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*Research Article*

# △-Convergence Problems for Asymptotically Nonexpansive Mappings in CAT(0) Spaces

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New △-convergence theorems of iterative sequences for asymptotically nonexpansive mappings in CAT(0) spaces are obtained. Consider an asymptotically nonexpansive self-mapping $T$ of a closed convex subset $C$ of a CAT(0) space $X$. Consider the iteration process $\{x_n\}$, where $x_0 \in C$ is arbitrary and $x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n)T^n y_n$ or $x_{n+1} = \alpha_n T^n x_n \oplus (1-\alpha_n) y_n$, $y_n = \beta_n x_n \oplus (1-\beta_n)T^n x_n$ for $n \geq 1$, where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$. It is shown that under certain appropriate conditions on $\alpha_n, \beta_n$, $\{x_n\}$ △-converges to a fixed point of $T$.

## 1. Introduction and Preliminaries

Let $C$ be a nonempty subset of a metric space $(X, d)$. A mapping $T : C \rightarrow C$ is a contraction if there exists $k \in [0,1)$ such that for all $x, y \in C$, we have $d(Tx, Ty) < kd(x, y)$. It is said to be nonexpansive if for all $x, y \in C$, we have $d(Tx, Ty) \leq d(x, y)$. $T$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \in [1, \infty)$ with $k_n \rightarrow 1$ such that $d(T^n x, T^n y) \leq k_n d(x, y)$ for all integers $n \geq 1$ and all $x, y \in C$. Clearly, every contraction mapping is nonexpansive and every nonexpansive mapping is asymptotically nonexpansive with sequence $k_n = 1$, for all $n \geq 1$. There are, however, asymptotically nonexpansive mappings which are not nonexpansive (see, e.g., [1]). As a generalization of the class of nonexpansive mappings, the class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [2] in 1972 and has been studied by several authors (see, e.g., [3–5]). Goebel and Kirk proved that if $C$ is a nonempty closed convex and bounded subset of a uniformly convex Banach space (more general than a Hilbert space, i.e., CAT(0) space), then every asymptotically nonexpansive self-mapping of $C$ has a fixed point. The weak and strong convergence problems to fixed points of nonexpansive and asymptotically nonexpansive mappings have been studied by many authors.

We will denote by $F(T)$ the set of fixed points of $T$. In 1967, Halpern [6] introduced an explicit iterative scheme for a nonexpansive mapping $T$ on a subset $C$ of a Hilbert space by taking any point $u, x_1 \in C$ and defined the iterative sequence $\{x_n\}$ by

$$x_{n+1} = \alpha_n u + (1-\alpha_n) T x_n, \quad \text{for } n \geq 1, \tag{1}$$

where $\alpha_n \in [0,1]$. He pointed out that under certain appropriate conditions on $\alpha_n$, $\{x_n\}$ converges strongly to a fixed point of $T$. In 1994, Tan and Xu [7] introduced the following iterative scheme for asymptotically nonexpansive mapping on uniformly convex Banach space:

$$\begin{gathered} x_0 \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T^n y_n, \quad n \geq 0, \\ y_n = \gamma_n x_n + (1-\gamma_n) T^n x_n, \quad n \geq 0, \end{gathered} \tag{2}$$

where $\{\alpha_n\}, \{\gamma_n\} \subseteq (0,1)$. They proved that under certain appropriate conditions on $\alpha_n, \gamma_n$, $\{x_n\}$ converges weakly to a fixed point of $T$.

In 2012, we [8] studied the viscosity approximation methods for nonexpansive mappings on CAT(0) space. For a contraction $f$ on $C$, consider the iteration process $\{x_n\}$, where $x_0 \in C$ is arbitrary and

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n) T x_n, \tag{3}$$

for $n \geq 1$, where $\{\alpha_n\} \subset (0,1)$. We proved that under certain appropriate conditions on $\alpha_n$, $\{x_n\}$ converges strongly to a fixed point of $T$ which solves some variational inequality.

