at the moment. In this expository paper we do not give an introduction to it or an ample historical background; for this we refer to [1–11]. Here we only want to attract the readers attention to some selected topics by presenting some new results and methods in several areas of the theory, which have not been treated at all or only marginally in those publications and which are somehow connected to the research interests of the authors of this paper. Also the number of references is significantly limited (otherwise the list of references would be the major part of the paper) and is only somehow representative (but certainly not fully) to the subjects discussed in this survey.

First we present a brief historical background for the stability of the Cauchy equation. Next we discuss some aspects of stability and nonstability of functional equations in single variable, some methods of proofs applied in that theory (the Forti method and the methods of fixed points), stability in non-Archmedean spaces, selected results on functional congruences, stability of composite type functional equations (in particular of the Gołąb-Schinzel equation and its generalizations), and finally the notion of hyperstability. We end the paper with remarks also on some other miscellaneous issues.

## 2. Some Classical Results Concerning the Cauchy Equation

Throughout this paper $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ denote, as usual, the sets of positive integers, integers, reals, and complex numbers, respectively. Moreover, $\mathbb{R}_+ := [0, \infty)$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

For the beginning let us mention that the first known result on stability of functional equations is due to Pólya and Szegő [12] and reads as follows.

*For every real sequence $(a_n)_{n\in\mathbb{N}}$ with*

$$\sup_{n,m\in\mathbb{N}} |a_{n+m} - a_n - a_m| \leq 1, \tag{2.1}$$

*there is a real number $\omega$ such that*

$$\sup_{n\in\mathbb{N}} |a_n - \omega n| \leq 1. \tag{2.2}$$

*Moreover,*

$$\omega = \lim_{n\to\infty} \frac{a_n}{n}. \tag{2.3}$$

But the main motivation for study of that subject is due to Ulam (cf. [13]), who in 1940 in his talk at the University of Wisconsin presented some unsolved problems and among them was the following question.

*Let $G_1$ be a group and $(G_2, d)$ a metric group. Given $\varepsilon > 0$, does there exist $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies*

$$d(f(xy), f(x)f(y)) < \delta, \quad x, y \in G_1, \tag{2.4}$$

*then a homomorphism $T : G_1 \to G_2$ exists with*

$$d(f(x), T(x)) < \varepsilon, \quad x \in G_1? \tag{2.5}$$

In 1941 Hyers [14] published the following answer to it.

Let $X$ and $Y$ be Banach spaces and $\varepsilon > 0$. Then for every $g : X \to Y$ with

$$\sup_{x,y\in X} \|g(x+y) - g(x) - g(y)\| \leq \varepsilon, \tag{2.6}$$

there exists a unique function $f : X \to Y$ such that

$$\sup_{x\in X} \|g(x) - f(x)\| \leq \varepsilon, \tag{2.7}$$

$$f(x+y) = f(x) + f(y), \quad x, y \in X. \tag{2.8}$$

We can describe that latter result saying that *the Cauchy functional equation* (2.8) *is Hyers-Ulam stable (or has the Hyers-Ulam stability) in the class of functions* $Y^X$. For examples of various possible definitions of stability for functional equations and some discussions on them we refer to [9].

The result of Hyers was extended by Aoki [15] (for $0 < p < 1$; see also [16–18]), Gajda [19] (for $p > 1$), and Rassias [20] (for $p < 0$; see also [21, p. 326] and [22]), in the following way.

**Theorem 2.1.** *Let $E_1$ and $E_2$ be two normed spaces, let $E_2$ be complete, $c \geq 0$, and let $p \neq 1$ be a real number. Let $f : E_1 \to E_2$ be an operator such that*

$$\|f(x+y) - f(x) - f(y)\| \leq c\left(\|x\|^p + \|y\|^p\right), \quad x, y \in E_1 \setminus \{0\}. \tag{2.9}$$

*Then there exists a unique additive operator $T : E_1 \to E_2$ with*

$$\|f(x) - T(x)\| \leq \frac{c\|x\|^p}{\left|1 - 2^{p-1}\right|}, \quad x \in E_1 \setminus \{0\}. \tag{2.10}$$

A further generalization was suggested by Bourgin [22] (see also [2, 6–8, 23]), without a proof, and next rediscovered and improved many years later by Găvruţa [24]. Below, we present the Găvruţa type result in a bit generalized form (on the restricted domain), which can be easily derived from [25, Theorem 1].

**Corollary 2.2.** *Let $X$ be a linear space over a field with $2 \neq 0$ and let $Y$ be a Banach space. Let $V \subset X$ be nonempty, $\varphi : V^2 \to \mathbb{R}$, and $f : V \to Y$ satisfy*

$$\|g(x+y) - g(x) - g(y)\| \leq \varphi(x, y), \quad x, y \in V,\ x + y \in V. \tag{2.11}$$

*Suppose that there is $\varepsilon \in \{-1, 1\}$ such that $2^\varepsilon V \subset V$ and*

$$H(x) := \sum_{i=0}^{\infty} 2^{-i\varepsilon} \varphi\left(2^{i\varepsilon} x, 2^{i\varepsilon} x\right) < \infty, \quad x \in V, \tag{2.12}$$

$$\liminf_{n \to \infty} \left|2^{-n\varepsilon} \varphi\left(2^{n\varepsilon} x, 2^{n\varepsilon} y\right)\right| = 0, \quad x, y \in V. \tag{2.13}$$

*Then there exists a unique $F : V \to Y$ such that*

$$\begin{gathered} F(x+y) = F(x) + F(y), \quad x, y \in V,\ x + y \in V, \\ \|F(x) - f(x)\| \leq H_0(x), \quad x \in V, \end{gathered} \tag{2.14}$$

*where*

$$H_0(x) := \begin{cases} 2^{-1} H(x), & \textit{if } \varepsilon = 1, \\ H\left(2^{-1} x\right), & \textit{if } \varepsilon = -1. \end{cases} \tag{2.15}$$

Corollary 2.2 generalizes several already classical results on stability of (2.8). In fact, if we take $\varepsilon = -1$ and

$$\varphi(x, y) := L_1\|x\|^p + L_2\|y\|^q + L_3\|x\|^r\|y\|^s, \quad x, y \in V \tag{2.16}$$

with some $L_1, L_2, L_3 \in \mathbb{R}_+$, $p, q \in (1, \infty)$, and $r, s \in \mathbb{R}$ with $r + s > 1$, then $H_0$ has the form

$$H_0(x) = \frac{L_1\|x\|^p}{2^p - 2} + \frac{L_2\|y\|^q}{2^q - 2} + \frac{\mathrm{L}_3\|y\|^{r+s}}{2^{r+s} - 2}, \quad x \in V. \tag{2.17}$$

On the other hand, if $\varepsilon = 1$, $V \subset X \setminus \{0\}$ and

$$\varphi(x, y) := \delta + L_1\|x\|^p + L_2\|y\|^q + L_3\|x\|^r\|y\|^s, \quad x, y \in V, \tag{2.18}$$

with some $\delta, L_1, L_2, L_3 \in \mathbb{R}_+$, $q, r \in (-\infty, 1)$, and $r, s \in \mathbb{R}$ with $r + s < 1$, then

$$H_0(x) = \delta + \frac{L_1\|x\|^p}{2 - 2^p} + \frac{L_2\|y\|^q}{2 - 2^q} + \frac{L_3\|y\|^{r+s}}{2 - 2^{r+s}}, \quad x \in V. \tag{2.19}$$

It is easily seen that, in this way, with $V = X$ and $L_1 = L_2 = L_3 = 0$ we get the result of Hyers [14], with $V = X$, $p = q$, $L_1 = L_2$ and $\delta = L_3 = 0$ we obtain Theorem 2.1, with $V = X$ and $\delta = L_1 = L_2 = 0$ we have the results of Rassias [26, 27].

*Remark 2.3.* Actually, as it is easily seen in the proof of [25, Theorem 1], it is enough to assume in Corollary 2.2 that $(X, +)$ is a commutative semigroup that is uniquely divisible by 2 (i.e., for each $x \in X$ there exists a unique $y \in X$ with $x = y + y$.)

For recent results on stability of some conditional versions of the Cauchy functional equation (2.8) we refer to, for example, [28–31].

## 3. Stability of the Linear Functional Equation in Single Variable

In this section $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $X$ stands for a Banach space over $\mathbb{K}$, $S$ is a nonempty set, $F : S \to X$, $m \in \mathbb{N}$, $f_1, \ldots, f_m : S \to S$, and $a_1, \ldots, a_m : S \to \mathbb{K}$, unless explicitly stated otherwise.

The functional equation

$$\varphi(x) = \sum_{i=1}^{m} a_i(x)\varphi(f_i(x)) + F(x), \tag{3.1}$$

for $\varphi : S \to X$, is known as the linear functional equation of order $m$. For some information on it we refer to [32, 33] and the references therein.

A simply particular case of functional equation (3.1), with $S \in \{\mathbb{N}_0, \mathbb{Z}\}$, is the difference equation:

$$y_n = \sum_{i=1}^{m} a_i(n) y_{n+i} + b_n, \quad n \in S, \tag{3.2}$$

for sequences $(y_n)_{n\in S}$ in $X$, where $(b_n)_{n\in S}$ is a fixed sequence in $X$, namely, (3.1) becomes difference equation (3.2) with

$$f_i(n) = n + i, \qquad y_n := \varphi(n) = \varphi(f_1(0)), \quad b_n := F(n), \quad n \in S. \tag{3.3}$$

There are only few results on stability of (3.1), and actually only of some particular cases of it. For example, [34, Corollary 4] (cf. [34, Remark 5]) yields the following stability result.

**Corollary 3.1.** *Assume that*

$$q(x) := \sum_{i=1}^{m} |a_i(x)| < 1, \quad x \in S, \tag{3.4}$$

*and $\varepsilon : S \to \mathbb{R}_+$ are such that*

$$\varepsilon(f_i(x)) \le \varepsilon(x), \qquad q(f_i(x)) \le q(x), \quad x \in S,\ i = 1, \ldots, m, \tag{3.5}$$

*(e.g., $\varepsilon$ and $q$ are constant). If a function $\varphi : S \to X$ satisfies the inequality*

$$\left\| \varphi(x) - \sum_{i=1}^{m} a_i(x)\varphi(f_i(x)) - F(x) \right\| \le \varepsilon(x), \quad x \in S, \tag{3.6}$$

*then there exists a unique solution $\psi : S \to X$ to* (3.1) *with*

$$\|\varphi(x) - \psi(x)\| \le \frac{\varepsilon(x)}{1 - q(x)}, \quad x \in S. \tag{3.7}$$

The assumption (3.4) seems to be quite restrictive. So far we only know that it can be avoided for some special cases of (3.1). For instance, this is the case when each function $a_i$ is constant, $a_m$ is nonzero, and $f_i = f^i$ for $i = 1, \ldots, m$ (with some function $f : S \to S$), where as usual, for each $p \in \mathbb{N}_0$, $f^p$ denotes the $p$th iterate of $f$, that is,

$$f^0(x) = x, \qquad f^{p+1}(x) = f(f^p(x)), \quad p \in \mathbb{N}_0,\ x \in S. \tag{3.8}$$

Then (3.1) can be written in the following form

$$\varphi(f^m(x)) = \sum_{i=0}^{m-1} d_i \varphi\left(f^i(x)\right) + F(x), \tag{3.9}$$

with some $d_0, \ldots, d_{m-1} \in \mathbb{K}$, and [35, Theorem 2] implies the following stability result.

