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Research Article

A Modified Halpern's Iterative Scheme for Solving Split Feasibility Problems

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The purpose of this paper is to introduce and study a modified Halpern's iterative scheme for solving the split feasibility problem (SFP) in the setting of infinite-dimensional Hilbert spaces. Under suitable conditions a strong convergence theorem is established. The main result presented in this paper improves and extends some recent results done by Xu (Iterative methods for the split feasibility problem in infinite-dimensional Hilbert space, Inverse Problem 26 (2010) 105018) and some others.

1. Introduction

Let C and Q be nonempty-closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let A be a linear-bounded operator from H_1 to H_2. The split feasibility problem (SFP) is finding a point $\hat{x}$ satisfying the following property:

$$\hat{x} \in C, \quad A\hat{x} \in Q. \tag{1.1}$$

The SFP was introduced by Censor and Elfving [1] for modeling inverse problems which arise from phase retrievals and medical image reconstruction [2], and very well-known iterative algorithms have been invented to solve it [2].

We use Γ to denote the solution set of SFP:

$$\Gamma = \{\hat{x} \in C : A\hat{x} \in Q\}, \tag{1.2}$$

and assume that the SFP (1.1) is consistent (i.e., (1.1) has a solution) so that Γ is closed, convex, and nonempty, it is not hard to see that $x \in C$ solves (1.1) if and only if it solves the following fixed point equation;

$$x = P_C\big(I - \gamma A^*(I - P_Q)A\big)x, \quad x \in C, \tag{1.3}$$

where P_C and P_Q are the (orthogonal) projections onto C and Q, respectively, $\gamma > 0$ is any positive constant and A^* denotes the adjoint of A. Moreover, for sufficiently small $\gamma > 0$, the operator $P_C(I-\gamma A^*(I-P_Q)A)$ which defines the fixed point equation in (1.3) is nonexpansive.

To solve the SFP (1.1), Byrne [2] proposed his CQ algorithm (see also [3]) which generates a sequence $\{x_n\}$ by

$$x_{n+1} = P_C\big(I - \gamma A^*(I - P_Q)A\big)x_n, \quad n \geq 0, \tag{1.4}$$

where $\gamma \in (0, 2/\lambda)$ with λ being the spectral radius of the operator A^*A.

Very recently, Xu [4] has viewed the CQ algorithm for averaged mappings and applied Mann's algorithm to solving the SFP, and he also proved that an averaged CQ algorithm is weakly convergent to a solution of the SFP.

In this paper, we also regard the CQ algorithm as a fixed point algorithm for averaged mappings and try to study the SFP by the following modified Halpern's iterative scheme;

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_C\big(I - \xi \mathrm{A}^*(I - P_Q)A\big)x_n, \quad n \geq 0, \tag{1.5}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are three sequences in $[0,1]$ satisfying $\alpha_n+\beta_n+\gamma_n = 1$. Furthermore, our result extends and improves the result of Xu [4] from weak to strong convergence theorems.

2. Preliminaries

Throughout the paper, we adopt the following notation.

Let x_n be a sequence and x be a point in a normed space X. We use $x_n \to x$ and $x_n \rightharpoonup x$ to denote strong and weak convergence to x of the sequence $\{x_n\}$, respectively. In addition, we use $\omega_w(x_n)$ to denote the weak ω-limit set of the sequence $\{x_n\}$; namely,

$$\omega_{\mathrm{w}}(x_n) := \big\{x : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\big\}. \tag{2.1}$$

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively, and let K be a nonempty-closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in K, denoted by $P_K x$, such that

$$\|x - P_K x\| \leq \big\|x - y\big\|, \quad \forall y \in K, \tag{2.2}$$

P_K is called the metric projection of H onto K. It is well known that P_K is a nonexpansive mapping of H onto K and satisfies

$$\langle x-y, P_K x - P_K y\rangle \geq \|P_K x - P_K y\|^2, \tag{2.3}$$

for every $x, y \in H$. Moreover, $P_K x$ is characterized by the following properties: $P_K x \in K$ and

$$\begin{gathered}\langle x - P_K x, y - P_K x\rangle \leq 0, \\ \|x-y\|^2 \geq \|x - P_K x\|^2 + \|y - P_K x\|^2,\end{gathered} \tag{2.4}$$

for all $x \in H, y \in K$.

Some important properties of projections are gathered in the following proposition.

Proposition 2.1. *Given $x \in H$ and $z \in K$. Then $z = P_K x$ if and only if*

$$\langle x-z, y-z\rangle \leq 0, \quad \forall y \in K. \tag{2.5}$$

One also needs other sorts of nonlinear operators which are introduced below.

