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Research Article

Uniqueness of Weak Solutions to an Electrohydrodynamics Model

Yong Zhou[1] and Jishan Fan[2]

[1] *Department of Mathematics, Zhejiang Normal University, Jinhua 321004, China*
[2] *Department of Applied Mathematics, Nanjing Forestry University, Nanjing 210037, China*

Correspondence should be addressed to Yong Zhou, yzhoumath@zjnu.edu.cn

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This paper studies uniqueness of weak solutions to an electrohydrodynamics model in $\mathbb{R}^d (d = 2, 3)$. When $d = 2$, we prove a uniqueness without any condition on the velocity. For $d = 3$, we prove a weak-strong uniqueness result with a condition on the vorticity in the homogeneous Besov space.

1. Introduction

We consider the following model of electrokinetic fluid in $\mathbb{R}^d \times (0, \infty)$ [1, 2]:

$$\partial_t u + u \cdot \nabla u + \nabla \pi - \Delta u = \Delta\phi \nabla\phi, \tag{1.1}$$

$$\operatorname{div} u = 0, \tag{1.2}$$

$$\partial_t n + u \cdot \nabla n = \nabla \cdot (\nabla n - n \nabla\phi), \tag{1.3}$$

$$\partial_t p + u \cdot \nabla p = \nabla \cdot (\nabla p + p \nabla\phi), \tag{1.4}$$

$$\Delta\phi = n - p, \tag{1.5}$$

$$(u, n, p)(x, 0) = (u_0, n_0, p_0)(x), \quad x \in \mathbb{R}^d \ (d = 2, 3). \tag{1.6}$$

The unknowns u, π, ϕ, n, and p denote the velocity, pressure, electric potential, anion concentration, and cation concentration, respectively.

Equations (1.3)–(1.5) are known as the electrochemical equations [3] or semiconductor equations [4, 5], and electro-rheological systems [2, 6] when formally setting $u = 0$. (1.1) and (1.2) are Navier-Stokes equations with the Lorentz force $\Delta\phi\nabla\phi$.

The uniqueness of weak solutions to the Navier-Stokes equations is still open. In 1962, Serrin [7] gave the first uniqueness condition:

$$u \in L^r\left(0, T; L^s\left(\mathbb{R}^3\right)\right) \text{ with } \frac{2}{r} + \frac{3}{s} = 1, \quad 3 < s \leq \infty. \tag{1.7}$$

Kozono and Taniuchi [8] proved the following uniqueness criterion:

$$u \in L^2\left(0, T; \mathrm{BMO}\left(\mathbb{R}^3\right)\right). \tag{1.8}$$

Here BMO denotes the functions of bounded mean oscillation. Ogawa and Taniuchi [9] obtained the uniqueness criterion:

$$\nabla u \in L \operatorname{Log} L\left(0, T; \dot{B}^0_{\infty,\infty}\left(\mathbb{R}^3\right)\right) \tag{1.9}$$

with

$$L \operatorname{Log} L\left(0, T; \dot{B}^0_{\infty,\infty}\right) := \left\{ f; \int_0^T \|f\|_{\dot{B}^0_{\infty,\infty}} \log\left(e + \|f\|_{\dot{B}^0_{\infty,\infty}}\right) dt < \infty \right\}. \tag{1.10}$$

Here it should be noted that Kozono et al. [10] proved that u is smooth if

$$\nabla u \in L^1\left(0, T; \dot{B}^0_{\infty,\infty}\left(\mathbb{R}^3\right)\right). \tag{1.11}$$

Here $\dot{B}^0_{\infty,\infty}$ is the homogeneous Besov space.

Kurokiba and Ogawa [4] considered the semiconductor equations (1.3)–(1.5) when $u = 0$ and proved that the existence and uniqueness of weak solutions with L^p initial data (n_0, p_0) when $p = (d/2)(d \geq 3)$ and $1 < p < 2 \ \ (d = 2)$.

Note that the system (1.1)–(1.5) holds its form under the scaling $(u, \pi, \phi, n, p) \to (u_\lambda, \pi_\lambda, \phi_\lambda, n_\lambda, p_\lambda) := (\lambda u, \lambda^2 \pi, \phi, \lambda^2 n, \lambda^2 p)(\lambda^2 t, \lambda x)$. Under this scaling, the space $L^r(0, T; L^s)$ is invariant for u when $2/r + d/s = 1$ and the space $L^r(0, T; L^s)$ is invariant for (n, p) when $2/r + d/s = 2$. Furthermore, L^d for u_0 and $L^{d/2}$ for (n_0, p_0) are invariant spaces under this scaling. Fan and Gao [11], Ryham [12], and Schmuck [13] proved the existence, uniqueness, and regularity of global weak solutions to system (1.1)–(1.6) in a bounded domain Ω. When $\Omega = \mathbb{R}^d$, Jerome [14] established the first existence result in Kato's semigroup framework. Zhao et al. [15] obtained global well-posedness for small initial data in Besov spaces with negative index.

The aim of this paper is to generalize the results of [4, 9]. We will prove the following results.