**Theorem 3.2.** *Let $\delta \in \mathbb{R}_+$, $d_0, \ldots, d_{m-1} \in \mathbb{K}$, $\varphi_s : S \to X$ satisfy*

$$\left\| \varphi_s(f^m(x)) - \sum_{j=0}^{m-1} d_j \varphi_s\left(f^j(x)\right) - F(x) \right\| \leq \delta, \quad x \in S, \tag{3.10}$$

*and $r_1, \ldots, r_m \in \mathbb{C}$ denote the roots of the characteristic equation*

$$r^m - \sum_{j=0}^{m-1} d_j r^j = 0. \tag{3.11}$$

*Assume that one of the following three conditions is valid.*

$1^\circ$ $|r_j| > 1$ *for* $j = 1, \ldots, m$.

$2^\circ$ $|r_j| \in (1, \infty) \cup \{0\}$ *for* $j = 1, \ldots, m$ *and* $f$ *is injective.*

$3^\circ$ $|r_j| \neq 1$ *for* $j = 1, \ldots, m$ *and* $f$ *is bijective.*

*Then there is a solution $\varphi : S \to X$ of* (3.9) *with*

$$\|\varphi_s(x) - \varphi(x)\| \leq \frac{\delta}{|1 - |r_1|| \cdot \ldots \cdot |1 - |r_m||}, \quad x \in S. \tag{3.12}$$

*Moreover, in the case where $1^\circ$ or $3^\circ$ holds, $\varphi$ is the unique solution of* (3.9) *such that*

$$\sup_{x \in S} \|\varphi_s(x) - \varphi(x)\| < \infty. \tag{3.13}$$

The following example (see [35, Example 1]) shows that the statement of Theorem 3.2 need not to be valid in the general situation if $|r_j| = 1$ for some $j \in \{1, \ldots, m\}$.

*Example 3.3.* Fix $\delta > 0$. Let $S = X = \mathbb{K}$ and let the functions $f$ and $\varphi_s$ be given by

$$f(x) = x + 1, \quad \varphi_s(x) := \frac{\delta}{2} x^2, \quad x \in \mathbb{K}. \tag{3.14}$$

Then it is easily seen that

$$\begin{aligned} &\left| \varphi_s\left(f^2(x)\right) - 2\varphi_s(f(x)) + \varphi_s(x) \right| \\ &\quad = \left| \frac{\delta}{2}(x+2)^2 - \delta(x+1)^2 + \frac{\delta}{2} x^2 \right| = \delta, \quad x \in \mathbb{K}. \end{aligned} \tag{3.15}$$

Suppose that $\varphi : \mathbb{K} \to \mathbb{K}$ is a solution of

$$\varphi\left(f^2(x)\right) = 2\varphi(f(x)) - \varphi(x). \tag{3.16}$$

Clearly,

$$r^2 - 2r + 1 = 0 \tag{3.17}$$

is the characteristic equation of (3.16) with the roots $r_1 = r_2 = 1$. Let

$$\psi(x) := \varphi(x+1) - \varphi(x), \qquad x \in \mathbb{K}. \tag{3.18}$$

Then it is easily seen that $\psi(x+1) = \psi(x)$ for $x \in \mathbb{K}$, whence by a simple induction on $n \in \mathbb{N}$ we get

$$\varphi(n) = \varphi(0) + n\psi(0), \quad n \in \mathbb{N}. \tag{3.19}$$

Consequently

$$\lim_{n\to\infty} \left|\varphi_s(n) - \varphi(n)\right| = \lim_{n\to\infty} \left|\frac{\delta}{2}n^2 - \varphi(0) - n\psi(0)\right| = \infty, \tag{3.20}$$

which means that

$$\sup_{x\in\mathbb{K}} \left|\varphi_s(x) - \varphi(x)\right| = \infty. \tag{3.21}$$

Thus we have shown that the statement of Theorem 3.2 is not valid in this case.

Estimation (3.12) is not optimal at least in some cases; for details we refer to [36, Remark 1.5, and Theorem 3.1] (see also [37]).

For some investigations of stability of the functional equation

$$\varphi(f^m(x)) = \sum_{j=1}^{m} a_j(x)\varphi\left(f^{m-j}(x)\right) + F(x), \tag{3.22}$$

with $m > 1$, we refer to [38] (note that the equation is a special case of (3.1) and a generalization of (3.9)). Here we only present one simplified result from there.

To this end we need a hypothesis concerning the roots of the equations

$$z^m - \sum_{j=1}^{m} a_j(x) z^{m-j} = 0, \tag{3.23}$$

with $x \in S$, which reads as follows.

($\mathscr{H}$) Functions $r_1, \ldots, r_m : S \to \mathbb{C}$ satisfy the condition

$$\prod_{i=1}^{m}(z - r_i(x)) = z^m - \sum_{j=1}^{m} a_j(x) z^{m-j}, \quad x \in S,\ z \in \mathbb{C}. \tag{3.24}$$

Hypothesis $(\mathscr{H})$ means that, for every $x \in S$, $r_1(x), \ldots, r_m(x) \in \mathbb{C}$ are the complex roots of (3.23). Clearly, the functions $r_1, \ldots, r_m$ are not unique, but for every $x \in S$ the sequence

$$(r_1(x), \ldots, r_m(x)) \tag{3.25}$$

is uniquely determined up to a permutation. Moreover, $0 \notin a_m(S)$ if and only if $0 \notin r_j(S)$ for each $j = 1, \ldots, m$ (see [38, Remark 1]).

We say that a function $g : S \to X$ is $f$-invariant provided

$$g(f(x)) = g(x), \quad x \in S. \tag{3.26}$$

Now we are in a position to present a result that can de deduced from [38, Theorem 1].

**Theorem 3.4.** *Let $\varepsilon_0 : S \to \mathbb{R}_+$, let $(\mathscr{H})$ be valid, and let $r_j$ be $f$-invariant for $j > 1$.*

*Assume that $0 \notin a_m(S)$ and $\varphi_s : S \to X$ fulfills the inequality*

$$\left\| \varphi_s(f^m(x)) - \sum_{j=1}^{m} a_j(x) \varphi_s\left(f^{m-j}(x)\right) - F(x) \right\| \leq \varepsilon_0(x), \quad x \in S. \tag{3.27}$$

*Further, suppose that*

$$\begin{aligned} \varepsilon_1(x) &:= \sum_{k=0}^{\infty} \frac{\varepsilon_0(f^k(x))}{\prod_{p=0}^{k} |r_1(f^p(x))|} < \infty, \quad x \in S, \\ \varepsilon_j(x) &:= \sum_{k=0}^{\infty} \frac{\varepsilon_{j-1}(f^k(x))}{|r_j(x)|^{k+1}} < \infty, \quad x \in S,\ j > 1. \end{aligned} \tag{3.28}$$

*Then* (3.22) *has a solution $\varphi : S \to X$ with*

$$\|\varphi_s(x) - \varphi(x)\| \leq \varepsilon_m(x), \quad x \in S. \tag{3.29}$$

As it folows from [38, Remark 8], the form of $\varphi$ in Theorem 3.4 can be explicitly described in some recurrent way.

Some further results on stability of (3.9), particular cases of it and some other similar equations in single variable can be found in [1, 35, 39–51]. For instance, it has been shown in [34, 52, 53] that stability of numerous functional equations of this kind is a direct consequence of some fixed point results. We deal with that issue in the section on the fixed point methods.

At the end of this part we would like to suggest some terminology that might be useful in the investigation of stability also for some other equations (as before, $B^D$ denotes the class of functions mapping a nonempty set $D$ into a nonempty set $B$). Moreover, that terminology could be somehow helpful in clarification of the notion of nonstability, which is very briefly discussed in the next section.

*Definition 3.5.* Let $\mathcal{C} \subset \mathbb{R}_+^S$ be nonempty and let $\mathcal{T}$ be an operator mapping $\mathcal{C}$ into $\mathbb{R}_+^S$. We say that (3.1) is $\mathcal{T}$-stable (with uniqueness, resp.) provided for every $\varepsilon \in \mathcal{C}$ and $\varphi : S \to X$ with

$$\left\| \varphi_s(x) - \sum_{i=1}^{m} a_i(x)\varphi_s(f_i(x)) - F(x) \right\| \le \varepsilon(x), \quad x \in S \tag{3.30}$$

there exists a (unique, resp.) solution $\widetilde{\varphi} : S \to X$ of (3.1) such that

$$\|\varphi(x) - \widetilde{\varphi}(x)\| \le \mathcal{T}\varepsilon(x), \quad x \in S. \tag{3.31}$$

In connection with the original statement of Ulam's problem we might think of yet another definition that seems to be quite natural and useful sometimes.

*Definition 3.6.* Let $\varepsilon : S \to \mathbb{R}_+$ and $L \in \mathbb{R}_+$. We say that functional equation (3.1) is $(\varepsilon, L)$-stable (with uniqueness, resp.,) provided for every function $\varphi : S \to X$ satisfying (3.30), there exists a (unique, resp.,) solution $\widetilde{\varphi} : S \to X$ to (3.1) such that

$$\|\varphi(x) - \widetilde{\varphi}(x)\| \le L\varepsilon(x), \quad x \in S. \tag{3.32}$$

Given $a : S \to \mathbb{R}_+ \setminus \{0\}$, for each $\phi : S \to \mathbb{R}_+$ we write

$$\begin{gathered} \mathcal{A}_a^f \phi(x) := \sum_{j=0}^{\infty} \frac{\phi(f^j(x))}{\prod_{k=0}^{j} |a(f^k(x))|}, \quad x \in S, \\ \mathfrak{D} := \left\{ \varepsilon : S \to \mathbb{R}_+^0 : \mathcal{A}_a^f \varepsilon(x) < \infty, x \in S \right\}. \end{gathered} \tag{3.33}$$

Then $\mathcal{A}_a^f$ is an operator mapping $\mathfrak{D}$ into $\mathbb{R}^S$ and, according to Theorem 3.4, the functional equation

$$\varphi(f(x)) = a(x)\varphi(x) + F(x), \quad x \in S \tag{3.34}$$

(i.e., (3.22) with $m = 1$) is $\mathcal{A}_a^f$-stable with uniqueness (cf. [48, Theorem 2.1]).

Further, note that for every $\varepsilon \in \mathbb{R}_+^S$ with

$$\begin{gathered} \varepsilon(f(t)) \le \varepsilon(t), \quad t \in S, \\ s := \inf_{t \in S} |a(t)| > 1, \end{gathered} \tag{3.35}$$

we have

$$\mathcal{A}_a^f \varepsilon(x) \le \sum_{j=0}^{\infty} \frac{\varepsilon(x)}{s^k} = \frac{\varepsilon(x)}{s-1}, \quad x \in S, \tag{3.36}$$

and consequently (3.34) is $(\varepsilon, L)$-stable with

$$L := \frac{1}{s-1}. \tag{3.37}$$

## 4. Nonstability

There are only few outcomes of which we could say that they concern nonstability of functional equation. The first well-known one is due to Gajda [19] and answers a question raised by Rassias [54]. Namely, he gave an example of a function showing that a result analogous to that described in Theorem 2.1 cannot be obtained for $p = 1$ (for further such examples see [21]; cf. also, e.g., [55, 56]).

