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*Research Article*

# $f$-Orthomorphisms and $f$-Linear Operators on the Order Dual of an $f$-Algebra

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We consider the $f$-orthomorphisms and $f$-linear operators on the order dual of an $f$-algebra. In particular, when the $f$-algebra has the factorization property (not necessarily unital), we prove that the orthomorphisms, $f$-orthomorphisms, and $f$-linear operators on the order dual are precisely the same class of operators.

## 1. Introduction

Let $A$ be an $f$-algebra with ${}^{\circ}(A^{\sim}) = \{0\}$. Recall that we can define a multiplication on $(A^{\sim})_n^{\sim}$, the order continuous part of the order bidual of $A$, with respect to which $(A^{\sim})_n^{\sim}$ can also be made an $f$-algebra. This is done in three steps:

(1) $A \times A^{\sim} \to A^{\sim}$

$(a, f) \mapsto f \cdot a : (f \cdot a)(b) = f(ab)$ for $b \in A$,

(2) $(A^{\sim})_n^{\sim} \times A^{\sim} \to A^{\sim}$

$(F, f) \mapsto F \cdot f : (F \cdot f)(a) = F(f \cdot a)$ for $a \in A$,

(3) $(A^{\sim})_n^{\sim} \times (A^{\sim})_n^{\sim} \to (A^{\sim})_n^{\sim}$

$(F, G) \mapsto F \cdot G : (F \cdot G)(f) = F(G \cdot f)$ for $f \in A^{\sim}$.

With the so-called Arens multiplication defined in step (3), $(A^{\sim})_n^{\sim}$ is an Archimedean (and hence commutative) $f$-algebra. Moreover, if $A$ has a multiplicative unit, then $(A^{\sim})_n^{\sim} = (A^{\sim})^{\sim}$, the whole order bidual of $A$. The mapping $V : (A^{\sim})_n^{\sim} \to \mathrm{Orth}(A^{\sim})$ defined by $V(F) = V_F$ for all $F \in (A^{\sim})_n^{\sim}$, where $V_F(f) = F \cdot f$ for every $f \in A^{\sim}$, is an algebra and Riesz isomorphism. See [1, 2] for details.

Let $A$ be an $f$-algebra. A Riesz space $L$ with $^{\circ}(L^{\sim}) = \{0\}$ is said to be an (left) $f$-module over $A$ (cf. [2, 3]) if $L$ is a left module over $A$ and satisfies the following two conditions:

(i) for each $a \in A^+$ and $x \in L^+$, we have $ax \in L^+$,

(ii) if $x \perp y$, then for each $a \in A$, we have $a \cdot x \perp y$.

When $A$ is an $f$-algebra with unit $e$, saying $L$ is a unital $f$-module over $A$ implies that the left multiplication satisfies $e \cdot x = x$ for all $x \in L$. From Corollary 2.3 in [2], we know that if $L$ is an $f$-module over $A$, then $L^{\sim}$ is an $f$-module over $A$ (and $(A^{\sim})_n^{\sim}$). The $f$-module $L$ over $A$ with unit $e$ is said to be *topologically full* with respect to $A$ if for two arbitrary vectors $x$, $y$ satisfying $0 \leq y \leq x$ in $L$, there exists a net $0 \leq a_\alpha \leq e$ in $A$ such that $a_\alpha \cdot x \rightarrow y$ in $\sigma(L, L^{\sim})$. If $L$ is topologically full with respect to $A$, then $L^{\sim}$ is topologically full with respect to $(A^{\sim})_n^{\sim}$ [2, Proposition 3.12].