Theorem 1.1. *Let* $(n_0, p_0) \in L^1(\mathbb{R}^2) \cap L\log L(\mathbb{R}^2)$, $n_0, p_0 \geq 0$ *in* $\mathbb{R}^2$, $\int n_0\,dx = \int p_0\,dx$, $\nabla\phi_0 \in L^2$, *and* $u_0 \in L^2$. *Then there exists a unique weak solution* (u, n, p, ϕ) *to the problem* (1.1)–(1.6) *satisfying*

$$\begin{gathered}(n,p) \in L^\infty(0,T;L^1 \cap L\log L) \cap L^2(0,T;L^2) \cap L^{4/3}(0,T;W^{1,4/3}), \quad n,p \geq 0 \text{ in } \mathbb{R}^2 \times (0,T)\\ (\partial_t n, \partial_t p) \in L^{4/3}(0,T;W^{-1,4/3}),\\ \nabla\phi \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1) \cap L^4(0,T;L^4),\\ u \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1) \cap L^4(0,T;L^4),\\ \partial_t u \in L^{4/3}(0,T;H^{-1}) \quad \text{for any } T>0.\end{gathered} \tag{1.12}$$

Remark 1.2. We can assume $n_0 - p_0 \in \mathscr{H}^1$ (Hardy space) and $\Delta\phi_0 = n_0 - p_0$ gives $\nabla\phi_0 \in L^2(\mathbb{R}^2)$.

Theorem 1.3 ($d = 3$). *Let* $(n_0, p_0) \in L^{3/2}$, $n_0, p_0 \geq 0$ *in* $\mathbb{R}^3$, $\int n_0\,dx = \int p_0\,dx$, *and* $u_0 \in L^2$. *Suppose that* (1.9) *holds true, then there exists a unique weak solution* (u, n, p, ϕ) *to the problem* (1.1)–(1.6) *satisfying*

$$\begin{gathered}(n^{3/4}, p^{3/4}) \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1), \quad n,p \geq 0 \text{ in } \mathbb{R}^3 \times (0,T),\\ (n,p) \in L^\infty(0,T;L^{3/2}) \cap L^{5/2}(0,T;L^{5/2}) \cap L^{5/3}(0,T;W^{1,5/3}) \cap L^4(0,T;L^2),\\ (\partial_t n, \partial_t p) \in L^{5/3}(0,T;W^{-1,3/2}),\\ \nabla\phi \in L^\infty(0,T;W^{1,3/2}) \cap L^{5/2}(0,T;W^{1,5/2}),\\ \nabla\phi \in L^\infty(0,T;L^3) \cap L^{5/2}(0,T;L^{15}),\\ u \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1), \qquad \partial_t u \in L^2(0,T;W^{-1,3/2})\end{gathered} \tag{1.13}$$

for any $T > 0$.

Let η_j, $j = 0, \pm 1, \pm 2, \pm 3, \ldots$, be the Littlewood-Paley dyadic decomposition of unity that satisfies $\hat{\eta} \in C_0^\infty(B_2 \setminus B_{1/2})$, $\hat{\eta}_j(\xi) = \hat{\eta}(2^{-j}\xi)$, and $\sum_{j=-\infty}^{\infty} \hat{\eta}_j(\xi) = 1$ except $\xi = 0$. To fill the origin, we put a smooth cut off $\varphi \in \mathcal{S}(\mathbb{R}^3)$ with $\hat{\varphi}(\xi) \in C_0^\infty(B_1)$ such that

$$\hat{\varphi} + \sum_{j=0}^{\infty} \hat{\eta}_j(\xi) = 1. \tag{1.14}$$

The homogeneous Besov space $\dot{B}^s_{p,q} := \{f \in \mathcal{S}' : \|f\|_{\dot{B}^s_{p,q}} < \infty\}$ is introduced by the norm

$$\|f\|_{\dot{B}^s_{p,q}} := \left(\sum_{j=-\infty}^{\infty} \left\|2^{js}\eta_j * f\right\|_{L^p}^q\right)^{1/q}, \tag{1.15}$$

for $s \in \mathbb{R}, 1 \leq p, q \leq \infty$.

It is easy to prove the existence of weak solutions [14] and thus we omit the details here; we only need to derive the estimates (1.12) and (1.13) and prove the uniqueness.

2. Proof of Theorem 1.1

First, by the maximum principle, it is easy to prove that

$$n, p \geq 0 \quad \text{in } \mathbb{R}^d \times (0, \infty). \tag{2.1}$$

Testing (1.3) by $1 + \log n$ and testing (1.4) by $1 + \log p$, respectively, using (1.2), summing up the resulting equality, we obtain

$$\begin{aligned} &\int n \log n + p \log p \, dx + 4 \int_0^T \int |\nabla \sqrt{n}|^2 + |\nabla \sqrt{p}|^2 dx\, dt + \int_0^T \int |\Delta \phi|^2 dx\, dt \\ &\quad = \int n_0 \log n_0 + p_0 \log p_0 dx. \end{aligned} \tag{2.2}$$

Substracting (1.4) from (1.3), we see that

$$\partial_t (n - p) + u \cdot \nabla (n - p) = \nabla \cdot (\nabla (n - p) - (n + p) \nabla \phi). \tag{2.3}$$