The purpose of this paper is to study the iterative scheme defined as follows: consider an asymptotically nonexpansive self-mapping $T$ of a closed convex subset $C$ of a CAT(0) space $X$ with coefficient $k_n$. consider the iteration process $\{x_n\}$, where $x_0 \in C$ is arbitrary and

$$\begin{aligned} x_{n+1} &= \alpha_n x_n \oplus (1-\alpha_n) T^n y_n, \\ y_n &= \beta_n x_n \oplus (1-\beta_n) T^n x_n, \end{aligned} \tag{4}$$

or

$$\begin{aligned} x_{n+1} &= \alpha_n T^n x_n \oplus (1-\alpha_n) y_n, \\ y_n &= \beta_n x_n \oplus (1-\beta_n) T^n x_n, \end{aligned} \tag{5}$$

for $n \geq 1$, where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$. We show that $\{x_n\}$ $\triangle$-converges to a fixed point of $T$ under certain appropriate conditions on $\alpha_n$, $\beta_n$, and $k_n$.

We now collect some elementary facts about CAT(0) spaces which will be used in the proofs of our main results.

**Lemma 1.** *Let $X$ be a CAT(0) space. Then, one has the following:*

(i) *(see [9, Lemma 2.4]) for each $x, y, z \in X$ and $t \in [0,1]$, one has*

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x,z) + td(y,z), \tag{6}$$

(ii) *(see [10]) for each $x, y, z \in X$ and $t, s \in [0,1]$ one has*

$$d((1-t)x \oplus ty, (1-s)x \oplus sy) \leq |t-s|\, d(x,y), \tag{7}$$

(iii) *(see [5, Lemma 3]) for each $x, y, z \in X$ and $t \in [0,1]$, one has*

$$d((1-t)z \oplus tx, (1-t)z \oplus ty) \leq td(x,y), \tag{8}$$

(iv) *(see [9]) for each $x, y, z \in X$ and $t \in [0,1]$, one has*

$$\begin{aligned} &d^2((1-t)x \oplus ty, z) \\ &\quad \leq td^2(x,z) + (1-t)d^2(y,z) - t(1-t)d^2(x,y). \end{aligned} \tag{9}$$

Let $X$ be a complete CAT(0) space and let $\{x_n\}$ be a bounded sequence in a complete $X$ and for $x \in X$ set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n). \tag{10}$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}, \tag{11}$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}. \tag{12}$$

It is known (see, e.g., [11, Proposition 7]) that in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point.

A sequence $\{x_n\}$ in $X$ is said to *$\triangle$-converge* to $x \in X$ if $x$ is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\triangle\text{-}\lim_n x_n = x$ and call $x$ the $\triangle$-limit of $\{x_n\}$.

**Lemma 2.** *Assume that $X$ is a CAT(0) space. Then, one has the following:*

(i) *(see [12]) every bounded sequence in $X$ has a $\triangle$-convergent subsequence;*

(ii) *(see [13]) if $K$ is a closed convex subset of $X$ and $T : K \to X$ is an asymptotically nonexpansive mapping, then the conditions $\{x_n\}$ $\triangle$-converge to $x$ and $d(x_n, T(x_n)) \to 0$, imply $x \in K$ and $x \in F(T)$.*

**Lemma 3** (see [14, 15]). *Let $\{a_n\}, \{b_n\}$, and $\{c_n\}$ be three nonnegative real sequences satisfying the following condition:*

$$a_{n+1} \leq (1+b_n)a_n + c_n, \quad \forall n \geq n_0, \tag{13}$$

*where $n_0$ is some nonnegative integer, $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} c_n < \infty$. Then the limit $\lim_{n\to\infty} a_n$ exists.*

## 2. $\triangle$-Convergence of the Iteration Sequences

In this section, we will study the $\triangle$-convergence of the iteration sequence for asymptotically nonexpansive mappings in CAT(0) spaces.

Suppose that $X$ be a CAT(0) space, $C$ a closed convex subset of $X$, and $T : C \to C$ an asymptotically nonexpansive mapping with coefficient $k_n$. Firstly, we consider the iteration process:

$$\begin{aligned} &x_0 \in C, \\ &x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n) T^n y_n, \quad n \geq 0, \\ &y_n = \beta_n x_n \oplus (1-\beta_n) T^n x_n, \quad n \geq 0, \end{aligned} \tag{14}$$

where $\{\alpha_n\}, \{\beta_n\} \subseteq (0,1)$ and $k_n$ satisfy the following.