Let $A$ be a unital $f$-algebra, and, $L$, $M$ be $f$-modules over $A$. $T \in L_b(L, M)$ is called an $f$-linear operator if $T(a \cdot x) = a \cdot Tx$ for each $a \in A$ and $x \in L$. The collection of all $f$-linear operators will be denoted by $L_b(L, M; A)$. For each $x \in L$ and $f \in L^{\sim}$, we can define $\psi_{x,f} \in A^{\sim}$ by $\psi_{x,f}(a) = f(a \cdot x)$ for all $a \in A$. Let $S(x) := \{\psi_{x,f} \ : \ f \in L^{\sim}\}$. Then $S(x)$ is an order ideal in $A^{\sim}$ [2]. $T \in L_b(L, M)$ is said to be an $f$-orthomorphism if $S(Tx) \subseteq S(x)$ for each $x \in L$. The collection of all $f$-orthomorphisms will be denoted by $\operatorname{Orth}(L, M; A)$. Turan [2] showed that $\operatorname{Orth}(L, M; A) = L_b(L, M; A)$ whenever $M$ is topologically full with respect to $A$.

Clearly, $A^{\sim}$ is an $f$-module over the $f$-algebras $A$ and $(A^{\sim})_n^{\sim}$, respectively. If $A$ is unital, then $A$ is topologically full with respect to itself ([2, Proposition 2.6]). From the above remarks we know that $A^{\sim}$ is topologically full with respect to $(A^{\sim})_n^{\sim}$, and hence, the $f$-orthomorphisms and $f$-linear operators are precisely the same class of operators, that is,

$$\operatorname{Orth}(A^{\sim}, A^{\sim}; (A^{\sim})_n^{\sim}) = L_b(A^{\sim}, A^{\sim}; (A^{\sim})_n^{\sim}). \tag{*}$$

An $f$-algebra $A$ is said to be *square-root closed* whenever for any $a \in A$ there exists $b \in A$ such that $|a| = b^2$. An immediate example is that a uniformly complete $f$-algebra with unit element is square-root closed [4]. However, a square-root closed $f$-algebra is not necessarily unital. For instance, $c_0$, with the familiar coordinatewise operations and ordering, is a square-root closed $f$-algebra without unit. We recall that an Archimedean $f$-algebra $A$ is said to have the *factorization property* if, given $a \in A$, there exist $b, c \in A$ such that $a = bc$. It should be noted that if $A$ is unital or square-root closed, then $A$ has the factorization property.

In this paper, we do not have to assume that the $f$-algebras are unital. We modify the definition of the $f$-orthomorphism introduced by Turan [2, Definition 3.7] and consider the $f$-orthomorphisms and $f$-linear operators on the order dual of an $f$-algebra. In particular, when the $f$-algebra with separating order dual has the factorization property, we prove that the orthomorphisms, $f$-orthomorphisms, and $f$-linear operators on the order dual are precisely the same class of operators, that is, the above equality (*) still holds.

Our notions are standard. For the theory of Riesz spaces, positive operators, and $f$-algebras, we refer the reader to the monographs [5–7].

## 2. $f$-Orthomorphisms on the Order Dual

Let $A$ be an $f$-algebra with separating order dual (and hence $A$ Archimedean!) and $f \in A^{\sim}$. We consider the mapping $T_f : (A^{\sim})_n^{\sim} \rightarrow A^{\sim}$ defined by $T_f(F) = F \cdot f$ for all $F \in (A^{\sim})_n^{\sim}$.

It should be noted that the mapping $V : (A^{\sim})_n^{\sim} \to \mathrm{Orth}(A^{\sim})$ defined by $V(F) = V_F$ for all $F \in (A^{\sim})_n^{\sim}$, where $V_F(f) = F \cdot f$ for every $f \in A^{\sim}$, is an algebra and Riesz isomorphism (cf. [2, Proposition 2.2]).

**Theorem 2.1.** *For $0 \le f \in A^{\sim}$, $T_f$ is an interval preserving lattice homomorphism.*

*Proof.* Clearly, $T_f$ is linear and positive. Since the mapping $V$ is a lattice homomorphism and $V_F, V_G \in \mathrm{Orth}(A^{\sim})$ for $F, G \in (A^{\sim})_n^{\sim}$, we have

$$\begin{aligned} T_f(F \vee G) = (F \vee G) \cdot f &= V_{F \vee G}(f) \\ &= (V(F \vee G))(f) \\ &= (V(F) \vee V(G))(f) \\ &= \big(V(F)(f)\big) \vee \big(V(G)(f)\big) \\ &= F \cdot f \vee G \cdot f \\ &= T_f(F) \vee T_f(G). \end{aligned} \tag{2.1}$$

Hence, $T_f$ is a lattice homomorphism.

