Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 124510, 6 pages
http://dx.doi.org/10.1155/2013/124510

Research Article

Some Operator Inequalities on Chaotic Order and Monotonicity of Related Operator Function

Changsen Yang and Yanmin Liu

College of Mathematics and Information Science, Henan Normal University, Xinxiang 453002, China

Correspondence should be addressed to Changsen Yang; yangchangsen0991@sina.com

Received 24 March 2013; Accepted 24 April 2013

Academic Editor: Yisheng Song

We will discuss some operator inequalities on chaotic order about several operators, which are generalization of Furuta inequality and show monotonicity of related Furuta type operator function.

1. Introduction

An operator T is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all vectors x in a Hilbert space, and T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible.

Theorem LH (Löwner-Heinz inequality, denoted by (LH) briefly). *If $A \geq B \geq 0$ holds, then $A^\alpha \geq B^\alpha$ for any $\alpha \in [0, 1]$.*

This was originally proved in [1, 2] and then in [3]. Although (LH) asserts that $A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0, 1]$, unfortunately $A^\alpha \geq B^\alpha$ does not always hold for $\alpha > 1$. The following result has been obtained from this point of view.

Theorem F (Furuta inequality). *If $A \geq B \geq 0$, then for each $r \geq 0$,*

(i) $(B^{r/2}A^pB^{r/2})^{1/q} \geq (B^{r/2}B^pB^{r/2})^{1/q}$,

(ii) $(A^{r/2}A^pA^{r/2})^{1/q} \geq (A^{r/2}B^pA^{r/2})^{1/q}$

hold for $p \geq 0$ and $q \geq 1$ with $(1 + r)q \geq p + r$.

The original proof of Theorem F is shown in [4], an elementary one-page proof is in [5], and alternative ones are in [6, 7]. We remark that the domain of the parameters p, q, and r in Theorem F is the best possible for the inequalities (i) and (ii) under the assumption $A \geq B \geq 0$; see [8].

We write $A \gg B$ if $\log A \geq \log B$ for $A, B > 0$, which is called the chaotic order.

Theorem A. *For $A, B > 0$, the following (i) and (ii) hold:*

(i) *$A \gg B$ holds if and only if $A^r \geq (A^{r/2}B^pA^{r/2})^{r/(p+r)}$ for $p, r \geq 0$;*

(ii) *$A \gg B$ holds if and only if for any fixed $\delta \geq 0$, $F_{A,B}(p, r) = A^{-r/2}(A^{r/2}B^pA^{r/2})^{(\delta+r)/(p+r)}A^{-r/2}$ is a decreasing function of $p \geq \delta$ and $r \geq 0$.*

(i) in Theorem A is shown in [9, 10], an excellent proof in [11], a proof in the case $p = r$ in [12], (ii) in [9, 10], and so forth.

Lemma B (see [11]). *Let A be a positive invertible operator, and let B be an invertible operator. For any real number λ,*

$$(BAB^*) = BA^{1/2}\left(A^{1/2}B^*BA^{1/2}\right)^{\lambda-1}A^{1/2}B^*. \qquad (1)$$

Definition 1. Let $A_n, A_{n-1}, \ldots, A_2, A_1$, $B \geq 0$, $r_1, r_2, \ldots, r_n \geq 0$, and $p_1, p_2, \ldots, p_n \geq 0$ for a natural number n.

Let $C_{A_i,B}[n]$ be defined by

$$C_{A_i,B}[n] = A_n^{r_n/2}\left\{A_{n-1}^{r_{n-1}/2}\left[\cdots A_3^{r_3/2}\left\{A_2^{r_2/2}\left(A_1^{r_1/2}B^{p_1}A_1^{r_1/2}\right)^{p_2}A_2^{r_2/2}\right\}^{p_3} \times A_3^{r_3/2}\cdots\right]A_{n-1}^{r_{n-1}/2}\right\}^{p_n}A_n^{r_n/2}. \qquad (2)$$

For example,

$$C_{A_i,B}[2] = A_2^{r_2/2}\left(A_1^{r_1/2}B^{p_1}A_1^{r_1/2}\right)^{p_2}A_2^{r_2/2},$$
$$C_{A_i,B}[4] = A_4^{r_4/2}\left\{A_3^{r_3/2}\left[A_2^{r_2/2}\left(A_1^{r_1/2}B^{p_1}A_1^{r_1/2}\right)^{p_2}A_2^{r_2/2}\right]^{p_3} \times A_3^{r_3/2}\right\}^{p_4}A_4^{r_4/2}. \tag{3}$$