Let $T, A : H \to H$ be the nonlinear operators.

(1) *T is nonexpansive if $\|Tx - Ty\| \leq \|x-y\|$ for all $x, y \in H$.*

(2) *T is firmly nonexpansive if $2T - I$ is nonexpansive. Equivalent, $T = (I+S)/2$, where $S : H \to H$ is nonexpansive. Alternatively, T is firmly nonexpansive if and only if*

$$\langle x-y, Tx-Ty\rangle \geq \|Tx-Ty\|^2, \quad x, y \in H. \tag{2.6}$$

(3) *T is averaged if $T = (1-\alpha)I + \alpha S$, where $\alpha \in (0,1)$ and $S : H \to H$ is nonexpansive. In this case, one also says that T is α-averaged. A firmly nonexpansive mapping is $(1/2)$-averaged.*

(4) *A is monotone if $\langle Ax - Ay, x-y\rangle \geq 0$ for $x, y \in H$.*

(5) *A is β-strongly monotone, with $\beta > 0$, if*

$$\langle x-y, Ax-Ay\rangle \geq \beta\|x-y\|^2, \quad x, y \in H. \tag{2.7}$$

(6) *A is ν-inverse strongly monotone (ν-ism), with $\nu > 0$, if*

$$\langle x-y, Ax-Ay\rangle \geq \nu\|Ax-Ay\|^2, \quad x, y \in H. \tag{2.8}$$

It is well known that both P_K and $I - P_K$ are firmly nonexpansive and $(1/2)$-ism.

Denote by $\mathrm{Fix}(T)$ *the set of fixed points of a self-mapping T defined on H, (i.e.,* $\mathrm{Fix}(T) = x \in H : Tx = x$*).*

Proposition 2.2 (see [2, 5]). *One has the following assertions.*

(1) *T is nonexpansive if and only if the complement* $I - T$ *is* $(1/2)$*-ism.*

(2) *If T is* ν*-ism and* $\gamma > 0$*, then* γT *is* (ν/γ)*-ism.*

(3) *T is averaged if and only if the complement* $I - T$ *is* ν*-ism, for some* $\nu > (1/2)$.

Indeed, for $\alpha \in (0,1)$, *T is* α*-averaged if and only if* $I - T$ *is* $(1/2\alpha)$*-ism.*

(4) *If* T_1 *is* α_1*-averaged and* T_2 *is* α_2*-averaged, where* $\alpha_1, \alpha_2 \in (0,1)$*, then the composite* T_1T_2 *is* α*-averaged, where* $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$.

(5) *If* T_1 *and* T_2 *are averaged and have a common fixed point, then* $\mathrm{Fix}(T_1T_2) = \mathrm{Fix}(T_1) \cap \mathrm{Fix}(T_2)$.

Lemma 2.3 (see [6]). *Let K be a nonempty-closed convex subset of a real Hilbert space H and T be nonexpansive mapping on K with* $\mathrm{Fix}(T) \neq \emptyset$*. If* $\{x_n\}$ *is a sequence in K which converges weakly to x and if* $\{(I - T)x_n\}$ *converges strongly to y, then* $y = (I - T)x$*. In particular, if* $y = 0$*, then* $x \in \mathrm{Fix}(T)$.

Lemma 2.4 (see [7]). *Let* $(E, \langle\cdot,\cdot\rangle)$ *be an inner product space. Then for all* $x, y, z \in E$ *and* $\alpha_n, \beta_n, \gamma_n \in [0,1]$ *with* $\alpha_n + \beta_n + \gamma_n = 1$*, one has*

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 - \alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2. \quad (2.9)$$

Lemma 2.5 (see [8]). *Assume that* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \quad (2.10)$$

where $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence such that*

(1) $\sum_{n=0}^{\infty} \gamma_n = \infty$;

(2) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

3. Main Result

Let C be a nonempty closed and convex subset of a Hilbert space H. For any $u, x_0 \in C$, we define the sequence $\{x_n\}$ by

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_C(I - \xi A^*(I - P_Q)A)x_n, \quad n \geq 0, \quad (3.1)$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are three sequences in $[0,1]$ and satisfy $\alpha_n + \beta_n + \gamma_n = 1$.

Theorem 3.1. *Suppose that the SFP is consistent and* $0 < \xi < (2/\|A\|^2)$*. Let* $\{x_n\}$ *be a sequence defined as in* (3.1). *If the following assumptions are satisfied:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$ *but* $\sum_{n=1}^{\infty}\alpha_n = \infty$,

(C2) $\limsup_{n\to\infty}\beta_n < 1$,

(C3) *the sums* $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|$, $\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n|$ *and* $\sum_{n=1}^{\infty}|\gamma_{n+1}-\gamma_n|$ *are finite.*

Then $\{x_n\}$ *converges strongly to a solution of the SFP* (1.1).