Next, we show that $T_f$ is an interval preserving operator. We identify $x$ with its canonical image $x''$ in $(A^{\sim})_n^{\sim}$ and denote the restriction of $T_f$ to $A$ by $T_f|_A$. Then

$$T_F|_A(x) = T_f\left(x''\right) = x'' \cdot f = f \cdot x. \tag{2.2}$$

Thus, for each $F \in (A^{\sim})_n^{\sim}$ and $x \in A$, we see that

$$\Big((T_f|_A)'(F)\Big)(x) = F\big((T_f|_A)(x)\big) = F\big(f \cdot x\big) = \big(F \cdot f\big)(x) = \big(T_f(F)\big)(x), \tag{2.3}$$

which implies that $(T_f|_A)'$ is the same as $T_f$ on $(A^{\sim})_n^{\sim}$. Since $(T_f|_A)'$ is interval preserving (cf. [5, Theorem 7.8]), $T_f$ is likewise an interval preserving operator. □

**Corollary 2.2.** *For $f \in A^{\sim}, F \in (A^{\sim})_n^{\sim}$, one has $|F \cdot f| = |F| \cdot |f|$. Furthermore, if $f \perp g$ in $A^{\sim}$, $F \cdot f \perp G \cdot g$ holds for any $F, G \in (A^{\sim})_n^{\sim}$.*

*Proof.* Since $V_F$ is an orthomorphism on $A^{\sim}$, we have $V_F(f^+) \perp V_F(f^-)$ for each $f \in A^{\sim}$, that is, $F \cdot (f^+) \perp F \cdot (f^-)$. From Theorem 2.1, we know that

$$\begin{aligned} \left|F \cdot f\right| &= \left|F \cdot f^+\right| + \left|F \cdot f^-\right| \\ &= \left|T_{f^+}(F)\right| + \left|T_{f^-}(F)\right| \\ &= T_{f^+}(|F|) + T_{f^-}(|F|) \\ &= |F| \cdot f^+ + |F| \cdot f^- = |F| \cdot \left|f\right|. \end{aligned} \tag{2.4}$$

Let $f \perp g$ in $A^\sim$. Then we have

$$\begin{aligned}|F\cdot f| \wedge |G\cdot g| &= |F|\cdot|f| \wedge |G|\cdot|g| \\ &\le ((|F|+|G|)\cdot|f|) \wedge ((|F|+|G|)\cdot|g|) = 0,\end{aligned} \tag{2.5}$$

which implies that $F\cdot f \perp G\cdot g$ for all $F,G \in (A^\sim)^\sim_n$. □

Following the above discussion, we now consider $R(f) = \{F\cdot f : F \in (A^\sim)^\sim_n\}$, the image of $(A^\sim)^\sim_n$ under $T_f$.

**Corollary 2.3.** *If $A$ is an $f$-algebra and $f \in (A^\sim)$, then $R(f) = R(|f|)$, and $R(f)$ is an order ideal in $A^\sim$.*

*Proof.* First, since $T_{|f|}$ is an interval preserving lattice homomorphism, we can easily see that $R(|f|)$ is an order ideal in $A^\sim$. By Corollary 2.2 we conclude that $R(f) \subseteq R(|f|)$.

Now, to complete the proof we only need to prove that $R(|f|) \subseteq R(f)$. To this end, let $P_1 : A^\sim \to B_{f^+}$, $P_2 : A^\sim \to B_{f^-}$ be band projections, where $B_{f^+}$ and $B_{f^-}$ are the bands generated by $f^+$ and $f^-$ in $A^\sim$, respectively. If $\pi = P_1 - P_2$, we have

$$\pi \in \operatorname{Orth}(A^\sim), \qquad \pi(f) = |f|, \qquad \pi(|f|) = f. \tag{2.6}$$