Let $q[n]$ be defined by

$$q[n] = \{\cdots[(p_1+r_1)p_2+r_2]p_3+\cdots+r_{n-1}\}p_n+r_n. \tag{4}$$

For example,

$$q[1] = p_1+r_1, \qquad q[2] = (p_1+r_1)p_2+r_2,$$
$$q[4] = \{[(p_1+r_1)p_2+r_2]p_3+r_3\}p_4+r_4. \tag{5}$$

For the sake of convenience, we define

$$C_{A_i,B}[0] = B, \qquad q[0] = 1, \tag{6}$$

and these definitions in (6) may be reasonable by (2) and (4).

Lemma 2. *For $A_n, A_{n-1}, \ldots, A_2, A_1, B \geq 0$ and any natural number n, we have*

(i) $C_{A_i,B}[n] = A_n^{r_n/2}C_{A_i,B}[n-1]^{p_n}A_n^{r_n/2}$,

(ii) $q[n] = q[n-1]p_n + r_n$.

Proof. (i) and (ii) can be easily obtained by definitions (2) and (4). □

2. Basic Results Associated with $C_{A_i,B}[n]$ and $q[n]$

We will give some operator inequalities on chaotic order, and Theorem 5 is further extension of Theorem 3.1 in [13].

Lemma 3. *If $A \gg B$, for $p \geq 0$ and $r \geq 0$, then $A \gg (A^{r/2}B^pA^{r/2})^{1/(p+r)}$.*

Proof. Since $A \gg B$, we can obtain the following inequality.

$A^r \geq (A^{r/2}B^pA^{r/2})^{r/(p+r)}$ holds for $p \geq 0$ and $r \geq 0$ by (i) of Theorem A.

Take the logarithm on both sides of the previous inequality; that is,

$$\log A^r \geq \log\left(A^{r/2}B^pA^{r/2}\right)^{r/(p+r)}, \tag{7}$$

therefor we have

$$A \gg \left(A^{r/2}B^pA^{r/2}\right)^{1/(p+r)}. \tag{8}$$

□

Theorem 4. *If $A_n \gg A_{n-1} \gg \cdots \gg A_2 \gg A_1 \gg B$ and $r_1, r_2, \ldots, r_n \geq 0$, $p_1, p_2, \ldots, p_n \geq 0$ for a natural number n. Then the following inequality holds:*

$$A_n \gg C_{A_i,B}[n]^{1/q[n]}, \tag{9}$$

where $C_{A_i,B}[n]$ and $q[n]$ are defined in (2) and (4).

Proof. We will show (9) by mathematical induction. In the case $n = 1$.

Since $A_1 \gg B$ implies

$$A_1 \gg \left(A_1^{r_1/2}B^{p_1}A_1^{r_1/2}\right)^{r_1/(p_1+r_1)} \tag{10}$$

holds for any $p_1 \geq 0$ and $r_1 \geq 0$ by Lemma 3, whence (9) for $n = 1$.

Assume that (9) holds for a natural number k $(1 \leq k < n)$. We will show that (9) holds $r_1, r_2, \ldots, r_k, r_{k+1} \geq 0$ and $p_1, p_2, \ldots, p_k, p_{k+1} \geq 0$ for $k+1$.

Put $D = A_{k+1}$, $E = A_k$, and $F = C_{A_i,B}[k]^{1/q[k]}$, and (9) holds for $n = k$ implying

$$D \gg E \gg F > 0. \tag{11}$$

Equation (11) yields the following by Lemma 3, for $r \geq 0$ and $p \geq 0$:

$$D \gg \left(D^{r/2}F^pD^{r/2}\right)^{1/(p+r)}, \tag{12}$$

that is,

$$A_{k+1} \gg \left(A_{k+1}^{r/2}C_{A_i,B}[k]^{p/q[k]}A_{k+1}^{r/2}\right)^{1/(p+r)}. \tag{13}$$