In general it is not easy to define the notion of nonstability precisely, mostly because at the moment there are several notions of stability in use (see [9, 57]). For instance, we could understand nonstability as in Example 3.3. The other possibility is to refer to Definitions 3.5 and 3.6 and define $\mathfrak{T}$-nonstability and $(\varepsilon, L)$-nonstability, respectively. Finally, if there does not exist an $L \in \mathbb{R}_+$ such that the equation is $(\varepsilon, L)$-stable, then we could say that it is $\varepsilon$-nonstable.

For some further propositions of such definitions and preliminary results on nonstability we refer to [58–62]. As an example we present below the result from [60, Theorem 1] concerning nonstability of the difference equation

$$x_{n+1} = \overline{a}_n x_n + b_n, \quad n \in \mathbb{N}_0, \tag{4.1}$$

where $(x_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ are sequences in X and $(\overline{a}_n)_{n\geq 0}$ is a sequence in $\mathbb{K}$.

**Theorem 4.1.** *Let $(\varepsilon_n)_{n\geq 0}$ be a sequence of positive real numbers, $(b_n)_{n\geq 0}$ a sequence in X, and $(\overline{a}_n)_{n\geq 0}$ a sequence in $\mathbb{K}$ with the property*

$$\lim_{n\to\infty} \frac{\varepsilon_n |\overline{a}_{n+1}|}{\varepsilon_{n+1}} = 1. \tag{4.2}$$

*Then there exists a sequence $(y_n)_{n\geq 0}$ in X with*

$$\left\| y_{n+1} - \overline{a}_n y_n - b_n \right\| \leq \varepsilon_n, \quad n \in \mathbb{N}_0, \tag{4.3}$$

*such that, for every sequence $(x_n)_{n\geq 0}$ in X satisfying recurrence* (4.1),

$$\sup_{n\in\mathbb{N}} \frac{\left\| x_n - y_n \right\|}{\varepsilon_{n-1}} = \infty. \tag{4.4}$$

The issue of nonstability seems to be a new promising area for research.

## 5. Stability and Completeness

It is well known that the completeness of the target space is of great importance in the theory of Hyers-Ulam stability of functional equations; we could observe this fact for the stability of the Cauchy equation in the second section.

In [63], Forti and Schwaiger proved that if $X$ is a commutative group containing an element of infinite order, $Y$ is a normed space, and the Cauchy functional equation is Hyers-Ulam stable in the class $Y^X$, then the space $Y$ has to be complete (let us also mention here that Moszner [64] showed that all four assumptions are essential to get the completeness of $Y$).

The above-described effect, *stability implies completeness*, was recently proved for some other equations (see [65–68]). Here we present only one result of this kind. It concerns the quadratic equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{5.1}$$

and comes from [67].

**Theorem 5.1.** *Let $X$ be a finitely generated free commutative group and $Y$ be a normed space. If* (5.1) *is Hyers-Ulam stable in the class $Y^X$, then the space $Y$ is complete.*

## 6. The Method of Forti

As Forti [43] (see also, e.g., [69]) has clearly demonstrated, the stability of functional equations in single variable, in particular of the form:

$$\Psi \circ F \circ a = F \tag{6.1}$$

plays a basic role in many investigations of the stability of functional equations in several variables. Some examples presenting that method can be found in [25, 70, 71] (see also [72]). Here we give only one such example that corrects [70, Corollary 3.2], which unfortunately has been published with some details confused. The main tool is the following theorem (see [70, Theorem 2.1]; cf. [43]).

**Theorem 6.1.** *Assume that $(Y, d)$ is a complete metric space, $K$ is a nonempty set, $f : K \to Y$, $\Psi : Y \to Y$, $a : K \to K$, $h : K \to \mathbb{R}_+$, $\lambda \in \mathbb{R}_+$,*

$$\begin{aligned} &d(\Psi \circ f \circ a(x), f(x)) \le h(x), \quad x \in K, \\ &d(\Psi(x), \Psi(y)) \le \lambda d(x, y), \quad x, y \in Y, \\ &H(x) := \sum_{i=0}^{\infty} \lambda^i h\Big(a^i(x)\Big) < \infty, \quad x \in K. \end{aligned} \tag{6.2}$$

*Then, for every $x \in K$, the limit*

$$F(x) := \lim_{n \to \infty} \Psi^n \circ f \circ a^n(x) \tag{6.3}$$

*exists and* $F : K \to Y$ *is the unique function such that* (6.1) *holds and*

$$d(f(x), F(x)) \le H(x), \quad x \in K. \tag{6.4}$$

The next corollary presents the corrected version of [70, Corollary 3.2] and its proof. Let us make some preparations for it.

First, let us recall that a groupoid $(G, +)$ (i.e., a nonempty set $G$ endowed with a binary operation $+ : G^2 \to G$) is uniquely divisible by 2 provided, for each $x \in X$, there is a unique $y \in X$ with $x = 2y := y + y$; such $y$ we denote by $(1/2)x$. Next, we use the notion:

$$2^0 x := x, \qquad 2^n x = 2\left(2^{n-1} x\right), \tag{6.5}$$

and (only if the groupoid is uniquely divisible by 2)

$$2^{-n} x = \frac{1}{2}\left(2^{-n+1} x\right), \tag{6.6}$$

for every $x \in G$, $n \in \mathbb{N}$.

A groupoid $(G, +)$ is square symmetric provided the operation $+$ is square symmetric, that is, $2(x + y) = 2x + 2y$ for $x, y \in G$; it is easy to show by induction that, for each $n \in \mathbb{N}$ (for all $n \in \mathbb{Z}$, if the groupoid is uniquely divisible by 2), we have

$$2^n (x + y) = 2^n x + 2^n y, \quad x, y \in G. \tag{6.7}$$

Clearly every commutative semigroup is a square symmetric groupoid. Next, let $X$ be a linear space over a field $\mathbb{K}$, $a, b \in \mathbb{K}$, $z \in X$, and define a binary operation $* : X^2 \to X$ by

$$x * y := ax + by + z, \quad x, y \in X. \tag{6.8}$$

Then it is easy to check that $(X, *)$ provides a simple example of a square symmetric groupoid.

The square symmetric groupoids have been already considered in several papers investigating the stability of some functional equations (see, e.g., [73–79]). For a description of square symmetric operations we refer to [80].

Finally, we say that $(G, +, d)$ is a complete metric groupoid provided $(G, +)$ is a groupoid, $(G, d)$ is a complete metric space, and the operation $+ : G^2 \to G$ is continuous, in both variables simultaneously, with respect to the metric $d$.

Now we are in a position to present the mentioned above corrected version of [70, Corollary 3.2].

**Corollary 6.2.** *Let $(X,+)$ and $(Y,+)$ be square symmetric groupoids, $(Y,+)$ be uniquely divisible by 2, $(Y,+,d)$ be a complete metric groupoid, $K \subset X$, $2K \subset K$ (i.e., $2a \in K$ for $a \in K$), and $\chi : X^2 \to \mathbb{R}_+$. Suppose that there exist $\xi, \eta \in \mathbb{R}_+$ such that $\xi\eta < 1$,*

$$d\left(\frac{1}{2}x, \frac{1}{2}y\right) \le \xi d(x,y), \quad x,y \in Y,$$
$$\chi(2x,2y) \le \eta\chi(x,y), \quad x,y \in K, \tag{6.9}$$

*and $\varphi : K \to Y$ satisfies*

$$d(\varphi(x+y), \varphi(x)+\varphi(y)) \le \chi(x,y), \quad x,y \in K,\ x+y \in K. \tag{6.10}$$

*Then there is a unique function $F : K \to Y$ with*

$$F(x+y) = F(x) + F(y), \quad x,y \in K,\ x+y \in K, \tag{6.11}$$

$$d(\varphi(x), F(x)) \le \frac{\xi}{1-\xi\eta}\chi(x,x), \quad x \in K. \tag{6.12}$$

*Proof.* From (6.10), with $x = y$, we obtain $d(\varphi(2x), 2\varphi(x)) \le \chi(x,x)$ for $x \in K$, which yields

$$d\left(\frac{1}{2}\varphi(2x), \varphi(x)\right) \le \xi d(\varphi(2x), 2\varphi(x)) \le \xi\chi(x,x), \quad x \in K. \tag{6.13}$$

Hence, by Theorem 6.1 (with $\lambda = \xi$, $f = \varphi$, $\Psi(z) = (1/2)z$, $h(x) = \xi\chi(x,x)$, and $a(x) = 2x$) the limit

$$F(x) := \lim_{n\to\infty} 2^{-n}\varphi(2^n x) \tag{6.14}$$

exists for every $x \in K$ and

$$d(\varphi(x), F(x)) \le \xi\chi(x,x)\sum_{i=0}^{\infty}(\xi\eta)^i \le \frac{\xi}{1-\xi\eta}\chi(x,x), \quad x \in K. \tag{6.15}$$

Next, by (6.7) and (6.10), for every $x, y \in K$ with $x+y \in K$, we have

$$d(2^{-n}\varphi(2^n(x+y)), 2^{-n}\varphi(2^n x) + 2^{-n}\varphi(2^n y)) \le (\xi\eta)^n\chi(x,y), \tag{6.16}$$

for $n \in \mathbb{N}$, whence letting $n \to \infty$ we deduce that $F$ is a solution of (6.11).

It remains to show the uniqueness of $F$. So suppose that $G : K \to Y$,

$$d(\varphi(x), G(x)) \le \frac{\xi}{1-\xi\eta}\chi(x,x), \quad x \in K, \tag{6.17}$$

$$G(x+y) = G(x) + G(y), \quad x,y \in K,\ x+y \in K. \tag{6.18}$$

Then

$$d(F(x),G(x)) \le d\big(F(x),\varphi(x)\big) + d\big(\varphi(x),G(x)\big) \le \frac{2\xi}{1-\xi\eta}\chi(x,x), \quad x \in K, \tag{6.19}$$

and by induction it is easy to show that (6.11) and (6.18) yield $F(2^n x) = 2^n F(x)$ and $G(2^n x) = 2^n G(x)$ for every $x \in K$ and $n \in \mathbb{N}$. Hence, for each $x \in K$,

$$\begin{aligned} d(F(x),G(x)) &= d\big(2^{-n}F(2^n x), 2^{-n}G(2^n x)\big) \\ &\le \xi^n \chi(2^n x, 2^n x) \le (\xi\eta)^n \chi(x,x). \end{aligned} \tag{6.20}$$

Since $\xi\eta < 1$, letting $n \to \infty$ we get $F = G$. □

## 7. The Fixed Point Methods

Apart from the classical method applied by Hyers and its modification proposed by Forti (see also [72]), the fixed point methods seem to be the most popular at the moment in the investigations of the stability of functional equations, both in single and several variables. Although the fixed point method was used for the first time by Baker [39] who applied a variant of Banach's fixed point theorem to obtain the Hyers-Ulam stability of the functional equation

$$f(t) = F\big(t, f(\varphi(t))\big), \tag{7.1}$$

most authors follow Radu's approach (see [81], where a new proof of Theorem 2.1 for $p \in \mathbb{R}_+ \setminus \{1\}$ was given) and make use of a theorem of Diaz and Margolis. Here we only present one of the recent results obtained in this way.