Proof. We firstly show that the sequence $\{x_n\}$ is bounded. For our convenience, we take $T := P_C(I - \xi A^*(I - P_Q)A)$. Then, for any $x^* \in \Gamma$, we have $Tx^* = x^*$. Now, we observe that

$$\begin{aligned}\|x_{n+1} - x^*\| &\leq \left\|\alpha_n u + \beta_n x_n + \gamma_n T x_n - x^*\right\| \\ &\leq \alpha_n\|u - x^*\| + \beta_n\|x_n - x^*\| + \gamma_n\|Tx_n - x^*\|.\end{aligned} \tag{3.2}$$

Now, we note that the condition $0 < \xi < (2/\|A\|^2)$ implies that the operator $P_C(I - \xi A^*(I - P_Q)A)$ is averaged. Since $I - P_Q$ is firmly nonexpansive mappings and so is (1/2)-average, which is 1-ism. Also observe that $A^*(I - P_Q)A$ is $(1/\|A\|^2)$-ism so that $\xi A^*(I - P_Q)A$ is $(1/\xi\|A\|^2)$-ism. Further, from the fact that $I - \xi A^*(I - P_Q)A$ is $(\xi\|A\|^2/2)$-averaged and P_C is (1/2)-averaged, we may obtain that $P_C(I - \xi A^*(I - P_Q)A)$ is χ-averaged, where

$$\chi = \frac{1}{2} + \frac{\xi\|A\|^2}{2} - \frac{1}{2}\cdot\frac{\xi\|A\|^2}{2} = \frac{2 + \xi\|A\|^2}{4}. \tag{3.3}$$

This implies that $T = \chi I + (1 - \chi)S$, where $\chi = (2 + \xi\|A\|^2/4) \in (0, 1)$ for some nonexpansive mappings S. Note that T is also nonexpansive mappings. Hence, we have

$$\|Tx_n - x^*\| = \|Tx_n - Tx^*\| \leq \|x_n - x^*\|. \tag{3.4}$$

From the inequalities (3.2) and (3.4), we have

$$\begin{aligned}\|x_{n+1} - x^*\| &\leq \alpha_n\|u - x^*\| + (1 - \alpha_n)\|x_n - x^*\| \\ &\leq \max\{\|u - x^*\|, \|x_n - x^*\|\}.\end{aligned} \tag{3.5}$$

Continuing inductively, we may obtain that the inequality

$$\|x_{n+1} - x^*\| \leq \max\{\|u - x^*\|, \|x_0 - x^*\|\}, \tag{3.6}$$

holds for all $n \geq 0$. So, $\{x_n\}$ is bounded so does $\{Tx_n\}$.

Next, we will show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. Observe that

$$\begin{aligned}\|x_{n+1}-x_n\| &= \left\|\left(\alpha_n u+\beta_n x_n+\gamma_n T x_n\right)-\left(\alpha_{n-1}u+\beta_{n-1}x_{n-1}+\gamma_{n-1}Tx_{n-1}\right)\right\| \\ &\leq \left\|\left(\alpha_n u+\beta_n x_n+\gamma_n T x_n\right)-\left(\alpha_n u+\beta_n x_{n-1}+\gamma_n T x_{n-1}\right)\right\| \\ &\quad+\left\|\left(\alpha_n u+\beta_n x_{n-1}+\gamma_n T x_{n-1}\right)-\left(\alpha_{n-1}u+\beta_{n-1}x_{n-1}+\gamma_{n-1}Tx_{n-1}\right)\right\| \\ &\leq (1-\alpha_n)\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|u\|+\left|\beta_n-\beta_{n-1}\right|\|x_{n-1}\| \\ &\quad+\left|\gamma_n-\gamma_{n-1}\right|\|Tx_{n-1}\|.\end{aligned} \tag{3.7}$$

Since $\{x_n\}$ and $\{Tx_n\}$ are bounded, there exists $M=\sup(\|u\|,\|x_{n+1}\|,\|Tx_{n-1}\|)>0$ such that

$$\|x_{n+1}-x_n\|\leq(1-\alpha_n)\|x_n-x_{n-1}\|+M\left(|\alpha_n-\alpha_{n-1}|+\left|\beta_n-\beta_{n-1}\right|+\left|\gamma_n-\gamma_{n-1}\right|\right). \tag{3.8}$$

According to Lemma 2.5 and the condition (C3), we have $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.