Put $r = r_{k+1}$, $p = q[k]p_{k+1}$ in (13), then by (ii) of Lemma 2, the exponential power $1/(p+r)$ of the right hand side of (13) can be written as follows:

$$\frac{1}{p+r} = \frac{1}{q[k]p_{k+1}+r_{k+1}} = \frac{1}{q[k+1]}, \tag{14}$$

and we have the following desired (15) by (12) and (13):

$$A_{k+1} \gg \left\{A_{k+1}^{r_{k+1}/2}\left(C_{A_i,B}[k]\right)^{p_{k+1}}A_{k+1}^{r_{k+1}/2}\right\}^{1/q[k+1]} = C_{A_i,B}[k+1]^{1/q[k+1]}, \tag{15}$$

so that (15) shows that (9) holds for $k+1$. □

Theorem 5. *If $A_n \gg A_{n-1} \gg \cdots \gg A_2 \gg A_1 \gg B$ and $r_1, r_2, \ldots, r_n \geq 0$ for a natural number n. For any fixed $\delta \geq 0$, let $p_1, p_2, \ldots, p_n$ be satisfied by*

$$
\begin{aligned}
& p_1 \geq \delta, \\
& p_2 \geq \frac{\delta + r_1}{p_1 + r_1}, \\
& \vdots \\
& p_k \geq \frac{\delta + r_1 + r_2 + \cdots + r_{k-1}}{q[k-1]}, \\
& \vdots \\
& p_n \geq \frac{\delta + r_1 + r_2 + \cdots + r_{n-1}}{q[n-1]}.
\end{aligned} \tag{16}
$$

The operator function $I_k(p_k, r_k)$ for any natural number k such that $1 \leq k \leq n$ is defined by

$$I_k(p_k, r_k) = A_k^{-r_k/2} C_{A_i,B}[k]^{(\delta + r_1 + r_2 + \cdots + r_k)/q[k]} A_k^{-r_k/2}. \tag{17}$$

Then the following inequality holds:

$$A_{k-1}^{r_{k-1}/2} I_{k-1}(p_{k-1}, r_{k-1}) A_{k-1}^{r_{k-1}/2} \geq I_k(p_k, r_k) \tag{18}$$

for every natural number k such that $1 \leq k \leq n$, where $C_{A_i,B}[n]$ and $q[n]$ are defined in (2) and (4).

Proof. Since $C_{A_i,B}[0] = B$, $q[0] = 1$ in (6), we may define $I_0(p_0, r_0) = B^\delta$ for $p_0 = r_0 = 0$.

Because $A_1 \gg B$, then for any fixed $\delta \geq 0$,

$$B^\delta \geq A_1^{-r_1/2} \left(A_1^{r_1/2} B^{p_1} A_1^{r_1/2} \right)^{(\delta + r_1)/(p_1 + r_1)} A_1^{-r_1/2} \quad \text{for } p_1 \geq \delta, \; r_1 \geq 0, \tag{19}$$

since $F_{A_1,B}(\delta, r_0) \geq F_{A_1,B}(p_1, r_1)$ holds by (ii) of Theorem A. And (19) can be expressed as

$$B^\delta = A_0^{r_0/2} I_0(p_0, r_0) A_0^{r_0/2} \geq I_1(p_1, r_1). \tag{20}$$

We can apply Theorem 4, and we have the following (21) for any natural number k such that $1 \leq k \leq n$:

$$A_{k+1} \gg A_k \gg C_{A_i,B}[k]^{1/q[k]}. \tag{21}$$

Since $X \gg Y$ implies that $X^t \gg Y^t$ holds for any $t \geq 0$, (21) ensures

$$A_{k+1}^{\delta + r_1 + r_2 + \cdots + r_k} \gg C_{A_i,B}[k]^{(\delta + r_1 + r_2 + \cdots + r_k)/q[k]}. \tag{22}$$

Putting $A = A_{k+1}^{\delta + r_1 + r_2 + \cdots + r_k}$, $B_1 = C_{A_i,B}[k]^{(\delta + r_1 + r_2 + \cdots + r_k)/q[k]}$ and applying (19) for $\delta = 1$ and $A \gg B_1$, we have

$$B_1 \geq A^{-r/2} \left(A^{r/2} B_1^p A^{r/2} \right)^{(1+r)/(p+r)} A^{-r/2} \tag{23}$$

holds for $p \geq 1$ and $r \geq 0$.