(i) There exist positive integers $n_0, n_1$, and $\delta > 0$, $0 < b < \min\{1, 1/L\}$, where $L = \sup_n k_n$, such that

$$\begin{aligned} &0 < \delta < \alpha_n < 1-\delta, \quad n \geq n_0, \\ &0 < 1-\beta_n < b, \quad n \geq n_1, \end{aligned} \tag{15}$$

(ii) Consider $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$.

We will prove that $\{x_n\}$ $\triangle$-converges to a fixed point of $T$.

**Lemma 4.** *Let $X$ be a CAT(0) space, $C$ a closed convex subset of $X$, $T : C \to C$ an asymptotically nonexpansive mapping with coefficient $k_n$, and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. If $F(T) \neq \emptyset$, $\{\alpha_n\}, \{\beta_n\} \subseteq (0,1)$. Let $x_0 \in C$, $\{x_n\}$ be generated by $x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n)T^n y_n$, $y_n = \beta_n x_n \oplus (1-\beta_n)T^n x_n$, $n \geq 0$. Then the limit $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$.*

*Proof.* Taking $p \in F(T)$, we have

$$\begin{aligned} d(x_{n+1}, p) &= d(\alpha_n x_n \oplus (1-\alpha_n) T^n y_n, p) \\ &\le \alpha_n d(x_n, p) + (1-\alpha_n) d(T^n y_n, p) \\ &\le \alpha_n d(x_n, p) + (1-\alpha_n) k_n d(y_n, p) \\ &\le \alpha_n d(x_n, p) \\ &\quad + (1-\alpha_n) k_n \{\beta_n d(x_n, p) \\ &\qquad + (1-\beta_n) d(T^n x_n, p)\} \\ &\le \alpha_n d(x_n, p) \\ &\quad + (1-\alpha_n) k_n \{\beta_n d(x_n, p) \\ &\qquad + (1-\beta_n) k_n d(x_n, p)\} \\ &= \{1 + (1-\alpha_n)(k_n - 1) \\ &\quad \times [k_n(1-\beta_n) + 1]\} d(x_n, p) \\ &\le \left\{1 + \left(k_n^2 - 1\right)\right\} d(x_n, p). \end{aligned} \tag{16}$$

By Lemma 3, we can get that $\lim_{n\to\infty} d(x_n, p)$ exists. □

*Remark 5.* The above lemma implies that $\{x_n\}$ is bounded and so is the sequence $\{Tx_n\}$. Moreover, let $L = \sup_n k_n$, then we have

$$\begin{aligned} d(T^n x_n, p) &\le k_n d(x_n, p) \le L d(x_n, p), \\ d(y_n, p) &\le \beta_n d(x_n, p) + (1-\beta_n) d(T^n x_n, p) \\ &\le L d(x_n, p) \\ d(T^n y_n, p) &\le k_n d(y_n, p) \le L^2 d(x_n, p). \end{aligned} \tag{17}$$

It follows that the sequences $\{T^n x_n\}$, $\{y_n\}$, $\{T^n y_n\}$ are bounded.

**Proposition 6.** *Let $X$ be a CAT(0) space, $C$ a closed convex subset of $X$, and $T : C \to C$ an asymptotically nonexpansive mapping with coefficient $k_n$. If $F(T) \neq \emptyset$, $\{\alpha_n\}, \{\beta_n\} \subseteq (0, 1)$. Let $x_0 \in C$, $\{x_n\}$ be generated by $x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n) T^n y_n$, $y_n = \beta_n x_n \oplus (1-\beta_n) T^n x_n$, $n \ge 0$. Then under the hypotheses (i) and (ii), one can get that $\lim_{n\to\infty} d(x_n, T^n y_n) = 0$.*

