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Research Article

On Fixed Point Theory of Monotone Mappings with Respect to a Partial Order Introduced by a Vector Functional in Cone Metric Spaces

Zhilong Li[1] **and Shujun Jiang**[2]

[1] *School of Statistics, Jiangxi University of Finance and Economics, Nanchang 330013, China*
[2] *Department of Mathematics, Jiangxi University of Finance and Economics, Nanchang 330013, China*

Correspondence should be addressed to Zhilong Li; lzl771218@sina.com

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We presented some maximal and minimal fixed point theorems of set-valued monotone mappings with respect to a partial order introduced by a vector functional in cone metric spaces. In addition, we proved not only the existence of maximal and minimal fixed points but also the existence of the largest and the least fixed points of single-valued increasing mappings. It is worth mentioning that the results on single-valued mappings in this paper are still new even in the case of metric spaces and hence they indeed improve the recent results.

1. Introductions

Throughout this paper, let (X, d) be a complete cone metric space over a total minihedral and continuous cone P of a normed vector space E. A vector functional $\varphi : X \to E$ introduces a partial order $\prec$ on X as follows:

$$x \prec y \Longleftrightarrow d(x, y) \preceq \varphi(x) - \varphi(y), \tag{1}$$

for all $x, y \in X$, where $\preceq$ is the partial order on E determined by the cone P. Using the partial order introduced by the vector functional φ, Agarwal and Khamsi [1] extended Caristi's fixed point theorem [2] to the case of cone metric space and proved that all mapping $T : X \to X$ (resp., $T : X \to 2^X$) such that

$$\forall x \in X, \quad x \prec Tx \ (\text{resp., } \forall x \in X,\ \exists y \in Tx,\ x \prec y) \tag{2}$$

has a fixed point provided that φ is lower semicontinuous and bounded below on X. In [1, 3], the authors studied Kirk's problem [4, 5] in the case of cone metric spaces and obtained some generalized Caristi's fixed point theorems in cone metric spaces. For the researches on the generalization of primitive Caristi's result in the case of metric spaces, we refer the readers to [6–12]. For other references concerned with various fixed point results for one, two, three, or four self-mappings in the setting of metric, ordered metric, partial metric, Prešić-type mappings, cone metric, G-metric spaces, and so forth, we refer the readers to [13–24].

In particular, when $E = \mathbb{R}$, the partial order defined by (1) is reduced to the one defined by Caristi [2] who denote it by $\prec_1$. Zhang [25, 26] and Li [27] considered the existence of fixed points of a mapping $T : X \to X$ (resp., $T : X \to 2^X$) such that

$$x_0 \prec_1 Tx_0 \quad (\text{resp., } \exists y \in Tx_0,\ x_0 \prec_1 y), \tag{3}$$

for some $x_0 \in X$, and proved some maximal and minimal fixed point theorems at the expense that T is monotone with respect to the partial order $\prec_1$.

In this paper, we shall extend the results of Zhang [25, 26] and Li [27] to the case of cone metric spaces. Some maximal and minimal fixed point theorems of set-valued monotone mappings with respect to the partial order $\prec$ are established in cone metric spaces. In addition, not only the existence of maximal and minimal fixed points but also the existence of largest and least fixed points is proved for single-valued increasing mappings. It is worth mentioning that the results on single-valued mappings in this paper are still new even in the case of metric spaces and hence they indeed improve the results of Zhang [25] and Li [27].

2. Preliminaries

First, we recall some definitions and properties of cones and cone metric spaces; these can be found in [1, 3, 17–24, 28–30].

Let E be a topological vector space. A cone P of E is a nonempty closed subset of E such that $ax + by \in P$ for all $x, y \in P$ and all $a, b \geq 0$, and $P \cap (-P) = \{\theta\}$, where θ is the zero element of E. A cone P of E determines a partial order $\preceq$ on E by $x \preceq y \Leftrightarrow y - x \in P$ for all $x, y \in X$. For all $x, y \in E$ with $y - x \in \operatorname{int} P$, we write $x \ll y$, where $\operatorname{int} P$ is the interior of P.

Let P be a cone of a topological vector space. P is total order minihedral [29] if, for all upper bounded nonempty total ordered subset A of E, $\sup A$ exists in E. Equivalently, P is total order minihedral if, for all lower bounded nonempty total ordered subset A of E, $\inf A$ exists in E.