In addition, $\pi(f)\cdot a = \pi(f\cdot a)$ for all $a \in A$ (cf. Theorem 3.1). Since $\pi$ is an orthomorphism on $A^\sim$ and hence order continuous (cf. [5, Theorem 8.10]), we have $\pi'((A^\sim)^\sim_n) \subseteq (A^\sim)^\sim_n$. For all $a \in A$ and all $F \in (A^\sim)^\sim_n$, from

$$\begin{aligned}(F\cdot|f|)(a) &= (F\cdot\pi(f))(a) \\ &= F(\pi(f)\cdot a) \\ &= F(\pi(f\cdot a)) \\ &= (\pi'(F)\cdot f)(a),\end{aligned} \tag{2.7}$$

it follows that $F\cdot|f| = \pi'(F)\cdot f$ for all $F \in (A^\sim)^\sim_n$, which implies that $R(|f|) \subseteq R(f)$, as desired. □

Next, we give a necessary and sufficient condition for $R(f) \perp R(g)$ when $A$ has the factorization property. First, we need the following lemma.

**Lemma 2.4.** *Let $A$ be an $f$-algebra with the factorization property, and $f \in A^\sim$. If $f\cdot x = 0$ for each $x \in A$, then $f = 0$.*

*Proof.* Since $A$ has the factorization property, for each $a \in A^+$, there exist $x, y \in A$ such that $a = xy$. Hence, from

$$f(a) = f(xy) = (f\cdot x)(y) = 0, \tag{2.8}$$

it follows easily that $f = 0$ holds. □

**Theorem 2.5.** *Let $A$ be an $f$-algebra with the factorization property. If $f, g \in A^\sim$, then $f \perp g$ if and only if $R(f) \perp R(g)$.*

*Proof.* If $f \perp g$ in $A^\sim$, then it follows from Corollary 2.2 that $F \cdot f \perp G \cdot g$ for all $F, G \in (A^\sim)_n^\sim$. This implies that $R(f) \perp R(g)$.

Conversely, if $R(f)$ and $R(g)$ are disjoint, then for each $F \in ((A^\sim)_n^\sim)^+$ we have

$$\begin{aligned} F \cdot (|f| \wedge |g|) &= V_F(|f| \wedge |g|) \\ &= V_F(|f|) \wedge V_F(|g|) \\ &= F \cdot |f| \wedge F \cdot |g| \\ &= |F \cdot f| \wedge |F \cdot g| = 0. \end{aligned} \tag{2.9}$$

In particular, for any $x \in A$, its canonical image $x'' \in (A^\sim)_n^\sim$ also satisfies $x'' \cdot (|f| \wedge |g|) = (|f| \wedge |g|) \cdot x = 0$. By the preceding lemma, we have $|f| \wedge |g| = 0$, that is, $f \perp g$, as desired. □

Now, we give the definition of the so-called $f$-orthomorphism.

*Definition 2.6.* Let $A$ be an $f$-algebra and $T \in L_b(A^\sim)$. $T$ is called an $f$-orthomorphism on $A^\sim$ if $R(Tf) \subseteq R(f)$ for each $f \in A^\sim$. The collection of all $f$-orthomorphisms on $A^\sim$ will be denoted by $\mathrm{Orth}(A^\sim, A^\sim; (A^\sim)_n^\sim)$.

The next result deals with the relationship between the $f$-orthomorphisms and the orthomorphisms on the order dual of an $f$-algebra with the factorization property. Note that $\mathrm{Orth}(A^\sim)$ is a band in $L_b(A^\sim)$.