Putting $r_{k+1} = r(\delta + r_1 + r_2 + \cdots + r_k)$ in (23), then (23) can be rewritten by

$$
\begin{aligned}
B_1 \geq A_{k+1}^{-r_{k+1}/2} & \left(A_{k+1}^{r_{k+1}/2} C_{A_i,B}[k]^{((\delta + r_1 + r_2 + \cdots + r_k)/q[k])p} \right. \\
& \left. \times A_{k+1}^{r_{k+1}/2} \right)^{(1+r)/(p+r)} A_{k+1}^{-r_{k+1}/2}.
\end{aligned} \tag{24}
$$

Putting $p = (q[k] p_{k+1})/(\delta + r_1 + r_2 + \cdots + r_k) \geq 1$, since $p_{k+1} \geq (\delta + r_1 + r_2 + \cdots + r_k)/q[k]$ in (16), then we have

$$
\begin{aligned}
& A_k^{r_k/2} I_k(p_k, r_k) A_k^{r_k/2} \\
& = B_1 = C_{A_i,B}[k]^{(\delta + r_1 + r_2 + \cdots + r_k)/q[k]} \\
& \geq A_{k+1}^{-r_{k+1}/2} \\
& \quad \times \left(A_{k+1}^{r_{k+1}/2} C_{A_i,B}[k]^{((\delta + r_1 + r_2 + \cdots + r_k)/q[k])p} A_{k+1}^{r_{k+1}/2} \right)^{(1+r)/(p+r)} \\
& \quad \times A_{k+1}^{-r_{k+1}/2} \\
& = A_{k+1}^{-r_{k+1}/2} C_{A_i,B}[k+1]^{(\delta + r_1 + r_2 + \cdots + r_k + r_{k+1})/(q[k+1])} A_{k+1}^{-r_{k+1}/2} \\
& = I_{k+1}(p_{k+1}, r_{k+1}),
\end{aligned} \tag{25}
$$

and we have (18) for k such that $1 \leq k \leq n$ by (25) and (20) since (20) means (18) for $k = 1$. □

Corollary 6. *If $A_n \gg A_{n-1} \gg \cdots \gg A_2 \gg A_1 \gg B$ and $r_1, r_2, \ldots, r_n \geq 0$ for a natural number n. For any fixed $\delta \geq 0$, let $p_1, p_2, \ldots, p_n$ be satisfied by (16).*

Then the following inequalities hold:

$$
\begin{aligned}
B^\delta & \geq A_1^{-r_1/2} \left(A_1^{r_1/2} B^{p_1} A_1^{r_1/2} \right)^{(\delta + r_1)/(p_1 + r_1)} A_1^{-r_1/2} \\
& \geq A_1^{-r_1/2} A_2^{-r_2/2} \\
& \quad \times \left[A_2^{r_2/2} \left(A_1^{r_1/2} B^{p_1} A_1^{r_1/2} \right)^{p_2} A_2^{r_2/2} \right]^{(\delta + r_1 + r_2)/((p_1 + r_1)p_2 + r_2)} \\
& \quad \times A_2^{-r_2/2} A_1^{-r_1/2} \\
& \vdots \\
& \geq A_1^{-r_1/2} A_2^{-r_2/2} A_3^{-r_3/3} \cdots A_{n-1}^{-r_{n-1}/2} A_n^{-r_n/2} \\
& \quad \times C_{A_i,B}[n]^{(\delta + r_1 + r_2 + \cdots + r_n)/q[n]} \\
& \quad \times A_n^{-r_n/2} A_{n-1}^{-r_{n-1}/2} \cdots A_3^{-r_3/3} A_2^{-r_2/2} A_1^{-r_1/2},
\end{aligned} \tag{26}
$$

where $C_{A_i,B}[n]$, $q[n]$, and $I_k(p_k, r_k)$ $(1 \leq k \leq n)$ are defined in (2), (4), and (17).