Testing the above equation by $-\phi$, using (1.5) and (2.1), we see that

$$\frac{1}{2} \frac{d}{dt} \int |\nabla \phi|^2 \, dx - \int u \cdot \nabla \Delta \phi \cdot \phi \, dx + \int |\Delta \phi|^2 \, dx + \int (n + p) |\nabla \phi|^2 \, dx = 0. \tag{2.4}$$

Whence

$$\frac{1}{2} \frac{d}{dt} \int |\nabla \phi|^2 \, dx + \int u \Delta \phi \nabla \phi \, dx + \int |\Delta \phi|^2 \, dx + \int (n + p) |\nabla \phi|^2 \, dx = 0. \tag{2.5}$$

Testing (1.1) by u, using (1.2), we find that

$$\frac{1}{2} \frac{d}{dt} \int u^2 \, dx + \int |\nabla u|^2 \, dx = \int u \Delta \phi \nabla \phi \, dx. \tag{2.6}$$

Summing up (2.5) and (2.6), we get

$$\frac{1}{2} \frac{d}{dt} \int u^2 + |\nabla \phi|^2 \, dx + \int |\nabla u|^2 + |\Delta \phi|^2 + (n + p) |\nabla \phi|^2 \, dx = 0, \tag{2.7}$$

whence

$$\frac{1}{2} \int u^2 + |\nabla \phi|^2 \, dx + \int_0^T \int |\nabla u|^2 + |\Delta \phi|^2 + (n + p) |\nabla \phi|^2 dx\, dt \leq \frac{1}{2} \int u_0^2 + |\nabla \phi_0|^2 \, dx. \tag{2.8}$$

Integrating (1.3) and (1.4), we have

$$\int n\,dx = \int n_0\,dx = \int p_0\,dx = \int p\,dx. \tag{2.9}$$

Using the Gagliardo-Nirenberg inequality,

$$\|u\|_{L^4}^2 \leq C\|u\|_{L^2}\|\nabla u\|_{L^2}, \tag{2.10}$$

we deduce that

$$\|u\|_{L^4(0,T;L^4)} \leq C, \tag{2.11}$$

$$\left\|\nabla\phi\right\|_{L^4(0,T;L^4)} \leq C, \tag{2.12}$$

$$\left\|(n,p)\right\|_{L^2(0,T;L^2)} \leq C. \tag{2.13}$$

Since $\nabla n = 2\nabla\sqrt{n}\cdot\sqrt{n}, \nabla\sqrt{n} \in L^2(0,T;L^2), \sqrt{n} \in L^4(0,T;L^4)$, we easily infer that

$$\nabla n \in L^{4/3}\left(0,T;L^{4/3}\right), \tag{2.14}$$

by the Hölder inequality. Similarly, we have

$$\nabla p \in L^{4/3}\left(0,T;L^{4/3}\right). \tag{2.15}$$

It is easy to show that

$$(\partial_t n, \partial_t p) \in L^{4/3}\left(0,T;W^{-1,4/3}\right), \qquad \partial_t u \in L^{4/3}\left(0,T;H^{-1}\right). \tag{2.16}$$

Now we are in a position to prove the uniqueness. Let $(u_i,\pi_i,n_i,p_i,\phi_i)$ $(i=1,2)$ be two weak solutions to the problem (1.1)–(1.6). Also let us denote

$$u := u_1 - u_2, \qquad \pi := \pi_1 - \pi_2, \qquad n := n_1 - n_2, \qquad p := p_1 - p_2, \qquad \phi := \phi_1 - \phi_2. \tag{2.17}$$

We define N and P satisfying the following equations:

$$-\Delta N + N = n \text{ in } \mathbb{R}^d \times (0,\infty), \tag{2.18}$$

$$-\Delta P + P = p \text{ in } \mathbb{R}^d \times (0,\infty). \tag{2.19}$$

It is easy to verify that

$$\partial_t n + \nabla \cdot (u_1 n + u n_2) = \Delta n - \nabla \cdot (n \nabla \phi_1 + n_2 \nabla \phi), \tag{2.20}$$

$$\partial_t p + \nabla \cdot (u_1 p + u p_2) = \Delta p + \nabla \cdot (p \nabla \phi_1 + p_2 \nabla \phi), \tag{2.21}$$

$$\partial_t (n-p) + \nabla \cdot (u_1 (n-p) + u(n_2 - p_2)) = \Delta (n-p) - \nabla \cdot ((n+p)\nabla \phi_1 + (n_2 + p_2)\nabla \phi). \tag{2.22}$$

Testing (2.20) by N, we derive

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int N^2 + |\nabla N|^2 \, dx + \int |\nabla N|^2 + |\Delta N|^2 \, dx \\ &\quad = \int n \nabla \phi_1 \cdot \nabla N + n_2 \nabla \phi \nabla N + u_1 n \nabla N + u n_2 \nabla N \, dx =: I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{2.23}$$