**Theorem 2.7.** *Let $A$ be an $f$-algebra. Then $\mathrm{Orth}(A^\sim, A^\sim; (A^\sim)_n^\sim)$ is a linear subspace of $L_b(A^\sim)$ and $\mathrm{Orth}(A^\sim) \subseteq \mathrm{Orth}(A^\sim, A^\sim; (A^\sim)_n^\sim)$.*

*If $A$, in addition, has the factorization property, then $\mathrm{Orth}(A^\sim, A^\sim; (A^\sim)_n^\sim) = \mathrm{Orth}(A^\sim)$.*

*Proof.* First, we can easily see that $\mathrm{Orth}(A^\sim, A^\sim; (A^\sim)_n^\sim)$ is a linear subspace of $L_b(A^\sim)$. To prove $\mathrm{Orth}(A^\sim) \subseteq \mathrm{Orth}(A^\sim, A^\sim; (A^\sim)_n^\sim)$, let $\pi \in \mathrm{Orth}(A^\sim)$. We claim that $F \cdot \pi(f) = \pi'(F) \cdot f$ for all $F \in (A^\sim)_n^\sim$ and all $f \in A^\sim$. To this end, let $F \in (A^\sim)_n^\sim$, $f \in A^\sim$, and $x \in A$ be arbitrary. Since $(A^\sim)_n^\sim$ is a commutative $f$-algebra, by Theorem 3.1, we have

$$\begin{aligned} (\pi'(F) \cdot f)(x) = \pi'(F)(f \cdot x) = F(\pi(f \cdot x)) &= F(\pi(x'' \cdot f)) \\ &= F(x'' \cdot (\pi(f))) \\ &= (F \cdot x'')(\pi(f)) \\ &= (x'' \cdot F)(\pi(f)) \\ &= x''(F \cdot \pi(f)) = (F \cdot \pi(f))(x). \end{aligned} \tag{2.10}$$

Thus, $F \cdot \pi(f) = \pi'(F) \cdot f$. This implies that $R(\pi(f)) \subseteq R(f)$ for each $f \in A^\sim$, that is, $\mathrm{Orth}(A^\sim) \subseteq \mathrm{Orth}(A^\sim, A^\sim; (A^\sim)_n^\sim)$.

If $A$ has the factorization property, we prove that $\mathrm{Orth}(A^\sim, A^\sim; (A^\sim)_n^\sim) \subseteq \mathrm{Orth}(A^\sim)$ holds. To this end, take $T \in \mathrm{Orth}(A^\sim, A^\sim; (A^\sim)_n^\sim)$ and $f, g \in A^\sim$ satisfying $f \perp g$ in $A^\sim$. Then, it follows from Theorem 2.5 that $R(f) \perp R(g)$. Since $T \in \mathrm{Orth}(A^\sim, A^\sim; (A^\sim)_n^\sim)$, we have

$R(T(f)) \subset R(f)$. Therefore, $R(T(f)) \perp R(g)$, which implies that $T(f) \perp g$, and hence $T$ is an orthomorphism on $A^\sim$, as desired. □

## 3. $f$-Linear Operators on the Order Dual

Let $A$ be an $f$-algebra with separating order dual and $T \in L_b(A^\sim)$. Recall that $T$ is called to be $f$-linear with respect to $(A^\sim)_n^\sim$ if $T(G \cdot f) = G \cdot T(f)$ for all $f \in A^\sim$ and $G \in (A^\sim)_n^\sim$. The set of all $f$-linear operators on $A^\sim$ will be denoted by $L_b(A^\sim, A^\sim; (A^\sim)_n^\sim)$. It follows from [3, Lemma 4.4] that $L_b(A^\sim, A^\sim; (A^\sim)_n^\sim)$ is a band in $L_b(A^\sim)$.

**Theorem 3.1.** *Let $A$ be an $f$-algebra with separating order dual. Then $\mathrm{Orth}(A^\sim) \subseteq L_b(A^\sim, A^\sim; (A^\sim)_n^\sim)$.*

*Proof.* Clearly $\mathrm{Orth}(A^\sim)$ is commutative since $\mathrm{Orth}(A^\sim)$ is an Archimedean $f$-algebra. To complete the proof, let $\pi \in \mathrm{Orth}(A^\sim)$. We have

$$\pi(G \cdot f) = \pi(V_G(f)) = V_G(\pi(f)) = G \cdot (\pi(f)), \tag{3.1}$$

for all $f \in A^\sim$ and $G \in (A^\sim)_n^\sim$. Hence, $\pi \in L_b(A^\sim, A^\sim; (A^\sim)_n^\sim)$. □

The following result deals with the order adjoint of an $f$-linear operator on the order dual of an $f$-algebra. It should be noted that the order adjoint of an order-bounded operator is order continuous (cf. [5, Theorem 5.8]).