Let us recall that a mapping $f : V^n \to W$, where $V$ is a commutative group, $W$ is a linear space, and $n$ is a positive integer, is called *multiquadratic* if it is quadratic in each variable. Similarly we define *multiadditive* and *multi-Jensen* mappings. Some basic facts on multiadditive functions can be found for instance in [82] (where their application to the representation of polynomial functions is also presented), whereas for the general form of multi-Jensen mappings and their connection with generalized polynomials we refer to [83].

The stability of multiadditive, multi-Jensen, and multiquadratic mappings was recently investigated in [68, 84–93]. In particular, in [88] Radu's approach was applied to the proof of the following theorem.

**Theorem 7.1.** *Let $W$ be a Banach space and for every $i \in \{1,\ldots,n\}$, let $\varphi_i : V^{n+1} \to \mathbb{R}_+$ be a mapping such that*

$$\begin{aligned}
\lim_{j\to\infty} \frac{1}{4^j}\varphi_i\left(2^j x_1, x_2,\ldots,x_{n+1}\right) = \cdots \\
&= \lim_{j\to\infty}\frac{1}{4^j}\varphi_i\left(x_1,\ldots,x_{i-2},2^j x_{i-1},x_i,\ldots,x_{n+1}\right)\\
&= \lim_{j\to\infty}\frac{1}{4^j}\varphi_i\left(x_1,\ldots,x_{i-1},2^j x_i,2^j x_{i+1},x_{i+2},\ldots,x_{n+1}\right)\\
&= \lim_{j\to\infty}\frac{1}{4^j}\varphi_i\left(x_1,\ldots,x_{i+1},2^j x_{i+2},x_{i+3},\ldots,x_{n+1}\right) = \cdots\\
&= \lim_{j\to\infty}\frac{1}{4^j}\varphi_i\left(x_1,\ldots,x_n,2^j x_{n+1}\right) = 0, \quad (x_1,\ldots,x_{n+1})\in V^{n+1},
\end{aligned} \tag{7.2}$$

$$\begin{aligned}
&\varphi_i(x_1,\ldots,x_{i-1},2x_i,2x_i,x_{i+1},\ldots,x_n)\\
&\quad\le 4L_i\varphi_i(x_1,\ldots,x_i,x_i,x_{i+1},\ldots,x_n), \quad (x_1,\ldots,x_n)\in V^n,
\end{aligned} \tag{7.3}$$

*for an $L_i \in (0,1)$. If $f : V^n \to W$ is a mapping satisfying, for any $i \in \{1,\ldots,n\}$,*

$$f(x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n) = 0, \quad (x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)\in V^{n-1}, \tag{7.4}$$

$$\begin{aligned}
&\left\| f(x_1,\ldots,x_{i-1},x_i+x_i',x_{i+1},\ldots,x_n) + f(x_1,\ldots,x_{i-1},x_i-x_i',x_{i+1},\ldots,x_n)\right.\\
&\quad \left. - 2f(x_1,\ldots,x_n) - 2f(x_1,\ldots,x_{i-1},x_i',x_{i+1},\ldots,x_n)\right\|\\
&\quad\le \varphi_i(x_1,\ldots,x_i,x_i',x_{i+1},\ldots,x_n), \quad (x_1,\ldots,x_i,x_i',x_{i+1},\ldots,x_n)\in V^{n+1},
\end{aligned} \tag{7.5}$$

*then for every $i \in \{1,\ldots,n\}$ there exists a unique multiquadratic mapping $F_i : V^n \to W$ such that*

$$\begin{aligned}
\left\| f(x_1,\ldots,x_n) - F_i(x_1,\ldots,x_n)\right\| &\le \frac{1}{4-4L_i}\varphi_i(x_1,\ldots,x_i,x_i,x_{i+1},\ldots,x_n),\\
&\quad (x_1,\ldots,x_n)\in V^n.
\end{aligned} \tag{7.6}$$

Baker's idea (to prove his result it is enough to define suitable (complete) metric space and (contractive) operator, which form follows from the considered equation (in this case $T(a)(t) := F(t,a(\varphi(t)))$), and apply the (Banach) fixed point theorem) was used by several mathematicians, who applied other fixed point theorems to extend and generalize Baker's result. Now, we present some of these recent outcomes.

To formulate the first of them, let us recall that a mapping $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is called a *comparison function* if it is nondecreasing and

$$\lim_{n\to\infty} \gamma^n(t) = 0, \quad t \in (0,\infty). \tag{7.7}$$

In [94], Matkowski's fixed point theorem was applied to the proof of the following generalization of Baker's result.

**Theorem 7.2.** *Let $S$ be a nonempty set, let $(X,d)$ be a complete metric space, $\varphi : S \to S$, and $F : S \times X \to X$. Assume also that*

$$d(F(t,u),F(t,v)) \le \gamma(d(u,v)), \quad t \in S,\ u,v \in X, \tag{7.8}$$

*where $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function, and let $g : S \to X$, $\delta > 0$ be such that*

$$d\big(g(t),F\big(t,g(\varphi(t))\big)\big) \le \delta, \quad t \in S. \tag{7.9}$$

*Then there is a unique function $f : S \to X$ satisfying* (7.1) *and*

$$\rho(f,g) := \sup\{d(f(t),g(t)), t \in S\} < \infty. \tag{7.10}$$

*Moreover, $\rho(f,g) - \gamma(\rho(f,g)) \le \delta$.*

On the other hand, in [95], Baker's idea and a variant of Ćirić's fixed point theorem were used to obtain the following result concerning the stability of (7.1).

**Theorem 7.3.** *Let $S$ be a nonempty set, let $(X,d)$ be a complete metric space, $\varphi : S \to S$, and $F : S \times X \to X$ and*

$$\begin{aligned} d(F(t,x),F(t,y)) \le\ & \alpha_1(x,y)d(x,y) + \alpha_2(x,y)d(x,F(t,x)) \\ & + \alpha_3(x,y)d(y,F(t,y)) + \alpha_4(x,y)d(x,F(t,y)) \\ & + \alpha_5(x,y)d(y,F(t,x)), \quad t \in S,\ x,y \in X, \end{aligned} \tag{7.11}$$

*where $\alpha_1,\dots,\alpha_5 : X \times X \to \mathbb{R}_+$ satisfy*

$$\sum_{i=1}^{5} \alpha_i(x,y) \le \lambda, \tag{7.12}$$

*for all $x,y \in X$ and a $\lambda \in [0,1)$. If $g : S \to X$, $\delta > 0$, and* (7.9) *holds, then there is a unique function $f : S \to X$ satisfying* (7.1) *and*

$$d(f(t),g(t)) \le \frac{(2+\lambda)\delta}{2(1-\lambda)}, \quad t \in S. \tag{7.13}$$

A consequence of Theorem 7.3 is the following result on the stability of the linear functional equation of order 1.

**Corollary 7.4.** *Let $S$ be a nonempty set, let $E$ be a real or complex Banach space, $\varphi : S \to S$, $\alpha : S \to E$, $B : S \to \mathfrak{L}(E)$ (here $\mathfrak{L}(E)$ denotes the Banach algebra of all bounded linear operators on $E$), $\lambda \in [0,1)$, and*

$$\|B(t)\| \leq \lambda, \quad t \in S. \tag{7.14}$$

*If $g : S \to E$, $\delta > 0$, and*

$$\left\| g(t) - \left(\alpha(t) + B(t)\left(g(\varphi(t))\right)\right) \right\| \leq \delta, \quad t \in S, \tag{7.15}$$

*then there exists a unique function $f : S \to E$ satisfying*

$$f(t) = \alpha(t) + B(t)\left(f(\varphi(t))\right), \quad t \in S, \tag{7.16}$$

*and the condition*

$$\| f(t) - g(t) \| \leq \frac{\delta}{1-\lambda}, \quad t \in S. \tag{7.17}$$

In [96], Miheţ gave one more generalization of Baker's result. In order to do this he proved a fixed point alternative and used it in the proof of this generalization. To formulate Miheţ's theorem, let us recall that a mapping $\gamma : [0,\infty] \to [0,\infty]$ is called a *generalized strict comparison function* if it is nondecreasing, $\gamma(\infty) = \infty$,

$$\begin{gathered} \lim_{n\to\infty} \gamma^n(t) = 0, \quad t \in (0,\infty), \\ \lim_{t\to\infty} \left(t - \gamma(t)\right) = \infty. \end{gathered} \tag{7.18}$$

**Theorem 7.5.** *Let $S$ be a nonempty set, let $(X,d)$ be a complete metric space, $\varphi : S \to S$, and $F : S \times X \to X$. Assume also that*

$$d(F(t,u), F(t,v)) \leq \gamma(d(u,v)), \quad t \in S,\ u,v \in X, \tag{7.19}$$

*where $\gamma : [0,\infty] \to [0,\infty]$ is a generalized strict comparison function and let $g : S \to X$, $\delta > 0$ be such that (7.9) holds. Then there is a unique function $f : S \to X$ satisfying (7.1) and*

$$d\left(f(t), g(t)\right) \leq \sup\{s > 0 : s - \gamma(s) \leq \delta\}, \quad t \in S. \tag{7.20}$$

A somewhat different fixed point approach to the Hyers-Ulam stability of functional equations, in which the stability results are simple consequences of some new fixed point theorems, can be found in [34, 52, 53, 97].

Given a nonempty set $S$ and a metric space $(X,d)$, we define $\Delta : (X^S)^2 \to \mathbb{R}_+{}^S$ by

$$\Delta(\xi,\mu)(t) := d(\xi(t),\mu(t)), \quad \xi,\mu \in X^S,\ t \in S. \tag{7.21}$$

Now, we are in a position to present the following fixed point theorem from [34].

**Theorem 7.6.** *Let $S$ be a nonempty set, let $(X,d)$ be a complete metric space, $k \in \mathbb{N}$, $f_1,\dots,f_k : S \to S$, $L_1,\dots,L_k : S \to \mathbb{R}_+$, and let $\Lambda : \mathbb{R}_+^S \to \mathbb{R}_+^S$ be given by*

$$(\Lambda\delta)(t) := \sum_{i=1}^{k} L_i(t)\delta(f_i(t)), \quad \delta \in \mathbb{R}_+^S,\ t \in S. \tag{7.22}$$

*If $\mathcal{T} : X^S \to X^S$ is an operator satisfying the inequality*

$$\Delta(\mathcal{T}\xi,\mathcal{T}\mu)(t) \le \Lambda(\Delta(\xi,\mu))(t), \quad \xi,\mu \in X^S,\ t \in S \tag{7.23}$$

*and functions $\varepsilon : S \to \mathbb{R}_+$ and $g : S \to X$ are such that*

$$\Delta(\mathcal{T}g,g)(t) \le \varepsilon(t), \quad t \in S, \tag{7.24}$$

$$\sum_{n=0}^{\infty}(\Lambda^n\varepsilon)(t) =: \sigma(t) < \infty, \quad t \in S, \tag{7.25}$$

*then for every $t \in S$ the limit*

$$\lim_{n\to\infty}(\mathcal{T}^n g)(t) =: f(t) \tag{7.26}$$

*exists and the function $f : S \to X$, defined in this way, is a unique fixed point of $\mathcal{T}$ with*

$$\Delta(g,f)(t) \le \sigma(t), \quad t \in S. \tag{7.27}$$

A consequence of Theorem 7.6 is the following result on the stability of a quite wide class of functional equations in a single variable.