Using (2.10), (2.18) and (2.19), each term I_i $(i = 1,2,3,4)$ can be bounded as follows:

$$\begin{aligned} I_1 &\le \|\Delta N\|_{L^2} \|\nabla \phi_1\|_{L^4} \|\nabla N\|_{L^4} + \|N\|_{L^4} \|\nabla \phi_1\|_{L^4} \|\nabla N\|_{L^2} \\ &\le C\|\Delta N\|_{L^2} \|\nabla \phi_1\|_{L^4} \|\nabla N\|_{L^2}^{1/2} \|\Delta N\|_{L^2}^{1/2} + C\|\nabla \phi_1\|_{L^4} \|N\|_{H^1}^2 \\ &\le \frac{1}{18}\|\Delta N\|_{L^2}^2 + C\|\nabla \phi_1\|_{L^4}^4 \|\nabla N\|_{L^2}^2 + C\|\nabla \phi_1\|_{L^4} \|N\|_{H^1}^2, \\ I_2 &\le \|n_2\|_{L^2} \|\nabla \phi\|_{L^4} \|\nabla N\|_{L^4} \\ &\le \|n_2\|_{L^2} \|\nabla \phi\|_{L^4}^2 + \|n_2\|_{L^2} \|\nabla N\|_{L^4}^2 \\ &\le C\|n_2\|_{L^2} \|\nabla \phi\|_{L^2} \|\Delta \phi\|_{L^2} + C\|n_2\|_{L^2} \|\nabla N\|_{L^2} \|\Delta N\|_{L^2} \\ &\le \frac{1}{18}\|\Delta \phi\|_{L^2}^2 + \frac{1}{18}\|\Delta N\|_{L^2}^2 + C\|n_2\|_{L^2}^2 \left(\|\nabla \phi\|_{L^2}^2 + \|\nabla N\|_{L^2}^2\right), \\ I_3 &\le \|u_1\|_{L^4} \|\Delta N\|_{L^2} \|\nabla N\|_{L^4} \\ &\le C\|u_1\|_{L^4} \|\Delta N\|_{L^2} \|\nabla N\|_{L^2}^{1/2} \|\Delta N\|_{L^2}^{1/2} \\ &\le \frac{1}{18}\|\Delta N\|_{L^2}^2 + C\|u_1\|_{L^4}^4 \|\nabla N\|_{L^2}^2, \\ I_4 &\le \|n_2\|_{L^2} \|u\|_{L^4} \|\nabla N\|_{L^4} \\ &\le \|n_2\|_{L^2} \|u\|_{L^4}^2 + \|n_2\|_{L^2} \|\nabla N\|_{L^4}^2 \\ &\le C\|n_2\|_{L^2} \|u\|_{L^2} \|\nabla u\|_{L^2} + C\|n_2\|_{L^2} \|\nabla N\|_{L^2} \|\Delta N\|_{L^2} \\ &\le \frac{1}{18}\|\nabla u\|_{L^2}^2 + \frac{1}{18}\|\Delta N\|_{L^2}^2 + C\|n_2\|_{L^2}^2 \left(\|u\|_{L^2}^2 + \|\nabla N\|_{L^2}^2\right). \end{aligned} \tag{2.24}$$

Substituting these estimates into (2.23), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int N^2+|\nabla N|^2\,dx+\frac{1}{2}\int|\nabla N|^2+|\Delta N|^2\,dx\\
&\quad\le C\left(\|\nabla\phi_1\|_{L^4}^4+\|n_2\|_{L^2}^2+\|u_1\|_{L^4}^4+1\right)\left(\|N\|_{L^2}^2+\|\nabla N\|_{L^2}^2+\|u\|_{L^2}^2+\|\nabla\phi\|_{L^2}^2\right)\\
&\qquad+\frac{1}{18}\|\Delta\phi\|_{L^2}^2+\frac{1}{18}\|\nabla u\|_{L^2}^2.
\end{aligned}\tag{2.25}$$

Similarly for the p-equation, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int p^2+|\nabla p|^2\,dx+\frac{1}{2}\int|\nabla p|^2+|\Delta p|^2\,dx\\
&\quad\le C\left(\|\nabla\phi_1\|_{L^4}^4+\|p_2\|_{L^2}^2+\|u_1\|_{L^4}^4+1\right)\left(\|p\|_{L^2}^2+\|\nabla p\|_{L^2}^2+\|u\|_{L^2}^2+\|\nabla\phi\|_{L^2}^2\right)\\
&\qquad+\frac{1}{18}\|\Delta\phi\|_{L^2}^2+\frac{1}{18}\|\nabla u\|_{L^2}^2.
\end{aligned}\tag{2.26}$$

Testing (2.22) by $-\phi$, using (1.5), we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int|\nabla\phi|^2\,dx+\int|\Delta\phi|^2\,dx\\
&\quad=-\int(n+p)\nabla\phi_1\nabla\phi+(n_2+p_2)(\nabla\phi)^2+u_1\Delta\phi\nabla\phi+u(n_2-p_2)\nabla\phi\,dx\\
&\quad=:J_1+J_2+J_3+J_4.
\end{aligned}\tag{2.27}$$