*Proof.* By the assumption, $F(T)$ is nonempty. Take $p \in F(T)$, by Lemma 1(iv), we have

$$\begin{aligned} d^2(x_{n+1}, p) &= d^2(\alpha_n x_n \oplus (1-\alpha_n) T^n y_n, p) \\ &\le \alpha_n d^2(x_n, p) + (1-\alpha_n) d^2(T^n y_n, p) \\ &\quad - \alpha_n (1-\alpha_n) d^2(x_n, T^n y_n) \\ &\le d^2(x_n, p) + (1-\alpha_n) \left\{d^2(T^n y_n, p) \right. \\ &\qquad \left. - d^2(y_n, p)\right\} \\ &\quad + (1-\alpha_n) \left\{d^2(y_n, p) - d^2(x_n, p)\right\} \\ &\quad - \alpha_n (1-\alpha_n) d^2(x_n, T^n y_n), \\ d^2(y_n, p) - d^2(x_n, p) & \\ &= d^2(\beta_n x_n \oplus (1-\beta_n) T^n x_n, p) - d^2(x_n, p) \\ &\le \beta_n d^2(x_n, p) + (1-\beta_n) d^2(T^n x_n, p) \\ &\quad - \beta_n (1-\beta_n) d^2(x_n, T^n x_n) - d^2(x_n, p) \\ &\le \beta_n d^2(x_n, p) + (1-\beta_n) d^2(T^n x_n, p) \\ &\quad - d^2(x_n, p), \end{aligned} \tag{18}$$

which implies that

$$\begin{aligned} d^2(y_n, p) - d^2(x_n, p) &\le (1-\beta_n) \left[d^2(T^n x_n, p) - d^2(x_n, p)\right] \\ &\le (1-\beta_n) \left(k_n^2 - 1\right) d^2(x_n, p). \end{aligned} \tag{19}$$

Therefore, we have

$$\begin{aligned} d^2(x_{n+1}, p) &\le d^2(x_n, p) + (1-\alpha_n) \left(k_n^2 - 1\right) d^2(y_n, p) \\ &\quad + (1-\alpha_n)(1-\beta_n) \left(k_n^2 - 1\right) d^2(x_n, p) \\ &\quad - \alpha_n (1-\alpha_n) d^2(x_n, T^n y_n). \end{aligned} \tag{20}$$

Since $\{x_n\}$ and $\{y_n\}$ are bounded and $0 < \delta < \alpha_n < 1 - \delta$ for all $n \ge n_0$. we have

$$\begin{aligned} \delta^2 d^2(x_n, T^n y_n) &\le d^2(x_n, p) - d^2(x_{n+1}, p) \\ &\quad + (1-\alpha_n) \left(k_n^2 - 1\right) d^2(y_n, p) \\ &\quad + (1-\alpha_n)(1-\beta_n) \left(k_n^2 - 1\right) d^2(x_n, p). \end{aligned} \tag{21}$$

By the conditions (i) and (ii), we have

$$\sum_{n=1}^{\infty} \delta^2 d^2(x_n, T^n y_n) < \infty, \tag{22}$$

which implies that

$$\lim_{n\to\infty} d^2(x_n, T^n y_n) = 0. \tag{23}$$

□

**Theorem 7.** *Let $X$ be a CAT(0) space, $C$ a closed convex subset of $X$, and $T : C \to C$ an asymptotically nonexpansive mapping with coefficient $k_n$. If $F(T) \neq \emptyset$, $\{\alpha_n\}, \{\beta_n\} \subseteq (0, 1)$. Let $x_0 \in C$, $\{x_n\}$ be generated by $x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n) T^n y_n$, $y_n = \beta_n x_n \oplus (1-\beta_n) T^n x_n$, $n \ge 0$. Then under the hypotheses (i) and (ii), one can get that $\{x_n\}$ $\triangle$-converges to a fix point of $T$.*

*Proof.* We first show that $\lim_{n\to\infty} d(x_n, T^n x_n) = 0$. Indeed

$$\begin{aligned} d(x_n, y_n) &= d(x_n, \beta_n x_n \oplus (1-\beta_n) T^n x_n) \\ &\le (1-\beta_n) d(x_n, T^n x_n) \\ &\le (1-\beta_n)\{d(x_n, T^n y_n) + d(T^n y_n, T^n x_n)\} \\ &\le (1-\beta_n)\{d(x_n, T^n y_n) + L d(y_n, x_n)\}; \end{aligned} \tag{24}$$

it follows that

$$[1 - L(1-\beta_n)] d(x_n, y_n) \le (1-\beta_n) d(x_n, T^n y_n). \tag{25}$$

By the conditions (i) and (ii) and Proposition 6, we get $\lim_{n\to\infty} d(x_n, y_n) = 0$.