Let E be a normed vector space. A cone P of E is continuous [1, 3] if, for all subset A of E, $\inf A$ exists implies $\inf_{x\in A}\|x - \inf A\| = 0$, and $\sup A$ exists implies $\sup_{x\in A}\|x - \sup A\| = 0$. A cone P of E is normal [30] if there exists $N > 0$ such that for all $x, y \in P$, $x \preceq y$ implies $\|x\| \leq N\|y\|$, and the minimal N is called a normal constant of P. Equivalently, A cone P of E is normal provided that for all $\{x_n\}, \{y_n\}, \{z_n\} \subseteq E$ with $x_n \preceq y_n \preceq z_n$ for all n, $x_n \to x$ and $z_n \to x$ imply $y_n \to x$ for some $x \in X$.

Remark 1. A total order minihedral cone P of a normed space E is certainly normal see [29].

Let X be a nonempty set and P a cone of a topological vector space E. A cone metric [28] is a mapping $d : X \times X \to P$ such that for all $x, y, x \in X$,

($d1$) $d(x, y) = \theta$ if and only if $x = y$,

($d2$) $d(x, y) = d(y, x)$,

($d3$) $d(x, y) \preceq d(x, z) + d(z, y)$.

A pair (X, d) is called a cone metric space over P if $d : X \times X \to P$ is a cone metric. Let (X, d) be a cone metric space over a cone P of a topological vector space E. A sequence $\{x_n\}$ in (X, d) converges [28] to $x \in X$ (denote $x_n \xrightarrow{d} x$) if, for all $\varepsilon \in P$ with $\theta \ll \varepsilon$, there exists a positive integer n_0 such that $d(x_n, x) \ll \varepsilon$ for all $n \geq n_0$. A sequence $\{x_n\}$ in (X, d) is Cauchy [28] if, for all $\varepsilon \in P$ with $\theta \ll \varepsilon$, there exists a positive integer n_0 such that $d(x_n, x_m) \ll \varepsilon$ for all $m, n \geq n_0$. A cone metric space (X, d) is complete [28] if all Cauchy sequence $\{x_n\}$ in (X, d) converges to a point $x \in X$. A vector functional $\varphi : X \to E$ is sequentially continuous at some $x \in X$ if $\lim_{n\to\infty}\varphi(x_n) = \varphi(x)$ for all $\{x_n\} \subseteq X$ such that $x_n \xrightarrow{d} x$. If, for all $x \in X$, φ is sequentially continuous at x, then $\varphi : X \to E$ is sequentially continuous.

Remark 2. Let (X, d) be a cone metric space over a normal cone P of a normed vector space E and $\{x_n\}$ a sequence in (X, d). Then $x_n \xrightarrow{d} x$ if and only if $\lim_{n\to\infty} d(x_n, x) = \theta$, and $\{x_n\}$ is Cauchy if and only if $\lim_{m,n\to\infty} d(x_n, x_m) = \theta$ see [28].

Let X be a nonempty set and $\prec$ a partial order on X. For all $x, y \in X$ with $x \prec y$, set $[x, +\infty) = \{z \in X : x \prec z\}$, $(-\infty, x] = \{z \in X : z \prec x\}$, and $[x, y] = \{z \in X : x \prec z \prec y\}$. Let A be a nonempty subset of X. A set-valued mapping $T : X \to 2^X$ is increasing on A if, for all $x, y \in A$ with $x \prec y$ and all $u \in Tx$, there exists $v \in Ty$ such that $u \prec v$. A set-valued mapping $T : X \to 2^X$ is quasi-increasing if, for all $x, y \in A$ with $x \prec y$ and all $v \in Ty$, there exists $u \in Tx$ such that $u \prec v$. In particular, a single-valued mapping $T : X \to X$ is increasing on A if, for all $x, y \in A$ with $x \prec y$, $Tx \prec Ty$.

A point $x^* \in X$ is called a fixed point of a set-valued (resp., single-valued) mapping T if $x^* \in Tx^*$ (resp. $x^* = Tx^*$). Let A be a nonempty subset of X and let $x^* \in A$ be a fixed point of a mapping T. x^* is called a maximal (resp. minimal) fixed point of T in A if for all fixed point $x \in A$ of T, $x^* \prec x$ (resp., $x \prec x^*$) implies $x^* = x$. $x^* \in A$ is called a largest (resp., least) fixed point of T in A if, for all fixed point $x \in A$ of T, $x \prec x^*$ (resp., $x^* \prec x$). A largest (resp., least) fixed point of T in A is naturally a maximal (resp., minimal) fixed point in A, but the converse may not be true.