We note that

$$\begin{aligned}\|x_n-Tx_n\| &\leq \|x_n-x_{n+1}\|+\|x_{n+1}-Tx_n\| \\ &\leq \|x_n-x_{n+1}\|+\left\|\alpha_n u+\beta_n x_n+(1-\alpha_n-\beta_n)Tx_n-Tx_n\right\| \\ &= \|x_n-x_{n+1}\|+\left\|\alpha_n(u-Tx_n)+\beta_n(x_n-Tx_n)\right\| \\ &\leq \|x_n-x_{n+1}\|+\alpha_n\|(u-Tx_n)\|+\beta_n\|(x_n-Tx_n)\| \\ &= \frac{1}{1-\beta_n}\|x_{n+1}-x_n\|+\frac{\alpha_n}{1-\beta_n}\|u-Tx_n\|.\end{aligned} \tag{3.9}$$

Consequently, by the condition (C1) and (C2), we also have $\lim_{n\to\infty}\|x_n-Tx_n\|=0$. Next, we will show that

$$\limsup_{n\to\infty}\langle v-z_0,x_{n+1}-z_0\rangle\leq 0,\qquad \text{where } z_0=P_\Gamma v. \tag{3.10}$$

To show this, we can choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle v-z_0,Tx_n-z_0\rangle=\lim_{k\to\infty}\langle v-z_0,Tx_{n_k}-z_0\rangle. \tag{3.11}$$

As $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ which converges weakly to z. We may assume without loss of generality that $x_{n_k}\rightharpoonup z$. Since $\|Tx_n-x_n\|\to 0$, we obtain $Tx_{n_k}\rightharpoonup z$ as $k\to\infty$. By Lemma 2.3, we obtain that $z\in \mathrm{Fix}(T)=\Gamma$.

Now from (2.4), observe that

$$\begin{aligned}\limsup_{n\to\infty}\langle v-z_0,x_n-z_0\rangle &= \limsup_{n\to\infty}\langle v-z_0,Tx_n-z_0\rangle\\ &= \lim_{k\to\infty}\langle v-z_0,Tx_{n_k}-z_0\rangle\\ &= \langle v-z_0,z-z_0\rangle\le 0.\end{aligned}\tag{3.12}$$

Therefore, we compute

$$\begin{aligned}\|x_{n+1}-z_0\|^2 &= \langle\alpha_n v+\beta_n x_n+\gamma_n Tx_n-z_0,x_{n+1}-z_0\rangle\\ &= \alpha_n\langle v-z_0,x_{n+1}-z_0\rangle+\beta_n\langle x_n-z_0,x_{n+1}-z_0\rangle+\gamma_n\langle Tx_n-z_0,x_{n+1}-z_0\rangle\\ &\le \alpha_n\langle v-z_0,x_{n+1}-z_0\rangle+\frac{1}{2}\beta_n\left(\|x_n-z_0\|^2+\|x_{n+1}-z_0\|^2\right)\\ &\quad+\frac{1}{2}\gamma_n\left(\|x_n-z_0\|^2+\|x_{n+1}-z_0\|^2\right)\\ &\le \alpha_n\langle v-z_0,x_{n+1}-z_0\rangle+\frac{1}{2}(1-\alpha_n)\left(\|x_n-z_0\|^2+\|x_{n+1}-z_0\|^2\right)\end{aligned}\tag{3.13}$$

which implies that

$$\|x_{n+1}-z_0\|^2\le(1-\alpha_n)\left(\|x_n-z_0\|^2\right)+2\alpha_n\langle v-z_0,x_{n+1}-z_0\rangle.\tag{3.14}$$

Finally, by (3.12), (3.14), and Lemma 2.5, we conclude that $\{x_n\}$ converges to z_0. This completes the proof. □

Letting $\beta_n\equiv 0$ of iterative scheme (3.1) in Theorem 3.1, then we obtain the following corollary.

Corollary 3.2. *For any $u,x_0\in C$, one defines the sequence $\{x_n\}$ by*

$$x_{n+1}=\alpha_n u+(1-\alpha_n)P_C(I-\xi A^*(I-P_Q)A)x_n,\quad n\ge 0,\tag{3.15}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$. Suppose that the SFP is consistent and $0<\xi<(2/\|A\|^2)$.

Let $\{x_n\}$ be defined as in (3.15). If the following assumptions are satisfied:

(C1) $\lim_{n\to\infty}\alpha_n=0$ *but* $\sum_{n=1}^{\infty}\alpha_n=\infty$,

(C2) $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$.

Then $\{x_n\}$ converges to a solution of the SFP (1.1).

Remark 3.3. Theorem 3.1 and Corollary 3.2 extend and improve the result of Xu [4] from weak to strong convergence theorems by using the modified Halpern's iterative scheme.

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