And then,

$$\begin{aligned} d(x_n, T^n x_n) &\le d(x_n, T^n y_n) + d(T^n y_n, T^n x_n) \\ &\le d(x_n, T^n y_n) + L d(y_n, x_n). \end{aligned} \tag{26}$$

By Proposition 6, we get that $\lim_{n\to\infty} d(x_n, T^n x_n) = 0$.

We claim that $\lim_{n\to\infty} d(x_n, T x_n) = 0$. Indeed we have

$$\begin{aligned} d(y_n, T^n x_n) &= d(\beta_n x_n \oplus (1-\beta_n) T^n x_n, T^n x_n) \\ &\le \beta_n d(x_n, T^n x_n) \longrightarrow 0. \\ d(x_{n+1}, x_n) &= d(\alpha_n x_n \oplus (1-\alpha_n) T^n y_n, x_n) \\ &\le (1-\alpha_n) d(x_n, T^n y_n) \longrightarrow 0. \\ d\left(x_{n-1}, T^{n-1} x_n\right) &\le d\left(x_{n-1}, T^{n-1} x_{n-1}\right) \\ &\quad + d\left(T^{n-1} x_{n-1}, T^{n-1} x_n\right) \\ &\le d\left(x_{n-1}, T^{n-1} x_{n-1}\right) + L d(x_{n-1}, x_n) \longrightarrow 0. \\ d\left(x_n, T^{n-1} x_n\right) &\le d\left(\alpha_{n-1} x_{n-1} \right.\\ &\qquad \left. \oplus (1-\alpha_{n-1}) T^{n-1} y_{n-1}, T^{n-1} x_n\right) \\ &\le \alpha_{n-1} d\left(x_{n-1}, T^{n-1} x_n\right) \\ &\quad + (1-\alpha_{n-1}) d\left(T^{n-1} y_{n-1}, T^{n-1} x_n\right) \\ &\le \alpha_{n-1} d\left(x_{n-1}, T^{n-1} x_n\right) \\ &\quad + (1-\alpha_{n-1}) L d(y_{n-1}, x_n) \\ &\le \alpha_{n-1} d\left(x_{n-1}, T^{n-1} x_n\right) \\ &\quad + (1-\alpha_{n-1}) L [d(y_{n-1}, x_{n-1}) \\ &\qquad + d(x_{n-1}, x_n)] \longrightarrow 0. \end{aligned} \tag{27}$$

Thus,

$$\begin{aligned} d(x_n, T x_n) &\le d(x_n, T^n x_n) + d(T^n x_n, T x_n) \\ &\le d(x_n, T^n x_n) + L d\left(T^{n-1} x_n, x_n\right) \longrightarrow 0. \end{aligned} \tag{28}$$

Since $\{x_n\}$ is bounded, we may assume that $\{x_n\}$ $\triangle$-converges to a point $\hat{x}$. By Lemma 2, we have $\hat{x} \in F(T)$. □

Next we will consider another iteration process:

$$\begin{aligned} &x_0 \in C, \\ &x_{n+1} = \alpha_n T^n x_n \oplus (1-\alpha_n) y_n, \quad n \ge 0, \\ &y_n = \beta_n x_n \oplus (1-\beta_n) T^n x_n, \quad n \ge 0, \end{aligned} \tag{29}$$

where $\{\alpha_n\}, \{\beta_n\} \subseteq (0,1)$, and $k_n$ satisfy the following

(H1) There exist positive integers $n_0$ and $\delta > 0$, such that

$$\begin{aligned} &0 < \delta < \alpha_n < 1 - \delta, \quad n \ge n_0; \\ &1 - \beta_n \longrightarrow 0; \end{aligned} \tag{30}$$

(H2) $\sum_{n=1}^{\infty} (k_n - 1) < \infty$.

We will prove that $\{x_n\}$ also $\triangle$-converges to a fixed point of $T$.