**Corollary 7.7.** *Let $S$ be a nonempty set, let $(X,d)$ be a complete metric space, $k \in \mathbb{N}$, $f_1,\dots,f_k : S \to S$, $L_1,\dots,L_k : S \to \mathbb{R}_+$, and let a function $\Phi : S \times X^k \to X$ satisfy the inequality*

$$d(\Phi(t,y_1,\dots,y_k),\Phi(t,z_1,\dots,z_k)) \le \sum_{i=1}^{k} L_i(t)d(y_i,z_i), \tag{7.28}$$

*for any $(y_1,\dots,y_k),(z_1,\dots,z_k) \in X^k$ and $t \in S$, and $\mathcal{T} : X^S \to X^S$ be an operator defined by*

$$(\mathcal{T}\varphi)(t) := \Phi(t,\varphi(f_1(t)),\dots,\varphi(f_k(t))), \quad \varphi \in X^S,\ t \in S. \tag{7.29}$$

*Assume also that* $\Lambda$ *is given by* (7.22) *and functions* $g : S \to X$ *and* $\varepsilon : S \to \mathbb{R}_+$ *are such that*

$$d\big(g(t), \Phi\big(t, g\big(f_1(t)\big), \ldots, g\big(f_k(t)\big)\big)\big) \le \varepsilon(t), \quad t \in S \tag{7.30}$$

*and* (7.25) *holds. Then for every* $t \in S$ *limit* (7.26) *exists and the function* $f : S \to X$ *is a unique solution of the functional equation*

$$\Phi\big(t, f\big(f_1(t)\big), \ldots, f\big(f_k(t)\big)\big) = f(t), \quad t \in S \tag{7.31}$$

*satisfying inequality* (7.27).

Let us also mention here that very recently Cădariu et al. [97] improved the above two outcomes considering, instead of that given by (7.22), a more general operator $\Lambda$.

Next, following [53], we deal with the case of non-Archimedean metric spaces. In order to do this, we introduce some notations and definitions.

Let $S$ be a nonempty set. For any $\delta_1, \delta_2 \in \mathbb{R}_+^S$ we write $\delta_1 \le \delta_2$ provided

$$\delta_1(t) \le \delta_2(t), \quad t \in S, \tag{7.32}$$

and we say that an operator $\Lambda : \mathbb{R}_+^S \to \mathbb{R}_+^S$ is *nondecreasing* if it satisfies the condition

$$\Lambda\delta_1 \le \Lambda\delta_2, \quad \delta_1, \delta_2 \in \mathbb{R}_+^S, \ \delta_1 \le \delta_2. \tag{7.33}$$

Moreover, given a sequence $(g_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_+^S$, we write $\lim_{n\to\infty} g_n = 0$ provided

$$\lim_{n\to\infty} g_n(t) = 0, \quad t \in S. \tag{7.34}$$

We will also use the following hypothesis concerning operators $\Lambda : \mathbb{R}_+^S \to \mathbb{R}_+^S$:

($\mathcal{C}$) $\lim_{n\to\infty} \Lambda\delta_n = 0$ for every sequence $(\delta_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_+^S$ with $\lim_{n\to\infty}\delta_n = 0$.

Finally, recall that a metric $d$ on a nonempty set $X$ is called *non-Archimedean* (or an *ultrametric*) provided

$$d(x, z) \le \max\{d(x, y), d(y, z)\}, \quad x, y, z \in X. \tag{7.35}$$

We can now formulate the following fixed point theorem.

**Theorem 7.8.** *Let* $S$ *be a nonempty set, let* $(X, d)$ *be a complete non-Archimedean metric space, and let* $\Lambda : \mathbb{R}_+^S \to \mathbb{R}_+^S$ *be a nondecreasing operator satisfying hypothesis* ($\mathcal{C}$). *If* $\mathcal{T} : X^S \to X^S$ *is an operator satisfying inequality* (7.23) *and functions* $\varepsilon : S \to \mathbb{R}_+$ *and* $g : S \to X$ *are such that*

$$\lim_{n\to\infty} \Lambda^n \varepsilon = 0, \tag{7.36}$$

*and* (7.24) *holds, then for every* $t \in S$ *limit* (7.26) *exists and the function* $f : S \to X$*, defined in this way, is a fixed point of* $\mathcal{T}$ *with*

$$\Delta(g, f)(t) \le \sup_{n \in \mathbb{N}_0} (\Lambda^n \varepsilon)(t) =: \sigma(t), \quad t \in S. \tag{7.37}$$

*If, moreover,*

$$(\Lambda\sigma)(t) \le \sup_{n \in \mathbb{N}_0} \left(\Lambda^{n+1}\varepsilon\right)(t), \quad t \in S, \tag{7.38}$$

*then* $f$ *is the unique fixed point of* $\mathcal{T}$ *satisfying* (7.37)*.*

An immediate consequence of Theorem 7.8 is the following result on the stability of (7.31) in complete non-Archimedean metric spaces.

**Corollary 7.9.** *Let* $S$ *be a nonempty set,* $(X, d)$ *be a complete non-Archimedean metric space,* $k \in \mathbb{N}$*,* $f_1, \dots, f_k : S \to S$*,* $L_1, \dots, L_k : S \to \mathbb{R}_+$*, and a function* $\Phi : S \times X^k \to X$ *satisfy the inequality*

$$d(\Phi(t, y_1, \dots, y_k), \Phi(t, z_1, \dots, z_k)) \le \max_{i \in \{1,\dots,k\}} L_i(t) d(y_i, z_i), \tag{7.39}$$

*for any* $(y_1, \dots, y_k), (z_1, \dots, z_k) \in X^k$ *and* $t \in S$*, and* $\mathcal{T} : X^S \to X^S$ *be an operator defined by* (7.29)*. Assume also that* $\Lambda$ *is given by*

$$(\Lambda\delta)(t) := \max_{i \in \{1,\dots,k\}} L_i(t)\delta(f_i(t)), \quad \delta \in \mathbb{R}_+^S, \ t \in S, \tag{7.40}$$

*and functions* $g : S \to X$ *and* $\varepsilon : S \to \mathbb{R}_+$ *are such that* (7.30) *and* (7.36) *hold. Then for every* $t \in S$ *limit* (7.26) *exists and the function* $f : S \to X$ *is a solution of functional equation* (7.31) *satisfying inequality* (7.37)*.*

Given nonempty sets $S, Z$ and functions $\varphi : S \to S$, $F : S \times Z \to Z$, we define an operator $\mathcal{L}_\varphi^F : Z^S \to Z^S$ by

$$\mathcal{L}_\varphi^F(g)(t) := F(t, g(\varphi(t))), \quad g \in Z^S, \ t \in S, \tag{7.41}$$

and we say that $\mathcal{U} : Z^S \to Z^S$ is an *operator of substitution* provided $\mathcal{U} = \mathcal{L}_\psi^G$ with some $\psi : S \to S$ and $G : S \times Z \to Z$. Moreover, if $G(t, \cdot)$ is continuous for each $t \in S$ (with respect to a topology in $Z$), then we say that $\mathcal{U}$ is *continuous*.

The following fixed point theorem was proved in [52].

**Theorem 7.10.** *Let* $S$ *be a nonempty set, let* $(X, d)$ *be a complete metric space,* $\Lambda : S \times \mathbb{R}_+ \to \mathbb{R}_+$*,* $\mathcal{T} : X^S \to X^S$*,* $\varphi : S \to S$*, and*

$$\Delta(\mathcal{T}\alpha, \mathcal{T}\beta)(t) \le \Lambda(t, \Delta(\alpha \circ \varphi, \beta \circ \varphi)(t)), \quad \alpha, \beta \in X^S, \ t \in S. \tag{7.42}$$

*Assume also that for every $t \in S$, $\Lambda_t := \Lambda(t,\cdot)$ is nondecreasing, $\varepsilon : S \to \mathbb{R}_+$, $g : S \to X$,*

$$\sum_{n=0}^{\infty}\left(\left(\mathcal{L}_{\varphi}^{\Lambda}\right)^{n}\varepsilon\right)(t) =: \sigma(t) < \infty, \quad t \in S, \tag{7.43}$$

*and* (7.24) *holds. Then for every $t \in S$ limit* (7.26) *exists and inequality* (7.27) *is satisfied. Moreover, the following two statements are true.*

(i) *If $\mathcal{T}$ is a continuous operator of substitution or $\Lambda_t$ is continuous at* 0 *for each $t \in S$, then $f$ is a fixed point of $\mathcal{T}$.*

(ii) *If $\Lambda_t$ is subadditive (that is,*

$$\Lambda_t(a+b) \le \Lambda_t(a) + \Lambda_t(b), \tag{7.44}$$

*for all $a, b \in \mathbb{R}_+$) for each $t \in S$, then $\mathcal{T}$ has at most one fixed point $f \in X^S$ such that*

$$\Delta(g,f)(t) \le M\sigma(t), \quad t \in S, \tag{7.45}$$

*for a positive integer $M$.*

Theorem 7.10 with $\mathcal{T} = \mathcal{L}_{\varphi}^{F}$ immediately gives the following generalization of Baker's result.

**Corollary 7.11.** *Let $S$ be a nonempty set, let $(X,d)$ be a complete metric space, $F : S \times X \to X$, $\Lambda : S \times \mathbb{R}_+ \to \mathbb{R}_+$, and*

$$d(F(t,x),F(t,y)) \le \Lambda(t,d(x,y)), \quad t \in S, \quad x,y \in X. \tag{7.46}$$

*Assume also that $\varphi : S \to S$, $\varepsilon : S \to \mathbb{R}_+$,* (7.43) *holds, $g : S \to X$, for every $t \in S$, $\Lambda_t := \Lambda(t,\cdot)$ is nondecreasing, $F(t,\cdot)$ is continuous, and*

$$d(g(t),F(t,g(\varphi(t)))) \le \varepsilon(t), \quad t \in S. \tag{7.47}$$

*Then for every $t \in S$ the limit*

$$f(t) := \lim_{n\to\infty}\left(\mathcal{L}_{\varphi}^{F}\right)^{n}(g)(t) \tag{7.48}$$

*exists,* (7.27) *holds and $f$ is a solution of* (7.1)*. Moreover, if for every $t \in S$, $\Lambda_t$ is subadditive and $M \in \mathbb{N}$, then $f : S \to X$ is the unique solution of* (7.1) *fulfilling* (7.45).