Using (2.10), (2.18), and (2.19), each term J_i $(i=1,2,3,4)$ can be bounded as follows:

$$\begin{aligned}
J_1&\le\|n+p\|_{L^2}\|\nabla\phi_1\|_{L^4}\|\nabla\phi\|_{L^4}\\
&\le C(\|\Delta N\|_{L^2}+\|\Delta P\|_{L^2}+\|N\|_{L^2}+\|P\|_{L^2})\|\nabla\phi_1\|_{L^4}\|\nabla\phi\|_{L^2}^{1/2}\|\Delta\phi\|_{L^2}^{1/2}\\
&\le\frac{1}{18}\|\Delta N\|_{L^2}^2+\frac{1}{18}\|\Delta P\|_{L^2}^2+C\|N\|_{L^2}^2+C\|P\|_{L^2}^2\\
&\quad+\frac{1}{18}\|\Delta\phi\|_{L^2}^2+C\|\nabla\phi_1\|_{L^4}^4\|\nabla\phi\|_{L^2}^2,\\
J_2&\le 0,\\
J_3&\le\|u_1\|_{L^4}\|\Delta\phi\|_{L^2}\|\nabla\phi\|_{L^4}
\end{aligned}$$

$$\begin{aligned}
&\leq C\|u_1\|_{L^4}\|\Delta\phi\|_{L^2}\|\nabla\phi\|_{L^2}^{1/2}\|\Delta\phi\|_{L^2}^{1/2} \\
&\leq \frac{1}{18}\|\Delta\phi\|_{L^2}^2 + C\|u_1\|_{L^4}^4\|\nabla\phi\|_{L^2}^2, \\
J_4 &\leq \|u\|_{L^4}\|n_2 - p_2\|_{L^2}\|\nabla\phi\|_{L^4} \\
&\leq \|n_2 + p_2\|_{L^2}\|u\|_{L^4}^2 + \|n_2 + p_2\|_{L^2}\|\nabla\phi\|_{L^4}^2 \\
&\leq C\|n_2 + p_2\|_{L^2}\|u\|_{L^2}\|\nabla u\|_{L^2} + C\|n_2 + p_2\|_{L^2}\|\nabla\phi\|_{L^2}\|\Delta\phi\|_{L^2} \\
&\leq \frac{1}{18}\|\nabla u\|_{L^2}^2 + \frac{1}{18}\|\Delta\phi\|_{L^2}^2 + C\|n_2 + p_2\|_{L^2}^2\left(\|u\|_{L^2}^2 + \|\nabla\phi\|_{L^2}^2\right).
\end{aligned} \tag{2.28}$$

Substituting these estimates into (2.27), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int|\nabla\phi|^2\,dx + \frac{1}{2}\int|\Delta\phi|^2\,dx \\
&\quad\leq C\left(\|\nabla\phi_1\|_{L^4}^4 + \|u_1\|_{L^4}^4 + \|n_2 + p_2\|_{L^2}^2 + 1\right)\left(\|u\|_{L^2}^2 + \|\nabla\phi\|_{L^2}^2 + \|N\|_{L^2}^2 + \|P\|_{L^2}^2\right) \\
&\quad\quad + \frac{1}{18}\|\Delta N\|_{L^2}^2 + \frac{1}{18}\|\Delta P\|_{L^2}^2 + \frac{1}{18}\|\nabla u\|_{L^2}^2.
\end{aligned} \tag{2.29}$$

It is easy to find that u satisfies

$$\partial_t u + u_2\cdot\nabla u + u\cdot\nabla u_1 + \nabla\pi - \Delta u = \Delta\phi\nabla\phi_1 + \Delta\phi_2\nabla\phi. \tag{2.30}$$

Testing this equation by u, using (1.2), we have

$$\frac{1}{2}\frac{d}{dt}\int u^2\,dx + \int|\nabla u|^2\,dx = \int\Delta\phi\nabla\phi_1 u + \Delta\phi_2\nabla\phi\cdot u - u\cdot\nabla u_1\cdot u\,dx =: \ell_1 + \ell_2 + \ell_3. \tag{2.31}$$

Using (2.10), each term ℓ_i $(i = 1, 2, 3)$ can be bounded as follows:

$$\begin{aligned}
\ell_1 &\leq \|\Delta\phi\|_{L^2}\|\nabla\phi_1\|_{L^4}\|u\|_{L^4} \\
&\leq C\|\Delta\phi\|_{L^2}\|\nabla\phi_1\|_{L^4}\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2} \\
&\leq \frac{1}{18}\|\Delta\phi\|_{L^2}^2 + \frac{1}{18}\|\nabla u\|_{L^2}^2 + C\|\nabla\phi_1\|_{L^4}^4\|u\|_{L^2}^2, \\
\ell_2 &\leq \|\Delta\phi_2\|_{L^2}\|\nabla\phi\|_{L^4}\|u\|_{L^4}
\end{aligned}$$

$$
\begin{aligned}
&\le C\|\Delta\phi_2\|_{L^2}\|\nabla\phi\|_{L^2}^{1/2}\|\Delta\phi\|_{L^2}^{1/2}\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2} \\
&\le \frac{1}{18}\|\Delta\phi\|_{L^2}^2 + \frac{1}{18}\|\nabla u\|_{L^2}^2 + C\|\Delta\phi_2\|_{L^2}^2\left(\|u\|_{L^2}^2 + \|\nabla\phi\|_{L^2}^2\right), \\
\ell_3 &\le \int u\cdot\nabla u\cdot u_1\,dx \\
&\le \|u\|_{L^4}\|\nabla u\|_{L^2}\|u_1\|_{L^4} \\
&\le C\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{3/2}\|u_1\|_{L^4} \\
&\le \frac{1}{18}\|\nabla u\|_{L^2}^2 + C\|u_1\|_{L^4}^4\|u\|_{L^2}^2.
\end{aligned}
\tag{2.32}
$$