Proof. Applying (18) of Theorem 5 for k such that $1 \le k \le n$, we have

$$\begin{aligned}
B^{\delta} &= A^{r_0/2} I_0\left(p_0, r_0\right) A^{r_0/2} \\
&\ge I_1\left(p_1, r_1\right) \\
&= A_1^{-r_1/2}\left(A_1^{r_1/2} B^{p_1} A_1^{r_1/2}\right)^{(\delta+r_1)/(p_1+r_1)} A_1^{-r_1/2} \\
&\ge A_1^{-r_1/2} I_2\left(p_2, r_2\right) A_1^{-r_1/2} \\
&= A_1^{-r_1/2} A_2^{-r_2/2}\left[A_2^{r_2/2}\left(A_1^{r_1/2} B^{p_1} A_1^{r_1/2}\right)^{p_2}\right. \\
&\qquad \left. \times A_2^{r_2/2}\right]^{(\delta+r_1+r_2)/((p_1+r_1)p_2+r_2)} \\
&\quad \times A_2^{-r_2/2} A_1^{-r_1/2} \qquad (27)\\
&\vdots \\
&\ge A_1^{-r_1/2} A_2^{-r_2/2} A_3^{-r_3/3} \cdots A_{n-1}^{-r_{n-1}/2} I_n\left(p_n, r_n\right) \\
&\quad \times A_{n-1}^{-r_{n-1}/2} \cdots A_3^{-r_3/3} A_2^{-r_2/2} A_1^{-r_1/2} \\
&= A_1^{-r_1/2} A_2^{-r_2/2} A_3^{-r_3/3} \cdots A_{n-1}^{-r_{n-1}/2} A_n^{-r_n/2} \\
&\quad \times C_{A_i,B}[n]^{(\delta+r_1+r_2+\cdots+r_n)/q[n]} \\
&\quad \times A_n^{-r_n/2} A_{n-1}^{-r_{n-1}/2} \cdots A_3^{-r_3/3} A_2^{-r_2/2} A_1^{-r_1/2}.
\end{aligned}$$

□

3. Monotonicity Property on Operator Functions

We would like to emphasize that the condition of Theorem 7 is stronger than Theorem 5, and moreover when we discuss monotonicity property on operator functions, we can only apply Theorem 7.

Theorem 7. *If $A_n \gg A_{n-1} \gg \cdots \gg A_2 \gg A_1 \gg B$ and $r_1, r_2, \ldots, r_n \ge 0$, $p_1, p_2, \ldots, p_n \ge 0$ for a natural number n. Then the following inequality holds:*

$$A_n^{r_n} \ge C_{A_i,B}[n]^{r_n/q[n]}, \qquad (28)$$

where $C_{A_i,B}[n]$ and $q[n]$ are defined in (2) *and* (4).

Proof. We will show (28) by mathematical induction. In the case $n = 1$.

Since $A_1 \gg B$ implies

$$A_1 \ge \left(A_1^{r_1/2} B^{p_1} A_1^{r_1/2}\right)^{r_1/(p_1+r_1)} \qquad (29)$$

holds for any, $p_1 \ge 0$ and $r_1 \ge 0$ by (i) of Theorem A, whence (28) for $n = 1$.

Assume that (28) holds for a natural number k ($1 \le k < n$). We will show (28) for $r_1, r_2, \ldots, r_{k+1} \ge 0$ and $p_1, p_2, \ldots, p_k$, $p_{k+1} \ge 0$ for $k + 1$.

We can obtain the following inequality from the hypothesis (28) for the case $n = k$:

$$A_k^{r_k} \ge C_{A_i,B}[k]^{r_k/q[k]}, \qquad (30)$$

hence we have $A_{k+1} \gg A_k \gg C_{A_i,B}[k]^{1/q[k]}$, and (i) of Theorem A ensures

$$A_{k+1}^{r} \ge \left(A_{k+1}^{r/2} C_{A_i,B}[k]^{p/q[k]} A_{k+1}^{r/2}\right)^{r/(p+r)} \quad \text{for } p, r \ge 0. \qquad (31)$$

Putting $r = r_{k+1}$ and $p = q[k]p_{k+1}$, then we have the following inequality:

$$\begin{aligned}
A_{k+1}^{r_{k+1}} &\ge \left(A_{k+1}^{r_{k+1}/2} C_{A_i,B}[k]^{p_{k+1}} A_{k+1}^{r_{k+1}/2}\right)^{r_{k+1}/(q[k]p_{k+1}+r_{k+1})} \\
&= C_{A_i,B}[k+1]^{r_{k+1}/q[k+1]},
\end{aligned} \qquad (32)$$

so that (32) shows (28) for $k + 1$. □

Theorem 8. *If $A_n \gg A_{n-1} \gg \cdots \gg A_2 \gg A_1 \gg B$ and $r_1, r_2, \ldots, r_n \ge 0$ for a natural number n. For any fixed $\delta \ge 0$, let $p_1, p_2, \ldots, p_n$ be satisfied by* (16).