Let us finally mention that the fixed point method is also a useful tool for proving the Hyers-Ulam stability of differential (see [98, 99]) and integral equations (see for instance [100–102]). Some further details and information on the connections between the fixed point theory and the Hyers-Ulam stability can be found in [103].

## 8. Stability in Non-Archimedean Spaces

Let us recall that a *non-Archimedean valuation* in a field $\mathbb{K}$ is a function $|\cdot| : \mathbb{K} \to \mathbb{R}_+$ with

$$
\begin{gathered}
|r| = 0, \quad \text{iff } r = 0, \\
|rs| = |r||s|, \quad r, s \in \mathbb{K}, \\
|r + s| \leq \max\{|r|, |s|\}, \quad r, s \in \mathbb{K}.
\end{gathered}
\tag{8.1}
$$

A field endowed with a non-Archimedean valuation is said to be *non-Archimedean*. Let $X$ be a linear space over a field $\mathbb{K}$ with a non-Archimedean valuation that is nontrivial (i.e., we additionally assume that there is an $r_0 \in \mathbb{K}$ such that $0 \neq |r_0| \neq 1$). A function $\|\cdot\| : X \to \mathbb{R}_+$ is said to be a *non-Archimedean norm* if it satisfies the following conditions:

$$
\begin{gathered}
\|x\| = 0, \quad \text{iff } x = 0, \\
\|rx\| = |r|\|x\|, \quad r \in \mathbb{K},\ x \in X, \\
\|x + y\| \leq \max\{\|x\|, \|y\|\}, \quad x, y \in X.
\end{gathered}
\tag{8.2}
$$

If $\|\cdot\| : X \to \mathbb{R}_+$ is a non-Archimedean norm in $X$, then the pair $(X, \|\cdot\|)$ is called a *non-Archimedean normed space*.

If $(X, \|\cdot\|)$ is a non-Archimedean normed space, then it is easily seen that the function $d_X : X^2 \to \mathbb{R}_+$, given by $d_X(x, y) := \|x - y\|$, is a non-Archimedean metric on $X$. Therefore non-Archimedean normed spaces are special cases of metric spaces. The most important examples of non-Archimedean normed spaces are the $p$-adic numbers $\mathbb{Q}_p$ (here $p$ is any prime number), which have gained the interest of physicists because of their connections with some problems coming from quantum physics, $p$-adic strings, and superstrings (see, for instance, [104]).

In [105], correcting the mistakes in the proof given by the second author in 1968, Arriola and Beyer showed that the Cauchy functional equation is Hyers-Ulam stable in $\mathbb{R}^{\mathbb{Q}_p}$. Schwaiger [106] did the same in the class of functions from a commutative group which is uniquely divisible by $p$ to a Banach space over $\mathbb{Q}_p$. In 2007, Moslehian and Rassias [107] proved the generalized Hyers-Ulam stability of the Cauchy equation in a more general setting, namely, in complete non-Archimedean normed spaces. After their results a lot of papers (see, for instance, [87–89, 93] and the references given there) on the stability of other equations in such spaces have been published. Here we present only one example of these outcomes which is a generalization of the result of Moslehian and Rassias and was obtained in [87] (cf. also Theorem 7 in [106]).

**Theorem 8.1.** *Let $V$ be a commutative semigroup and $W$ be a complete non-Archimedean normed space over a non-Archimedean field of characteristic different from 2. Assume also that $n \in \mathbb{N}$ and for every $i \in \{1,\ldots,n\}$, $\varphi_i : V^{n+1} \to \mathbb{R}_+$ is a mapping such that for each $(x_1,\ldots,x_{n+1}) \in V^{n+1}$,*

$$\begin{aligned}
&\lim_{j\to\infty} \frac{1}{|2|^j}\varphi_i\Big(2^j x_1, x_2, \ldots, x_{n+1}\Big) = \cdots \\
&\qquad = \lim_{j\to\infty} \frac{1}{|2|^j}\varphi_i\Big(x_1, \ldots, x_{i-2}, 2^j x_{i-1}, x_i, \ldots, x_{n+1}\Big) \\
&\qquad = \lim_{j\to\infty} \frac{1}{|2|^j}\varphi_i\Big(x_1, \ldots, x_{i-1}, 2^j x_i, 2^j x_{i+1}, x_{i+2}, \ldots, x_{n+1}\Big) \\
&\qquad = \lim_{j\to\infty} \frac{1}{|2|^j}\varphi_i\Big(x_1, \ldots, x_{i+1}, 2^j x_{i+2}, x_{i+3}, \ldots, x_{n+1}\Big) = \cdots \\
&\qquad = \lim_{j\to\infty} \frac{1}{|2|^j}\varphi_i\Big(x_1, \ldots, x_n, 2^j x_{n+1}\Big) = 0,
\end{aligned} \tag{8.3}$$

*and the limit*

$$\lim_{k\to\infty} \max\left\{\frac{1}{|2|^j}\varphi_i\Big(x_1, \ldots, x_{i-1}, 2^j x_i, 2^j x_i, x_{i+1}, \ldots, x_n\Big) : 0 \le j < k\right\}, \tag{8.4}$$

*denoted by $\widetilde{\varphi}_i(x_1,\ldots,x_n)$, exists. If $f : V^n \to W$ is a function satisfying*

$$\begin{aligned}
&\big\| f(x_1, \ldots, x_{i-1}, x_i + x_i', x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_n) \\
&\quad - f(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n)\big\| \\
&\le \varphi_i(x_1, \ldots, x_i, x_i', x_{i+1}, \ldots, x_n), \\
&\quad (x_1, \ldots, x_i, x_i', x_{i+1}, \ldots, x_n) \in V^{n+1}, \quad i \in \{1,\ldots,n\},
\end{aligned} \tag{8.5}$$

*then for every $i \in \{1,\ldots,n\}$ there exists a multiadditive mapping $F_i : V^n \to W$ for which*

$$\big\| f(x_1,\ldots,x_n) - F_i(x_1,\ldots,x_n)\big\| \le \frac{1}{|2|}\widetilde{\varphi}_i(x_1,\ldots,x_n), \quad (x_1,\ldots,x_n) \in V^n. \tag{8.6}$$

*For every $i \in \{1,\ldots,n\}$ the function $F_i$ is given by*

$$F_i(x_1,\ldots,x_n) := \lim_{j\to\infty} \frac{1}{2^j} f\Big(x_1, \ldots, x_{i-1}, 2^j x_i, x_{i+1}, \ldots, x_n\Big), \quad (x_1,\ldots,x_n) \in V^n. \tag{8.7}$$

*If, moreover,*

$$\lim_{l\to\infty}\lim_{k\to\infty}\max\left\{\frac{1}{|2|^j}\varphi_i\left(x_1,\ldots,x_{i-1},2^jx_i,2^jx_i,x_{i+1},\ldots,x_n\right):l\le j<k+l\right\}=0,$$
$$i\in\{1,\ldots,n\},(x_1,\ldots,x_n)\in V^n, \tag{8.8}$$

*then for every $i\in\{1,\ldots,n\}$, $F_i$ is the unique multiadditive mapping satisfying condition* (8.6).

It seems that [53] was the first paper where the Hyers-Ulam stability was considered in the most general setting, namely, in complete non-Archimedean metric spaces. One of its results (Corollary 7.9) was mentioned in Section 6; the others, which can be also derived from Theorem 7.8, read as follows (from now on $X$ denotes a nonempty set and $(Y,d)$ stands for a complete non-Archimedean metric space).

**Corollary 8.2.** *Suppose that $(Y,*)$ is a groupoid and*

$$d(x*z,y*z)=d(x,y),\quad x,y,z\in Y. \tag{8.9}$$

*Let $k,m\in\mathbb{N}$, $L_1,\ldots,L_k:X\to\mathbb{R}_+$, $G:X\times Y^m\to Y$, $f_1,\ldots,f_k,g_1,\ldots,g_m:X\to X$, and $\Phi:X\times Y^k\to Y$ satisfy inequality* (7.39) *for any $(y_1,\ldots,y_k),(z_1,\ldots,z_k)\in Y^k$ and $t\in X$. Assume also that functions $\varphi,\mu_1,\ldots,\mu_m:X\to Y$, and $\varepsilon:X\to\mathbb{R}_+$ are such that*

$$d(\varphi(x),\Phi(x,\varphi(f_1(x)),\ldots,\varphi(f_k(x)))*G(x,\mu_1(g_1(x)),\ldots,\mu_m(g_m(x))))\le\varepsilon(x),\quad x\in X \tag{8.10}$$

*and* (7.36) *holds with $\Lambda$ given by* (7.40)*. Then the limit $\lim_{n\to\infty}(\mathcal{T}_0{}^n\varphi)(x)=:\psi(x)$ exists for every $x\in X$, where $\mathcal{T}_0:Y^X\to Y^X$ is defined by*

$$(\mathcal{T}_0\xi)(x):=\Phi(x,\xi(f_1(x)),\ldots,\xi(f_k(x)))*G(x,\mu_1(g_1(x)),\ldots,\mu_m(g_m(x))), \tag{8.11}$$

*and the functions $\mu_1,\ldots,\mu_m$, and $\psi:X\to Y$ fulfil*

$$\psi(x)=\Phi(x,\psi(f_1(x)),\ldots,\psi(f_k(x)))*G(x,\mu_1(g_1(x)),\ldots,\mu_m(g_m(x))),$$
$$d(\varphi(x),\psi(x))\le\sup\nolimits_{n\in\mathbb{N}_0}(\Lambda^n\varepsilon)(x),\quad x\in X. \tag{8.12}$$

**Corollary 8.3.** *Suppose that $(Y,+)$ is a commutative group and $d$ is invariant (i.e., $d(x+z,y+z)=d(x,y)$ for $x,y,z\in Y$). Let $k\in\mathbb{N}$, $\varphi_1,\ldots,\varphi_k:X\to Y$, $\Phi_1,\ldots,\Phi_k:X\times Y\to Y$, and $\varepsilon:X\to\mathbb{R}_+$ satisfy*

$$d\left(\sum_{i=1}^k\varphi_i(x),\sum_{i=1}^k\Phi_i(x,\varphi_i(x))\right)\le\varepsilon(x),\quad x\in X. \tag{8.13}$$

*Assume also that there is a number $j \in \{1,\ldots,k\}$ such that*

$$d\left(\Phi_j(x,y),\Phi_j(x,z)\right) \le L_j(x)d(y,z), \quad x \in X,\ y,z \in Y \tag{8.14}$$

*with a function $L_j : X \to [0,1)$. Then the limit $\lim_{n\to\infty}(\mathcal{T}^n\varphi_j)(x) =: \psi(x)$ exists for every $x \in X$, where $\mathcal{T} : Y^X \to Y^X$ is given by*

$$(\mathcal{T}\varphi)(x) := \Phi_j(x,\varphi(x)) + \sum_{i=1,i\ne j}^{k} \Phi_i(x,\varphi_i(x)) - \sum_{i=1,i\ne j}^{k} \varphi_i(x), \tag{8.15}$$