**Lemma 3.2.** *Let $T \in L_b(A^\sim, A^\sim; (A^\sim)_n^\sim)$. Then the order adjoint $T'$ of $T$ satisfies $T'(F) \cdot f = F \cdot T(f)$ for all $F \in (A^\sim)_n^\sim$ and $f \in A^\sim$. In particular, $G \cdot T'(F) = T'(G \cdot F)$ for all $F, G \in (A^\sim)_n^\sim$.*

*Proof.* Since $T \in L_b(A^\sim, A^\sim; (A^\sim)_n^\sim)$, and $(A^\sim)_n^\sim$ is a commutative $f$-algebra, we have

$$\begin{aligned}(T'(F) \cdot f)(x) = T'(F)(f \cdot x) &= F(T(f \cdot x)) \\ &= F(T(x'' \cdot f)) \\ &= F(x'' \cdot (T(f))) \\ &= (F \cdot x'')(T(f)) \\ &= (x'' \cdot F)(T(f)) \\ &= x''(F \cdot T(F)) = (F \cdot T(f))(x),\end{aligned} \tag{3.2}$$

for all $F \in (A^\sim)_n^\sim$, $f \in A^\sim$, and $x \in A$, which implies that $T'(F) \cdot f = F \cdot T(f)$.

Let $F,\ G \in (A^\sim)_n^\sim$ be given. Then for $f \in A^\sim$, from

$$\begin{aligned}(G \cdot T'(F))(f) = G(T'(F) \cdot f) &= G(F \cdot T(f)) \\ &= (G \cdot F)(T(f)) \\ &= (T'(G \cdot F))(f),\end{aligned} \tag{3.3}$$

it follows that $G \cdot T'(F) = T'(G \cdot F)$. This completes the proof. □

**Theorem 3.3.**

$$L_b(A^{\sim}, A^{\sim}; (A^{\sim})_n^{\sim}) \subseteq Orth(A^{\sim}, A^{\sim}; (A^{\sim})_n^{\sim}). \tag{3.4}$$

*Proof.* For $T \in L_b(A^{\sim}, A^{\sim}; (A^{\sim})_n^{\sim})$, we know that $|T|$ is also $f$-linear with respect to $(A^{\sim})_n^{\sim}$. Assume that $0 \leq G \in (A^{\sim})_n^{\sim}$ and $f \in A^{\sim}$. So by Lemma 3.2, we have

$$0 \leq G \cdot (|T(f)|) \leq G \cdot (|T||f|) = (|T|'(G)) \cdot |f| = T_{|f|}(|T|'(G)). \tag{3.5}$$

Since $T_{|f|}$ is interval preserving, there exists $F \in (A^{\sim})_n^{\sim}$ such that $0 \leq F \leq |T|'(G)$ and $G \cdot (|T(f)|) = F \cdot |f|$. It is now immediate that $R(|T(f)|) \subseteq R(|f|)$, and hence, $L_b(A^{\sim}, A^{\sim}; (A^{\sim})_n^{\sim}) \subseteq \text{Orth}(A^{\sim}, A^{\sim}; (A^{\sim})_n^{\sim})$, as desired. □

Combining Theorems 3.1, 3.3, and 2.7, we have the following result.

**Theorem 3.4.** *If $A$ is an $f$-algebra with separating order dual, then*

$$Orth(A^{\sim}) \subseteq L_b(A^{\sim}, A^{\sim}; (A^{\sim})_n^{\sim}) \subseteq Orth(A^{\sim}, A^{\sim}; (A^{\sim})_n^{\sim}). \tag{3.6}$$

*In particular, if, in addition, $A$ has the factorization property, then*

$$Orth(A^{\sim}) = L_b(A^{\sim}, A^{\sim}; (A^{\sim})_n^{\sim}) = Orth(A^{\sim}, A^{\sim}; (A^{\sim})_n^{\sim}). \tag{3.7}$$

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