Substituting these estimates into (2.31), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int u^2\,dx + \int|\nabla u|^2\,dx \\
&\quad\le \frac{1}{9}\|\Delta\phi\|_{L^2}^2 + C\left(\|\nabla\phi_1\|_{L^4}^4 + \|\Delta\phi_2\|_{L^2}^2 + \|u_1\|_{L^4}^4\right)\left(\|u\|_{L^2}^2 + \|\nabla\phi\|_{L^2}^2\right).
\end{aligned}
\tag{2.33}
$$

Combining (2.25), (2.26), (2.29), and (2.33), using (2.8), (2.11), (2.12), (2.13), and the Gronwall inequality, we conclude that

$$
N = P = 0,\qquad u = 0,\qquad \nabla\phi = 0, \tag{2.34}
$$

and thus

$$
n = p = 0. \tag{2.35}
$$

This completes the proof.

3. Proof of Theorem 1.3

By the same calculations as that in [11], we can prove (1.13) and thus we omit the details here.

Now we are in a position to prove the uniqueness. We still use the same notations as that in Section 2, and similarly we get (2.23). But each term I_i $(i = 1,2,3,4)$ can be bounded as follows:

$$
\begin{aligned}
I_1 &\le \|\Delta N\|_{L^2}\|\nabla N\|_{L^{30/13}}\|\nabla\phi_1\|_{L^{15}} + \|N\|_{L^6}\|\nabla\phi_1\|_{L^3}\|\nabla N\|_{L^2} \\
&\le C\|\Delta N\|_{L^2}\|\nabla N\|_{L^2}^{4/5}\|\Delta N\|_{L^2}^{1/5}\|\nabla\phi_1\|_{L^{15}} + C\|\nabla N\|_{L^2}^2 \\
&\le \frac{1}{18}\|\Delta N\|_{L^2}^2 + C\|\nabla\phi_1\|_{L^{15}}^{5/2}\|\nabla N\|_{L^2}^2 + C\|\nabla N\|_{L^2}^2,
\end{aligned}
\tag{3.1}
$$

by the Gagliardo-Nirenberg inequality,

$$
\begin{aligned}
\|\nabla N\|_{L^{30/13}} &\le C\|\nabla N\|_{L^2}^{4/5}\|\Delta N\|_{L^2}^{1/5}, \\
I_2 &\le \|n_2\|_{L^2}\left\|\nabla\phi\right\|_{L^6}\|\nabla N\|_{L^3} \\
&\le C\|n_2\|_{L^2}\left\|\Delta\phi\right\|_{L^2}\|\nabla N\|_{L^2}^{1/2}\|\Delta N\|_{L^2}^{1/2} \\
&\le C\|n_2\|_{L^2}\|-\Delta N + N + \Delta P - P\|_{L^2}\|\nabla N\|_{L^2}^{1/2}\|\Delta N\|_{L^2}^{1/2} \\
&\le \frac{1}{18}\|\Delta N\|_{L^2}^2 + \frac{1}{18}\|\Delta P\|_{L^2}^2 + C\|N\|_{L^2}^2 + C\|P\|_{L^2}^2 + C\|n_2\|_{L^2}^4\|\nabla N\|_{L^2}^2
\end{aligned}
\tag{3.2}
$$

by the Gagliardo-Nirenberg inequality,

$$
\begin{aligned}
\|\nabla N\|_{L^3}^2 &\le C\|\nabla N\|_{L^2}\|\Delta N\|_{L^2}, \\
I_3 &= \int u_1 n\nabla N\,dx = -\int u_1\Delta N\nabla N\,dx.
\end{aligned}
\tag{3.3}
$$

Now we decompose u_1 into three parts in the phase variable:

$$
\begin{aligned}
u_1 &= \sum_{j<-M}\eta_j * u_1 + \sum_{j=-M}^{M}\eta_j * u_1 + \sum_{j>M}\eta_j * u_1 \\
&=: u_1^{\ell} + u_1^{m} + u_1^{h}.
\end{aligned}
\tag{3.4}
$$

Thus

$$
\begin{aligned}
I_3 &= -\int u_1^{\ell}\Delta N\nabla N\,dx + \sum_i\int \partial_i u_1^{m}\cdot\partial_i N\nabla N\,dx - \int u_1^{h}\Delta N\nabla N\,dx \\
&=: I_{31} + I_{32} + I_{33}.
\end{aligned}
\tag{3.5}
$$