*and the function $\psi : X \to Y$, defined in this way, is the unique solution of the functional equation*

$$\Phi_j(x,\psi(x)) + \sum_{i=1,i\ne j}^{k} \Phi_i(x,\varphi_i(x)) = \psi(x) + \sum_{i=1,i\ne j}^{k} \varphi_i(x), \tag{8.16}$$

*such that $d(\varphi_j(x),\psi(x)) \le \varepsilon(x)$ for $x \in X$.*

**Corollary 8.4.** *Let $(X,*)$ be a groupoid, $k \in \mathbb{N}$, $d_1,\ldots,d_k \in X$, $c \in \mathbb{R}_+$, $\varphi : X \to Y$, $L_1,\ldots,L_k : X \to \mathbb{R}_+$, a function $\Phi : X\times Y^k \to Y$ satisfy inequality (7.39) for any $(y_1,\ldots,y_k),(z_1,\ldots,z_k) \in Y^k$ and $t \in X$, and $\mathcal{T} : Y^X \to Y^X$ be an operator defined by*

$$(\mathcal{T}\varphi)(x) := \Phi(x,\varphi(x*d_1),\ldots,\varphi(x*d_k)), \quad \varphi \in Y^X,\ x \in X. \tag{8.17}$$

*Assume also that a function $\sigma : X \to \mathbb{R}_+$ is such that*

$$\begin{gathered} q := \sup_{x\in X}\left(\max_{i\in\{1,\ldots,k\}} L_i(x)\sigma(d_i)\right) < 1, \\ \sigma(x*y) \le \sigma(x)\sigma(y), \qquad x,y \in X, \\ d\left(\varphi(x),\Phi(x,\varphi(x*d_1),\ldots,\varphi(x*d_k))\right) \le c\ \sigma(x), \quad x \in X. \end{gathered} \tag{8.18}$$

*Then there exists a function $\psi : X \to Y$ such that*

$$\begin{gathered} \psi(x) = \Phi(x,\psi(x*d_1),\ldots,\psi(x*d_k)), \quad x \in X, \\ d(\varphi(x),\psi(x)) \le c\sigma(x), \quad x \in X. \end{gathered} \tag{8.19}$$

## 9. Functional Congruences

In this section $Y$ denotes a real Banach space, $K$ stands for a subgroup of the group $(Y,+)$, and $E$ is a real linear space, unless explicitly stated otherwise. We write

$$D_1 + TD_2 := \{x + ty : x \in D_1, y \in D_2, t \in T\}, \tag{9.1}$$

for $T \subset \mathbb{R}$ and $D_1, D_2 \subset E$.

Baron et al. [108] have started the study of conditions on a convex set $C \subset Y$ and a function $h : E \to Y$ with

$$h(x+y) - h(x) - h(y) \in K + C, \quad \text{for } x, y \in E, \tag{9.2}$$

which guarantee that there exists an additive function $A : E \to Y$ (i.e., $A(x+y) = A(x) + A(y)$ for $x, y \in E$) such that

$$h(x) - A(x) \in K + C, \quad \text{for } x \in E, \tag{9.3}$$

or, in other words, that $h$ can be represented in the form

$$h = A + \gamma + \kappa, \tag{9.4}$$

with some $\gamma : E \to C$, $\kappa : E \to K$. That is a continuation and an extension of some earlier investigations in [109–111]. Here we present some examples of results from [112] (see also [113, 114]), which generalize those in [108].

They correspond simultaneously to the classical Ulam's problem of stability for the Cauchy equation (with $K = \{0\}$) and to the subjects considered, for example, in [115–128], where functions satisfying (9.2) with $C = \{0\}$ (mainly on restricted domains), have been investigated. The latter issue appears naturally in connection with descriptions of subgroups of some groups (see [129]) and some representations of characters (see, e.g., [109, 115, 116, 122–125]).

It is proved in [112, Example 1] that without any additional assumptions on $h$, the mentioned above decomposition of $h$ is not possible in general.

In what follows we say that two nonempty sets $D_1, D_2 \subset Y$ are separated provided

$$\inf\{\|x - y\| : x \in D_1, y \in D_2\} > 0. \tag{9.5}$$

In the rest of this section $C$ stands for a nonempty closed, symmetric (i.e., $-x \in C$ for each $x \in C$), and convex subset of $Y$. The next theorem (see [112, Theorem 10]) involves the notion of Christensen measurability and we refer to [130] (cf. [131, 132]) for the details concerning it.

**Theorem 9.1.** *Suppose that $E$ is a Polish real linear space, $h : E \to Y$ is Christensen measurable,* (9.2) *holds, and one of the following three conditions is valid.*

(i) *The sets $4C$ and $K \setminus \{0\}$ are separated and $Y$ is separable.*

(ii) *The sets $10C$ and $K \setminus \{0\}$ are separated, $K$ is countable, and $C$ is bounded.*

(iii) *The sets $(10 + \varepsilon)C$ and $K \setminus \{0\}$ are separated for some $\varepsilon \in (0, \infty)$ and $K$ is countable.*

*Then there exists an additive function $A : E \to Y$ satisfying* (9.3).

*Moreover, in the case where $C$ is bounded, $A$ is unique and continuous.*

*Remark 9.2.* There arises a natural question to what extent each of assumptions (i)–(iii) in Theorem 9.1, but also in Theorems 9.3 and 9.4, can be weakened (if at all)?

Certainly, the boundedness of $C$ in Theorem 9.1 is necessary for the uniqueness and continuity of $A$ as it follows from [112, Remark 4]. It is also the case for the uniqueness, linearity, and continuity of $A$ in Theorems 9.3 and 9.4.

For the next theorem we need the notion of Baire property. Let us recall that $h : E \to Y$ has the Baire property provided, for every open set $V \subset Y$, the set $h^{-1}(V)$ has the Baire property, that is, there are an open set $U \subset E$ and sets $T_1, T_2 \subset E$ of the first category, with

$$h^{-1}(V) = (U \cup T_1) \setminus T_2. \tag{9.6}$$

Let us yet recall that a topology in a real linear space $Z$ is called semilinear provided the mapping

$$\mathbb{R} \times Z \times Z \ni (\alpha, x, y) \longrightarrow \alpha x + y \in Z \tag{9.7}$$

is separately continuous with respect to each variable (see, e.g., [133]). A real linear space $Z$ endowed with a semilinear topology is called a semilinear topological space.

Now we are in a position to present [112, Theorem 13].

**Theorem 9.3.** *Suppose that $E$ is a real semilinear topological space of the second category of Baire (in itself), one of conditions (i)–(iii) of Theorem 9.1 is valid, and $h : E \to Y$ fulfills* (9.2) *and has the Baire property. Then there exists an additive function $A : E \to Y$ such that* (9.3) *holds.*

*Moreover, in the case where $C$ is bounded, $A$ is unique and linear; in the case where $C$ is bounded and $E$ is a linear topological space, $A$ is unique and continuous.*

For our last theorem (see [112, Theorem 15]), let us recall that $f$, mapping a topological space $X$ into $Y$, is universally measurable provided, for every open set $U \subset Y$, the set $f^{-1}(U)$ is universally measurable, that is, it is in the universal completion of the Borel field in $E$ (see e.g., [131, 132]); $f$ is Borel provided, for every Borel set $D \subset Y$, the set $f^{-1}(D)$ is Borel in $X$.

**Theorem 9.4.** *Let $E$ be endowed with a topology such that the mapping*

$$\mathbb{R} \ni t \longrightarrow tx \in E \tag{9.8}$$

*is Borel for every $x \in E$, one of conditions (i)–(iii) of Theorem 9.1 be valid, and $h : E \to Y$ fulfill* (9.2) *and be universally measurable. Then there exists an additive function $A : E \to Y$ such that* (9.3) *holds.*

*Moreover, if $C$ is bounded, then $A$ is unique and linear; if $C$ is bounded and the topology in $E$ is linear and metrizable with a complete metric, then $A$ is unique and continuous.*

## 10. Hyperstability

In this part, $X$ and $Y$ are normed spaces, $U \subset X$ is nonempty, and $\varphi : U^2 \to \mathbb{R}_+$. We say that the following conditional Cauchy functional equation

$$f(x+y) = f(x) + f(y), \quad x, y \in U, \ x+y \in U \tag{10.1}$$

is $\varphi$-hyperstable in the class of functions $f : U \to Y$ provided each $f : U \to Y$ satisfying the inequality

$$\|f(x+y) - f(x) - f(y)\| \le \varphi(x,y), \quad x, y \in U, \ x + y \in U, \tag{10.2}$$

must fulfil (10.1).

According to our best knowledge, the first hyperstability result was published in [134] (for the constant function $\varphi$) and concerned the ring homomorphisms. However, the term *hyperstability* has been used for the first time probably in [135].

Now we present two very elementary hyperstability results for (10.1). The first one is a simple consequence of Corollary 2.2.

**Corollary 10.1.** *Let $L$ and $p \neq 1$ be fixed positive real numbers, $2U = U$, $C : U \to X$, and $C(2x) = 2C(x)$ for $x \in U$. Assume that $f : U \to Y$ satisfies* (10.2) *with $\varphi : U^2 \to \mathbb{R}$ given by*

$$\varphi(x,y) = L\|C(x) - C(y)\|^p, \quad x, y \in U. \tag{10.3}$$

*Then $f$ is a solution to* (10.1).

*Proof.* It is easily seen that condition (2.13) is valid with $\varepsilon = 1$ for $p < 1$ and with $\varepsilon = -1$ for $p > 1$. Hence it is enough to use Corollary 2.2. □

We have as well the following.

**Proposition 10.2.** *Let $X > 2$ and let $g : X \to Y$. Suppose that there exist functions $\eta, \mu : \mathbb{R} \to \mathbb{R}$ with $\mu(0) = 0$ and*

$$\|g(x+y) - g(x) - g(y)\| \le \mu(\eta(\|x\|) - \eta(\|y\|)), \quad x, y \in X. \tag{10.4}$$

*Then $g$ is additive.*

*Proof.* Inequality (10.4) yields

$$g(x+y) = g(x) + g(y), \quad x, y \in X, \quad \|x\| = \|y\|. \tag{10.5}$$

Hence, by [136, Theorem 3.1], $g$ is additive. □

Below we provide two simple examples of applications of those hyperstability results; they correspond to the investigations in [137–149] concerning the inhomogeneous Cauchy equation and the cocycle equation.

**Corollary 10.3.** *Let $G : U^2 \to Y$ be such that $G(x_0, y_0) \neq 0$ for some $x_0, y_0 \in U$ with $x_0 + y_0 \in U$. Assume that one of the following two conditions is valid.*

(a) *$2U = U$ and there exist $C : U \to X$ and positive reals $L$ and $p \neq 1$ with*

$$\begin{gathered} C(2x) = 2C(x), \quad x \in U, \\ \|G(x,y)\| \le L\|C(x) - C(y)\|^p, \quad x, y \in U. \end{gathered} \tag{10.6}$$

(b) $U = X$, $X > 2$ *and there are functions* $\eta, \mu : \mathbb{R} \to \mathbb{R}$ *with* $\mu(0) = 0$, *and*

$$\|G(x,y)\| \le \mu(\eta(\|x\|) - \eta(\|y\|)), \quad x, y \in X. \tag{10.7}$$

*Then the conditional functional equation*

$$g(x+y) = g(x) + g(y) + G(x,y), \quad x, y \in U, \ x+y \in U \tag{10.8}$$

*has no solutions in the class of functions* $g : U \to Y$.