Then

$$I_n\left(p_n, r_n\right) = A_n^{-r_n/2} C_{A_i,B}[n]^{(\delta+r_1+r_2+\cdots+r_n)/q[n]} A_n^{-r_n/2} \qquad (33)$$

is a decreasing function of both $r_n \ge 0$ and p_n which satisfies

$$p_n \ge \frac{\delta + r_1 + r_2 + \cdots + r_{n-1}}{q\left[n-1\right]}, \qquad (34)$$

where $C_{A_i,B}[n]$ and $q[n]$ are defined in (2) *and* (4).

Proof. Since the condition (16) with $\delta \ge 0$ suffices (28) in Theorem 7, we have the following inequality by Theorem 7; see (28).

We state the following important inequality (35) for the forthcoming discussion which is the inequality in (16):

$$q\left[n\right] = q\left[n-1\right] p_n + r_n \ge \delta + r_1 + r_2 + \cdots + r_{n-1} + r_n \qquad (35)$$

because the inequality in (35) follows by (ii) of Lemma 2, and the inequality follows by

$$q\left[n-1\right] p_n \ge \delta + r_1 + r_2 + \cdots + r_{n-1} \qquad (36)$$

obtained by (34).

(a) Proof of the result that $I_n(p_n, r_n)$ is a decreasing function of p_n.

Without loss of generality, we can assume that $p_n > 0$. We can obtain the following inequality by (28) and by (i) of Lemma 2:

$$\begin{aligned}
A_n^{r_n} &\ge C_{A_i,B}[n]^{r_n/q[n]} = \left(A_n^{r_n/2} C_{A_i,B}[n-1]^{p_n} A_n^{r_n/2}\right)^{r_n/q[n]} \\
&= A_n^{r_n/2} C_{A_i,B}[n-1]^{p_n/2} \\
&\quad \times \left(C_{A_i,B}[n-1]^{p_n/2} A_n^{r_n} C_{A_i,B}[n-1]^{p_n/2}\right)^{(r_n-q[n])/q[n]} \\
&\quad \times C_{A_i,B}[n-1]^{p_n/2} A_n^{r_n/2},
\end{aligned} \qquad (37)$$

and (37) implies

$$\left(C_{A_i,B}[n-1]^{p_n/2}A_n^{r_n}C_{A_i,B}[n-1]^{p_n/2}\right)^{(q[n]-r_n)/q[n]} \geq C_{A_i,B}[n-1]^{p_n}. \quad (38)$$

Put $\alpha = \omega/p_n \in [0,1]$ for $p_n \geq \omega \geq 0$, then we raise each side of (38) to the power $\alpha = \omega/p_n \in [0,1]$, then

$$\left(C_{A_i,B}[n-1]^{p_n/2}A_n^{r_n}C_{A_i,B}[n-1]^{p_n/2}\right)^{((q[n]-r_n)\omega)/(q[n]p_n)} \geq C_{A_i,B}[n-1]^{\omega}. \quad (39)$$