Recalling the Bernstein inequality,

$$
\left\|\eta_j * u\right\|_{L^q} \le C2^{3j(1/p-1/q)}\left\|\eta_j * u\right\|_{L^p}, \quad 1 \le p \le q \le \infty,
\tag{3.6}
$$

the low-frequency part is estimated as

$$
\begin{aligned}
I_{31} &\le \|\nabla N\|_{L^6}\|\Delta N\|_{L^2}\left\|u_1^{\ell}\right\|_{L^3} \\
&\le C\|\Delta N\|_{L^2}^2 \sum_{j<-M} 2^{j/2}\left\|\eta_j * u_1\right\|_{L^2} \\
&\le C\|\Delta N\|_{L^2}^2 \left(\sum_{j<-M} 2^j\right)^{1/2}\left(\sum_{j=-\infty}^{\infty}\left\|\eta_j * u_1\right\|_{L^2}^2\right)^{1/2} \\
&\le C2^{-M/2}\|\Delta N\|_{L^2}^2\|u_1\|_{L^2} \\
&\le C2^{-M/2}\|\Delta N\|_{L^2}^2.
\end{aligned}
\tag{3.7}
$$

The second term can be bounded as follows:

$$
\begin{aligned}
I_{32} &\le \sum_i \|\partial_i N\|_{L^2}\|\nabla N\|_{L^2}\left\|\partial_i u_1^m\right\|_{L^\infty} \\
&\le C\|\nabla N\|_{L^2}^2\left\|\nabla u_1^m\right\|_{L^\infty} \\
&\le C\|\nabla N\|_{L^2}^2 \sum_{j=-M}^{M}\left\|\eta_j * \nabla u_1\right\|_{L^\infty} \\
&\le CM\|\nabla N\|_{L^2}^2\|\nabla u_1\|_{\dot{B}^0_{\infty,\infty}}.
\end{aligned}
\tag{3.8}
$$

On the other hand, the last term is simply bounded by the Hausdorff-Young inequality as

$$
\begin{aligned}
I_{33} &\le \|\Delta N\|_{L^2}\|\nabla N\|_{L^2}\left\|u_1^h\right\|_{L^\infty} \\
&\le \|\Delta N\|_{L^2}\|\nabla N\|_{L^2}\sum_{j>M}\left\|\left\{(-\Delta)^{-1/2}(\eta_{j-1}+\eta_j+\eta_{j+1})\right\} * \eta_j * (-\Delta)^{1/2}u_1\right\|_{L^\infty} \\
&\le C\|\Delta N\|_{L^2}\|\nabla N\|_{L^2}\sum_{j>M} 2^{-j}\left\|\eta_j * (-\Delta)^{1/2}u_1\right\|_{L^\infty} \\
&\le C\|\Delta N\|_{L^2}\|\nabla N\|_{L^2}\sum_{j>M} 2^{-j}\|\nabla u_1\|_{\dot{B}^0_{\infty,\infty}} \\
&\le C2^{-M}\|\Delta N\|_{L^2}\|\nabla N\|_{L^2}\|\nabla u_1\|_{\dot{B}^0_{\infty,\infty}} \\
&\le C2^{-M}\|\nabla N\|_{L^2}^2\|\nabla u_1\|_{\dot{B}^0_{\infty,\infty}}^2 + C2^{-M}\|\Delta N\|_{L^2}^2.
\end{aligned}
\tag{3.9}
$$

Choosing M properly large so that $C2^{-M/2} \leq 1/36$ and $C2^{-M}\|\nabla u_1\|_{\dot{B}^0_{\infty,\infty}} \leq 1$, we reach

$$I_3 \leq \frac{1}{8}\|\Delta N\|_{L^2}^2 + C\|\nabla N\|_{L^2}^2\|\nabla u_1\|_{\dot{B}^0_{\infty,\infty}}\left(1+\log\left(e+\|\nabla u_1\|_{\dot{B}^0_{\infty,\infty}}\right)\right),$$
$$\begin{aligned} I_4 &\leq \|u\|_{L^6}\|n_2\|_{L^2}\|\nabla N\|_{L^3} \\ &\leq C\|\nabla u\|_{L^2}\|n_2\|_{L^2}\|\nabla N\|_{L^2}^{1/2}\|\Delta N\|_{L^2}^{1/2} \\ &\leq \frac{1}{18}\|\Delta N\|_{L^2}^2 + \frac{1}{18}\|\nabla u\|_{L^2}^2 + C\|n\|_{L^2}^4\|\nabla N\|_{L^2}^2. \end{aligned} \tag{3.10}$$

Substituting the above estimates into (2.23), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int N^2 + |\nabla N|^2\,dx + \frac{1}{2}\int |\nabla N|^2 + |\Delta N|^2 dx \\ &\quad\leq C\left(\|\nabla\phi_1\|_{L^{15}}^{5/2} + 1 + \|n_2\|_{L^2}^4\right)\left(\|N\|_{L^2}^2 + \|P\|_{L^2}^2 + \|\nabla N\|_{L^2}^2\right) \\ &\qquad + C\|\nabla N\|_{L^2}^2\|\nabla u_1\|_{\dot{B}^0_{\infty,\infty}}\left(1+\log\left(e+\|\nabla u_1\|_{\dot{B}^0_{\infty,\infty}}\right)\right) \\ &\qquad + \frac{1}{18}\|\Delta P\|_{L^2}^2 + \frac{1}{18}\|\nabla u\|_{L^2}^2. \end{aligned} \tag{3.11}$$