**Lemma 8.** *Let $X$ be a CAT(0) space, $C$ a closed convex subset of $X$, $T : C \to C$ an asymptotically nonexpansive mapping with coefficient $k_n$, and $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. If $F(T) \neq \emptyset$, $\{\alpha_n\}, \{\beta_n\} \subseteq (0,1)$. Let $x_0 \in C$, $\{x_n\}$ be generated by $x_{n+1} = \alpha_n T^n x_n \oplus (1-\alpha_n) y_n$, $y_n = \beta_n x_n \oplus (1-\beta_n) T^n x_n$, $n \ge 0$. Then the limit $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$.*

*Proof.* Taking $p \in F(T)$, we have

$$\begin{aligned} d(x_{n+1}, p) &= d(\alpha_n T^n x_n \oplus (1-\alpha_n) y_n, p) \\ &\le \alpha_n k_n d(x_n, p) + (1-\alpha_n) d(y_n, p) \\ &\le \alpha_n k_n d(x_n, p) \\ &\quad + (1-\alpha_n)\{\beta_n d(x_n, p) + (1-\beta_n) d(T^n x_n, p)\} \\ &\le \alpha_n k_n d(x_n, p) \\ &\quad + (1-\alpha_n)\{\beta_n d(x_n, p) + (1-\beta_n) k_n d(x_n, p)\} \\ &= \{1 + (k_n - 1)[1 - (1-\alpha_n)\beta_n]\} d(x_n, p). \end{aligned} \tag{31}$$

By Lemma 3, we can get that $\lim_{n\to\infty} d(x_n, p)$ exists. □

Next, we will prove $\lim_{n\to\infty} d(T^n x_n, y_n) = 0$.

**Proposition 9.** *Let $X$ be a CAT(0) space, $C$ a closed convex subset of $X$, and $T : C \to C$ an asymptotically nonexpansive mapping with coefficient $k_n$. If $F(T) \neq \emptyset$, $\{\alpha_n\}, \{\beta_n\} \subseteq (0,1)$. Let $x_0 \in C$, $\{x_n\}$ be generated by $x_{n+1} = \alpha_n T^n x_n \oplus (1-\alpha_n) y_n$, $y_n = \beta_n x_n \oplus (1-\beta_n) T^n x_n$, $n \ge 0$. Then under the hypotheses (H1) and (H2), one can get that $\lim_{n\to\infty} d(T^n x_n, y_n) = 0$.*

*Proof.* By the assumption, $F(T)$ is nonempty. Take $p \in F(T)$, let $L = \sup_n k_n$, then we have

$$\begin{aligned} d(T^n x_n, p) &\leq k_n d(x_n, p) \leq L d(x_n, p), \\ d(y_n, p) &\leq \beta_n d(x_n, p) + (1-\beta_n) d(T^n x_n, p) \\ &\leq L d(x_n, p) \\ d(T^n y_n, p) &\leq k_n d(y_n, p) \leq L^2 d(x_n, p). \end{aligned} \tag{32}$$

It follows that the sequences $\{x_n\}, \{T^n x_n\}, \{y_n\}, \{T^n y_n\}$ are bounded.

By Lemma 1, we have

$$\begin{aligned} d^2(x_{n+1}, p) &= d^2(\alpha_n T^n x_n \oplus (1-\alpha_n) y_n, p) \\ &\leq \alpha_n k_n^2 d^2(x_n, p) + (1-\alpha_n) d^2(y_n, p) \\ &\quad - \alpha_n (1-\alpha_n) d^2(T^n x_n, y_n) \\ &\leq d^2(x_n, p) + (1-\alpha_n) \{d^2(y_n, p) - d^2(x_n, p)\} \\ &\quad + \alpha_n (k_n^2 - 1) d^2(x_n, p) \\ &\quad - \alpha_n (1-\alpha_n) d^2(T^n x_n, y_n). \end{aligned} \tag{33}$$

Similar to the proof of Proposition 6, we can get

$$d^2(y_n, p) - d^2(x_n, p) \leq (1-\beta_n)(k_n^2 - 1) d^2(x_n, p). \tag{34}$$

Therefore, we have

$$\begin{aligned} d^2(x_{n+1}, p) \leq d^2(x_n, p) &+ (1-\alpha_n)(1-\beta_n) \\ &\times (k_n^2 - 1) d^2(x_n, p) \\ &+ \alpha_n (k_n^2 - 1) d^2(x_n, p) \\ &- \alpha_n (1-\alpha_n) d^2(T^n x_n, y_n). \end{aligned} \tag{35}$$