Whence we have

$$\begin{aligned}
&I_n(p_n,r_n)\\
&= A_n^{-r_n/2}\left(A_n^{r_n/2}C_{A_i,B}[n-1]^{p_n}A_n^{r_n/2}\right)^{(\delta+r_1+r_2+\cdots+r_n)/q[n]}A_n^{-r_n/2}\\
&= A_n^{-r_n/2}\\
&\quad\times\left\{\left(A_n^{r_n/2}C_{A_i,B}[n-1]^{p_n}\right.\right.\\
&\qquad\left.\left.\times A_n^{r_n/2}\right)^{(q[n]+q[n-1]\omega)/q[n]}\right\}^{(\delta+r_1+r_2+\cdots+r_n)/(q[n]+q[n-1]\omega)}\\
&\quad\times A_n^{-r_n/2}\\
&= A_n^{-r_n/2}\left\{A_n^{r_n/2}C_{A_i,B}[n-1]^{p_n/2}\right.\\
&\qquad\times\left(C_{A_i,B}[n-1]^{p_n/2}A_n^{r_n}\right.\\
&\qquad\quad\left.\times C_{A_i,B}[n-1]^{p_n/2}\right)^{(q[n-1]\omega)/q[n]}\\
&\qquad\left.\times C_{A_i,B}[n-1]^{p_n/2}A_n^{r_n/2}\right\}^{(\delta+r_1+r_2+\cdots+r_n)/(q[n]+q[n-1]\omega)}\\
&\quad\times A_n^{-r_n/2} \text{ by Lemma B}\\
&= A_n^{-r_n/2}\left\{A_n^{r_n/2}C_{A_i,B}[n-1]^{p_n/2}\right.\\
&\qquad\times\left(C_{A_i,B}[n-1]^{p_n/2}A_n^{r_n}\right.\\
&\qquad\quad\left.\times C_{A_i,B}[n-1]^{p_n/2}\right)^{((q[n]-r_n)\omega)/(q[n]p_n)}\\
&\qquad\times C_{A_i,B}[n-1]^{p_n/2}\\
&\qquad\left.\times A_n^{r_n/2}\right\}^{(\delta+r_1+r_2+\cdots+r_n)/(q[n]+q[n-1]\omega)}A_n^{-r_n/2}\\
&\geq A_n^{-r_n/2}\left(A_n^{r_n/2}C_{A_i,B}[n-1]^{p_n/2}C_{A_i,B}[n-1]^{\omega}\right.\\
&\qquad\left.\times C_{A_i,B}[n-1]^{p_n/2}A_n^{r_n/2}\right)^{(\delta+r_1+r_2+\cdots+r_n)/(q[n-1](p_n+\omega)+r_n)}\\
&\quad\times A_n^{-r_n/2}\\
&= I_n(p_n+\omega,r_n),
\end{aligned} \quad (40)$$

and the last inequality holds by LH because (39) and $(\delta + r_1 + r_2 + \cdots + r_n)/(q[n-1](p_n+\omega)+r_n) \in [0,1]$ which is ensured by (35) and $q[n] + q[n-1]\omega = q[n-1](p_n+\omega) + r_n \geq q[n]$ by (4), so that $I_n(p_n, r_n)$ is a decreasing function of p_n.

(b) Proof of the result that $I_n(p_n, r_n)$ is a decreasing function of r_n.

Without loss of generality, we can assume that $r_n > 0$. Raise each side of (28) to the power $\mu/r_n \in [0,1]$ for $r_n \geq \mu \geq 0$ by LH, then

$$A_n^{\mu} \geq \left(A_n^{r_n/2}C_{A_i,B}[n-1]^{p_n}A_n^{r_n/2}\right)^{\mu/q[n]}. \quad (41)$$

We state the following inequality by (ii) of Lemma 3 and (35):

$$\begin{aligned}
&q[n] - (\delta + r_1 + r_2 + \cdots + r_n)\\
&\quad= q[n-1]\,p_n + r_n - (\delta + r_1 + r_2 + \cdots + r_n)\\
&\quad= q[n-1]\,p_n - (\delta + r_1 + r_2 + \cdots + r_{n-1}) \geq 0.
\end{aligned} \quad (42)$$

Then we have

$$\begin{aligned}
&I_n(p_n,r_n)\\
&= A_n^{-r_n/2}C_{A_i,B}[n]^{(\delta+r_1+r_2+\cdots+r_n)/q[n]}A_n^{-r_n/2}\\
&= A_n^{-r_n/2}\left(A_n^{r_n/2}C_{A_i,B}[n-1]^{p_n}A^{r_n/2}\right)^{(\delta+r_1+r_2+\cdots+r_n)/q[n]}A_n^{-r_n/2}\\
&= C_{A_i,B}[n-1]^{p_n/2}\\
&\quad\times\left(C_{A_i,B}[n-1]^{p_n/2}A_n^{r_n}\right.\\
&\qquad\left.\times C_{A_i,B}[n-1]^{p_n/2}\right)^{(\delta+r_1+r_2+\cdots+r_n-q[n])/q[n]}C_{A_i,B}[n-1]^{p_n/2}\\
&= C_{A_i,B}[n-1]^{p_n/2}\\
&\quad\times\left\{\left(C_{A_i,B}[n-1]^{p_n/2}A_n^{r_n}\right.\right.\\
&\qquad\left.\left.\times C_{A_i,B}[n-1]^{p_n/2}\right)^{(q[n]+\mu)/q[n]}\right\}^{(\delta+r_1+r_2+\cdots+r_n-q[n])/(q[n]+\mu)}\\
&\quad\times C_{A_i,B}[n-1]^{p_n/2}\\
&= C_{A_i,B}[n-1]^{p_n/2}\\
&\quad\times\left\{C_{A_i,B}[n-1]^{p_n/2}A_n^{r_n/2}\right.\\
&\qquad\times\left(A_n^{r_n/2}C_{A_i,B}[n-1]^{p_n}A_n^{r_n/2}\right)^{\mu/q[n]}A_n^{r_n/2}\\
&\qquad\left.\times C_{A_i,B}[n-1]^{p_n/2}\right\}^{(\delta+r_1+r_2+\cdots+r_n-q[n])/(q[n]+\mu)}\\
&\quad\times C_{A_i,B}[n-1]^{p_n/2}
\end{aligned}$$