Similarly for the p-equation, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int P^2 + |\nabla P|^2\,dx + \int |\nabla P|^2 + |\Delta P|^2\,dx \\ &\quad\leq C\left(\|\nabla\phi_1\|_{L^{15}}^{5/2} + 1 + \|p_2\|_{L^2}^4\right)\left(\|N\|_{L^2}^2 + \|P\|_{L^2}^2 + \|\nabla P\|_{L^2}^2\right) \\ &\qquad + C\|\nabla P\|_{L^2}^2\|\nabla u_1\|_{\dot{B}^0_{\infty,\infty}}\left(1+\log\left(e+\|\nabla u_1\|_{\dot{B}^0_{\infty,\infty}}\right)\right) \\ &\qquad + \frac{1}{18}\|\Delta N\|_{L^2}^2 + \frac{1}{18}\|\nabla u\|_{L^2}^2. \end{aligned} \tag{3.12}$$

As in Section 2, we still have (2.31). But each term ℓ_i $(i = 1,2,3)$ can be bounded as follows:

$$\begin{aligned} \ell_1 &\leq \|\nabla\phi_1\|_{L^{15}}\|\Delta\phi\|_{L^2}\|u\|_{L^{30/13}} \\ &\leq C\|\nabla\phi_1\|_{L^{15}}(\|\Delta N\|_{L^2} + \|\Delta P\|_{L^2} + \|N\|_{L^2} + \|P\|_{L^2})\|u\|_{L^2}^{4/5}\|\nabla u\|_{L^2}^{1/5} \\ &\leq \frac{1}{18}\|\Delta N\|_{L^2}^2 + \frac{1}{18}\|\Delta P\|_{L^2}^2 + C\|N\|_{L^2}^2 + C\|P\|_{L^2}^2 \\ &\quad + \frac{1}{18}\|\nabla u\|_{L^2}^2 + C\|\nabla\phi_1\|_{L^{15}}^{5/2}\|u\|_{L^2}^2, \end{aligned} \tag{3.13}$$

by the Gagliardo-Nirenberg inequality,

$$\begin{aligned}
\|u\|_{L^{30/13}} &\le C\|u\|_{L^2}^{4/5}\|\nabla u\|_{L^2}^{1/5}, \\
\ell_2 &\le \left\|\Delta\phi_2\right\|_{L^2}\left\|\nabla\phi\right\|_{L^6}\|u\|_{L^3} \\
&\le C\left\|\Delta\phi_2\right\|_{L^2}\left\|\Delta\phi\right\|_{L^2}\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2} \\
&\le C\left\|\Delta\phi_2\right\|_{L^2}(\|\Delta N\|_{L^2}+\|\Delta P\|_{L^2}+\|N\|_{L^2}+\|P\|_{L^2})\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2} \\
&\le \frac{1}{18}\|\Delta N\|_{L^2}^2+\frac{1}{18}\|\Delta P\|_{L^2}^2+C\|N\|_{L^2}^2+C\|P\|_{L^2}^2 \\
&\quad+\frac{1}{18}\|\nabla u\|_{L^2}^2+C\left\|\Delta\phi_2\right\|_{L^2}^4\|u\|_{L^2}^2,
\end{aligned} \tag{3.14}$$

by the Gagliardo-Nirenberg inequality

$$\|u\|_{L^3}^2 \le C\|u\|_{L^2}\|\nabla u\|_{L^2}. \tag{3.15}$$

By the similar calculations as that of I_3, ℓ_3 can be bounded as follows:

$$\ell_3 \le \frac{1}{18}\|\nabla u\|_{L^2}^2+C\|u\|_{L^2}^2\|\nabla u_1\|_{\dot{B}^0_{\infty,\infty}}\left(1+\log\left(e+\|\nabla u_1\|_{\dot{B}^0_{\infty,\infty}}\right)\right). \tag{3.16}$$

Substituting the above estimates into (2.31), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int u^2\,dx+\frac{1}{2}\int|\nabla u|^2\,dx \\
&\quad\le \frac{1}{9}\|\Delta N\|_{L^2}^2+\frac{1}{9}\|\Delta P\|_{L^2}^2+C\|N\|_{L^2}^2+C\|P\|_{L^2}^2 \\
&\qquad+\left(\left\|\nabla\phi_1\right\|_{L^{15}}^{5/2}+\left\|\Delta\phi_2\right\|_{L^2}^4\right)\|u\|_{L^2}^2 \\
&\qquad+C\|u\|_{L^2}^2\|\nabla u_1\|_{\dot{B}^0_{\infty,\infty}}\left(1+\log\left(e+\|\nabla u_1\|_{\dot{B}^0_{\infty,\infty}}\right)\right).
\end{aligned} \tag{3.17}$$

Combining (3.11), (3.12), and (3.17), using (1.13) and the Gronwall inequality, we arrive at

$$N=P=0, \qquad u=0, \tag{3.18}$$

as thus

$$n=p=0, \qquad \nabla\phi=0. \tag{3.19}$$

This completes the proof.

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