*Proof.* Let $g : U \to Y$ be a solution to (10.8). Then

$$\|g(x+y) - g(x) - g(y)\| \le \|G(x,y)\|, \quad x, y \in U, \ x+y \in U. \tag{10.9}$$

Hence, by Corollary 10.1 (if (a) holds) and Proposition 10.2 (if (b) holds), $g$ is a solution to (10.1). This means that $G(x_0, y_0) = 0$, which is a contradiction. □

**Corollary 10.4.** *Let* $U = X$, *and* $G : X^2 \to Y$ *be a symmetric (i.e.,* $G(x,y) = G(y,x)$ *for* $x, y \in X$*) solution to the cocycle functional equation*

$$G(x,y) + G(x+y,z) = G(x,y+z) + G(y,z), \quad x, y, z \in X. \tag{10.10}$$

*Assume that one of conditions (a) and (b) holds. Then* $G(x,y) = 0$, *for* $x, y \in X$.

*Proof.* $G$ is coboundary (see [146] or [149]), that is, there is $g : X \to Y$ with $G(x,y) = g(x+y) - g(x) - g(y)$ for $x, y \in X$. Clearly $g$ is a solution to (10.8). Hence Corollary 10.3 implies the statement. □

For some further (more involved) examples of hyperstability results, concerning also some other functional equations, we refer to [150–153]. The issue of hyperstability seems to be a very promising subject to study within the theory of Ulam's type stability.

## 11. Stability of Composite Functional Equations

The problem of studying the stability of the composite functional equations was raised by Ger in 2000 (at the 38th International Symposium on Functional Equations) and in particular it concerned the Hyers-Ulam stability of the Gołąb-Schinzel equation

$$f(x + f(x)y) = f(x)f(y), \tag{11.1}$$

for the information concerning that equation and generalizations of it we refer to the survey paper [154].

In 2005, Chudziak [155] answered this question and proved that in the class of continuous real functions equation (11.1) is superstable. More precisely, he showed that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying

$$\left|f(x + f(x)y) - f(x)f(y)\right| \le \varepsilon, \quad x, y \in \mathbb{R}, \tag{11.2}$$

with a positive real number $\varepsilon$, then either $f$ is bounded or it is a solution of (11.1).

In [156], Chudziak and Tabor generalized this result. Namely, they proved that if $\mathbb{K}$ is a subfield of $\mathbb{C}$, $X$ is a vector space over $\mathbb{K}$ and $f : X \to \mathbb{K}$, is a function satisfying the inequality

$$\left|f(x + f(x)y) - f(x)f(y)\right| \le \varepsilon, \quad x, y \in X \tag{11.3}$$

and such that the limit

$$\lim_{t\to 0} f(tx) \tag{11.4}$$

exists (not necessarily finite) for every $x \in X \setminus f^{-1}(0)$, then either $f$ is bounded or it is a solution of (11.1) on $X$. Therefore, (11.1) is superstable also in this class of functions.

Later on, in [157, 158], the same results have been proved for the generalized Gołąb-Schinzel equation

$$f(x + f(x)^n y) = \lambda f(x)f(y), \tag{11.5}$$

where $n$ is a positive integer and $\lambda$ is a nonzero complex number. If $\lambda \in \mathbb{R}$, then functional equation (11.5) is superstable in the class of continuous real functions. If $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $\lambda \in \mathbb{K} \setminus \{0\}$, and $X$ is a vector space over $\mathbb{K}$, then (11.5) is superstable in the class of functions $f : X \to \mathbb{K}$ such that the limit (11.4) (not necessarily finite) exists for every $x \in X \setminus f^{-1}(0)$.

It is known (see [159]) that the phenomenon of superstability is caused by the fact that we mix two operations. Namely, on the right-hand side of (11.1) we have a product, but in (11.2) we measure the distance between the two sides of (11.1) using the difference. Therefore, it is more natural to measure the difference between 1 and the quotients of the sides of (11.1). In [159] it has been proved that for the exponential equation this approach leads to the traditional stability. The result is different in the case of the Gołąb-Schinzel equation.

In [160] it is proved that if $f : \mathbb{R} \to \mathbb{R}$ is continuous at 0 and satisfies the following two inequalities

$$\begin{gathered} \left|\frac{f(x)f(y)}{f(x + f(x)y)} - 1\right| \le \varepsilon, \quad \text{whenever } f(x + f(x)y) \ne 0, \\ \left|\frac{f(x + f(x)y)}{f(x)f(y)} - 1\right| \le \varepsilon, \quad \text{whenever } f(x)f(y) \ne 0 \end{gathered} \tag{11.6}$$

for a given $\varepsilon \in (0, 1)$, then either $f$ is close to 1 or it is a solution of (11.1). Therefore, with this definition of (quotient) stability, the Gołąb-Schinzel equation is also superstable in the class of real functions that are continuous at 0. This approach to stability, using quotients, is now called *the stability in the sense of* Ger.

Chudziak generalized this result in [161] where he proved that if $X$ is a vector space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $\lambda \in \mathbb{K} \setminus \{0\}$, $n$ is a positive integer and $f : X \to \mathbb{K}$ is such that $X \setminus f^{-1}(0)$ admits an algebraically interior point (i.e., a point $a$ such that, for every $x \in X \setminus \{0\}$, there exists $r_x > 0$ such that $a + sx \subset X \setminus f^{-1}(0)$ for $s \in \mathbb{K}$ with $|s| \le r_x$) and $f$ satisfies the following two inequalities

$$\left| \frac{\lambda f(x) f(y)}{f(x + f(x)^n y)} - 1 \right| \le \varepsilon, \quad \text{whenever } f(x + f(x)^n y) \ne 0,$$
$$\left| \frac{f(x + f(x)^n y)}{\lambda f(x) f(y)} - 1 \right| \le \varepsilon, \quad \text{whenever } f(x) f(y) \ne 0, \tag{11.7}$$

for a given $\varepsilon \in (0, 1)$, then either $f$ is bounded or it is a solution of (11.5). Thus, in the class of functions $f : X \to \mathbb{K}$ such that $X \setminus f^{-1}(0)$ admits an algebraically interior point, (11.5) is superstable in the sense of Ger.

In [162] those results were extended to a class of functional equations which includes (11.1), (11.4), and the exponential equation. Consider, namely, the functional equation

$$f(x + M(f(x)) y) = \lambda f(x) f(y), \tag{11.8}$$

where $\lambda \in \mathbb{R} \setminus \{0\}$ and $M : \mathbb{R} \to \mathbb{R}$ is a continuous nonzero multiplicative function. It turns out (see [162]) that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the inequality

$$\frac{1}{\varepsilon_1 + 1} \le \left| \frac{f(x + M(f(x)) y)}{\lambda f(x) f(y)} \right| \le \varepsilon_2 + 1, \quad \text{whenever } f(x + M(f(x)) y) f(x) f(y) \ne 0, \tag{11.9}$$

then either $f$ is a solution of the functional equation

$$f(x + M(f(x)) y) f(x) f(y) = 0, \tag{11.10}$$

or the following three conditions are valid.

(i) If $M$ is odd, then either $f$ is bounded or it is a solution of (11.8) with $\lambda = 1$.

(ii) If $M$ is even and $M(\mathbb{R}) \ne \{1\}$, then either $f$ is bounded or it is a solution of (11.8) with some $\lambda \in \{1, -1\}$.

(iii) If $M(\mathbb{R}) = \{1\}$, then there exists a unique $\alpha \in \mathbb{R}$ such that

$$\left| \frac{\lambda f(x)}{e^{\alpha x}} \right| \in \left[ \frac{1}{\varepsilon_1 + 1}, \varepsilon_2 + 1 \right], \quad x \in \mathbb{R}. \tag{11.11}$$

(For some results on (11.10) see [163]).

In [164], the stability in the sense of Ger of (11.8) was studied in the following more general setting.

**Theorem 11.1.** *Let $X$ be a real linear space and let $M$ be multiplicative and continuous at a point $x_0 \in \mathbb{R}$. Assume also that $f : X \to \mathbb{R}$ with $f(X) \neq \{0\}$ satisfies the inequalities*

$$\left|\frac{f(x+M(f(x))y)}{\lambda f(x)f(y)}-1\right| \leq \varepsilon_1, \quad \text{whenever } f(x)f(y) \neq 0,$$
$$\left|\frac{\lambda f(x)f(y)}{f(x+M(f(x))y)}-1\right| \leq \varepsilon_2, \quad \text{whenever } (x+M(f(x))y) \neq 0, \tag{11.12}$$

*for some $\varepsilon_1, \varepsilon_2 \in (0,1)$. If $M(\mathbb{R}) \subset \{-1,0,1\}$, then there exists a unique function $g : X \to \mathbb{R}$ with $g^{-1}(0) = f^{-1}(0)$ satisfying* (11.8) *and*

$$\left|\frac{f(x)}{g(x)}\right| \in \left[\frac{1}{\varepsilon_1+1}, \varepsilon_2+1\right], \quad x \in X \setminus g^{-1}(0). \tag{11.13}$$

*If $M(\mathbb{R}) \not\subset \{-1,0,1\}$ and the set $X \setminus f^{-1}(0)$ has an algebraically interior point, then either $f$ is bounded or it is solution of* (11.8) *with $\lambda$ replaced by sign($\lambda$).*

In view of the above result, some questions arise. Can we obtain analogous results in the complex case? Are the assumptions on $M$ and the set $X \setminus f^{-1}(0)$ necessary?

The results related to the stability of composite functional equations which have been obtained up to now and which have been described previously concern essentially the Gołąb-Schinzel type functional equations. A few other equations have been investigated in [165–167]. For instance, another very important example of composite functional equations is the translation equation

$$F(t, F(s,x)) = F(s+t, x), \tag{11.14}$$

(see [168–171] for more information on it and its applications) and its stability has been studied in [172–177].

It would be interesting to study also the stability of other composite type functional equations such as the Baxter functional equation [178] and the Ebanks functional equation [179].

## 12. Miscellaneous

At the end of this survey we would like to attract the attention of the readers to the results and new techniques of proving the stability results in [77, 180–186]; those techniques involve the methods of multivalued function.

A new approach to the stability of functional equations has been proposed by Paneah (see, e.g., [187]) with some critique of the notions that have been commonly accepted so far. Another method, using the concept of shadowing, was presented in [188] and recently applied in [79, 189–192].

An approach to stability in the ring of formal power series is suggested in [173].

Stability of some conditional versions of the Cauchy equation has been studied in [193–197], for example, of the following Mikusiński functional equation

$$f(x+y)\left(f(x+y)-f(x)-f(y)\right)=0. \tag{12.1}$$

For some connections between Ulam's type stability and the number theory see [198–200].

For some recent results on stability of derivations in rings and algebras see, for example, [201, 202] and the references therein.

Stability for ODE and PDE has been studied, for example, in [98, 99, 203–225], for stability results for some integral equations see [100–102, 226].

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