Since $\{x_n\}, \{y_n\}$ are bounded and $0 < \delta < \alpha_n < 1 - \delta$ for all $n \geq n_0$. we have

$$\begin{aligned} \delta^2 d^2(T^n x_n, y_n) \leq d^2(x_n, p) &- d^2(x_{n+1}, p) \\ &+ (1-\alpha_n)(1-\beta_n)(k_n^2 - 1) d^2(x_n, p) \\ &+ \alpha_n (k_n^2 - 1) d^2(x_n, p). \end{aligned} \tag{36}$$

By the conditions (H1) and (H2), we have $\sum_{n=1}^{\infty} (k_n^2 - 1) < \infty$ and

$$\sum_{n=1}^{\infty} \delta^2 d^2(T^n x_n, y_n) < \infty, \tag{37}$$

which implies that

$$\lim_{n \to \infty} d^2(T^n x_n, y_n) = 0. \tag{38}$$

□

**Theorem 10.** *Let $X$ be a CAT(0) space, $C$ a closed convex subset of $X$, and $T : C \to C$ an asymptotically nonexpansive mapping with coefficient $k_n$. If $F(T) \neq \emptyset$, $\{\alpha_n\}, \{\beta_n\} \subseteq (0,1)$. Let $x_0 \in C$, $\{x_n\}$ be generated by $x_{n+1} = \alpha_n T^n x_n \oplus (1-\alpha_n) y_n$, $y_n = \beta_n x_n \oplus (1-\beta_n) T^n x_n$, $n \geq 0$. Then under the hypotheses (H1) and (H2), one can get that $\{x_n\}$ $\triangle$-converges to a fix point of $T$.*

*Proof.* We first show that $\lim_{n \to \infty} d(x_n, T^n x_n) = 0$. Indeed, by Lemma 1, and $\beta_n \to 1$, we can get

$$\begin{aligned} d(x_n, y_n) &= d(x_n, \beta_n x_n \oplus (1-\beta_n) T^n x_n) \\ &\leq (1-\beta_n) d(x_n, T^n x_n) \longrightarrow 0. \end{aligned} \tag{39}$$

And then,

$$d(x_n, T^n x_n) \leq d(x_n, y_n) + d(y_n, T^n x_n). \tag{40}$$

By Proposition 9, we obtain that $\lim_{n \to \infty} d(x_n, T^n x_n) = 0$.

We claim that $\lim_{n \to \infty} d(x_n, T x_n) = 0$. Indeed we have

$$\begin{aligned} d(x_{n+1}, x_n) &= d(\alpha_n T^n x_n \oplus (1-\alpha_n) y_n, x_n) \\ &\leq \alpha_n d(T^n x_n, x_n) + (1-\alpha_n) d(x_n, y_n) \longrightarrow 0. \\ d(x_n, T^{n-1} x_n) &\leq d(\alpha_{n-1} T^{n-1} x_{n-1} \oplus (1-\alpha_{n-1}) y_{n-1}, T^{n-1} x_n) \\ &\leq \alpha_{n-1} d(T^{n-1} x_{n-1}, T^{n-1} x_n) \\ &\quad + (1-\alpha_{n-1}) d(y_{n-1}, T^{n-1} x_n) \\ &\leq \alpha_{n-1} k_{n-1} d(x_{n-1}, x_n) \\ &\quad + (1-\alpha_{n-1}) [d(y_{n-1}, T^{n-1} x_{n-1}) \\ &\quad + d(T^{n-1} x_{n-1}, T^{n-1} x_n)] \\ &\leq \alpha_{n-1} k_{n-1} d(x_{n-1}, x_n) \\ &\quad + (1-\alpha_{n-1}) [d(y_{n-1}, T^{n-1} x_{n-1}) \\ &\quad + k_{n-1} d(x_{n-1}, x_n)] \longrightarrow 0. \end{aligned} \tag{41}$$

Thus,

$$\begin{aligned} d(x_n, T x_n) &\leq d(x_n, T^n x_n) + d(T^n x_n, T x_n) \\ &\leq d(x_n, T^n x_n) + L d(T^{n-1} x_n, x_n) \longrightarrow 0. \end{aligned} \tag{42}$$

Since $\{x_n\}$ is bounded, we may assume that $\{x_n\}$ $\triangle$-converges to a point $\hat{x}$. By Lemma 2, we have $\hat{x} \in F(T)$. □

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