$$\begin{aligned}
&\geq C_{A_i,B}[n-1]^{p_n/2} \\
&\quad \times \left\{C_{A_i,B}[n-1]^{p_n/2} A_n^{r_n+\mu} \right. \\
&\quad\quad \left. \times\, C_{A_i,B}[n-1]^{p_n/2}\right\}^{(\delta+r_1+r_2+\cdots+r_n-q[n])/(q[n]+\mu)} \\
&\quad \times C_{A_i,B}[n-1]^{p_n/2} \\
&= I_n\left(p_n, r_n+\mu\right),
\end{aligned} \tag{43}$$

and the last inequality holds by LH because (41) and

$$\begin{aligned}
&\frac{\delta+r_1+r_2+\cdots+r_n-q\,[n]}{q\,[n]+\mu} \\
&\quad = -\frac{q\,[n]-\left(\delta+r_1+r_2+\cdots+r_n\right)}{q\,[n]+\mu} \in [-1,0]\,,
\end{aligned} \tag{44}$$

so that $I_k(p_k, r_k)$ is a decreasing function of r_n. □

Acknowledgments

This work was supported by the National Natural Science Foundation of China (1127112; 11201127), Technology and Pioneering project in Henan Province (122300410110).

References

[1] E. Heinz, "Beiträge zur Störungstheorie der Spektralzerlegung," *Mathematische Annalen*, vol. 123, pp. 415–438, 1951.

[2] K. Löwner, "Über monotone Matrixfunktionen," *Mathematische Zeitschrift*, vol. 38, no. 1, pp. 177–216, 1934.

[3] G. K. Pedersen, "Some operator monotone functions," *Proceedings of the American Mathematical Society*, vol. 36, pp. 309–310, 1972.

[4] T. Furuta, "$A \geq B$ assures $(B^r A^p B^r)^{1/q} \geq B^{p+r/q}$ for $r \geq 0, p \geq 0, q \geq 1$ with $(1+2r)q \geq p+2r$," *Proceedings of the American Mathematical Society*, vol. 101, no. 1, pp. 85–88, 1987.

[5] T. Furuta, "An elementary proof of an order preserving inequality," *Proceedings of the Japan Academy*, vol. 65, no. 5, p. 126, 1989.

[6] M. Fujii, "Furuta's inequality and its mean theoretic approach," *Journal of Operator Theory*, vol. 23, no. 1, pp. 67–72, 1990.

[7] E. Kamei, "A satellite to Furuta's inequality," *Mathematica Japonica*, vol. 33, no. 6, pp. 883–886, 1988.

[8] K. Tanahashi, "Best possibility of the Furuta inequality," *Proceedings of the American Mathematical Society*, vol. 124, no. 1, pp. 141–146, 1996.

[9] T. Furuta, "Applications of order preserving operator inequality," *Operator Theory*, vol. 59, pp. 180–190, 1992.

[10] M. Uchiyama, "Some exponential operator inequalities," *Mathematical Inequalities and Applications*, vol. 2, no. 3, pp. 469–471, 1999.

[11] T. Furuta, "Extension of the Furuta inequality and Ando-Hiai log-majorization," *Linear Algebra and Its Applications*, vol. 219, pp. 139–155, 1996.

[12] M. Fujii, T. Furuta, and E. Kamei, "Furuta's inequality and its application to Ando's theorem," *Linear Algebra and Its Applications*, vol. 179, pp. 161–169, 1993.

[13] T. Furuta, "Operator functions on chaotic order involving order preserving operator inequalities," *Journal of Mathematical Inequalities*, vol. 6, no. 1, pp. 15–31, 2012.