3. Fixed Point Theorems

In this section, we always assume that the partial order $\prec$ is defined by (1).

Theorem 3. *Let $(X, d, \prec)$ be a complete partially ordered cone metric space over a total order minihedral and continuous cone P of a normed vector space E. Let $\varphi : X \to E$ be a sequentially continuous vector functional and let $T : X \to 2^X$ be a set-valued mapping such that Tx is compact for all $x \in X$. Assume that there exists $x_0 \in X$ such that φ is bounded below on $[x_0, +\infty)$, T is increasing on $[x_0, +\infty)$, and $Tx_0 \cap [x_0, +\infty) \neq \emptyset$. Then T has a maximal fixed point $x^* \in [x_0, +\infty)$.*

Proof. Since P is a total order minihedral cone and E is a normed space, then P is a normal cone by Remark 1. Set

$$Q_1 = \{x \in [x_0, +\infty) : Tx \cap [x, +\infty) \neq \emptyset\}. \tag{4}$$

Clearly, Q_1 is nonempty since $x_0 \in Q_1$. Let $\{x_\alpha\}_{\alpha\in\Gamma} \subseteq Q_1$ be an increasing chain, where Γ is a directed set. Then by (1) we have

$$d(x_\alpha, x_\beta) \preceq \varphi(x_\alpha) - \varphi(x_\beta), \tag{5}$$

for all $\alpha, \beta \in \Gamma$ with $\alpha \preceq \beta$. This implies that $\{\varphi(x_\alpha)\}$ is a decreasing chain in E. Since P is total order minihedral and φ is bounded below on $[x_0, +\infty)$, then $\inf_{\alpha\in\Gamma}\varphi(x_\alpha)$ exists in E. Moreover, $\inf_{\alpha\in\Gamma}\|\varphi(x_\alpha) - \inf_{\alpha\in\Gamma}\varphi(x_\alpha)\| = 0$ since P is continuous. Therefore there exists an increasing sequence $\{x_{\alpha_n}\} \subseteq \{x_\alpha\}$ such that $\lim_{n\to\infty}\|\varphi(x_{\alpha_n}) - \inf_{\alpha\in\Gamma}\varphi(x_\alpha)\| = 0$, that is,

$$\lim_{n\to\infty}\varphi(x_{\alpha_n}) = \inf_{\alpha\in\Gamma}\varphi(x_\alpha). \tag{6}$$

By (1) we have for all $m \in \mathbb{N}$ such that $m \geq n$,

$$d\left(x_{\alpha_n}, x_{\alpha_m}\right) \preceq \varphi\left(x_{\alpha_n}\right) - \varphi\left(x_{\alpha_m}\right). \tag{7}$$

Let $n \to \infty$, by (6) we have $\lim_{m,n\to\infty}[\varphi(x_{\alpha_n}) - \varphi(x_{\alpha_m})] = \theta$ and hence $\lim_{m,n\to\infty} d(x_{\alpha_n}, x_{\alpha_m}) = \theta$ by the normality of P. Moreover by Remark 2, $\{x_{\alpha_n}\}$ is a Cauchy sequence in X. Therefore by the completeness of X, there exists some $\overline{x} \in X$ such that

$$x_{\alpha_n} \xrightarrow{d} \overline{x}. \tag{8}$$

Note that $\{x_{\alpha_n}\}$ is an increasing sequence of Q_1, then by (1), we have for all n,

$$d\left(x_0, x_{\alpha_n}\right) \preceq \varphi\left(x_0\right) - \varphi\left(x_{\alpha_n}\right), \tag{9}$$

And, for all $n \geq n_0$,

$$d\left(x_{\alpha_{n_0}}, x_{\alpha_n}\right) \preceq \varphi\left(x_{\alpha_{n_0}}\right) - \varphi\left(x_{\alpha_n}\right), \tag{10}$$

where n_0 is an arbitrary integer. Let $n \to \infty$, then by (8) and the continuity of φ we have $d(x_0, \overline{x}) \preceq \varphi(x_0) - \varphi(\overline{x})$ and $d(x_{\alpha_{n_0}}, \overline{x}) \preceq \varphi(x_{\alpha_{n_0}}) - \varphi(\overline{x})$, that is, $\overline{x} \in [x_0, +\infty)$ and $x_{\alpha_{n_0}} \prec \overline{x}$. Moreover the arbitrary property of n_0 forces that

$$x_{\alpha_n} \prec \overline{x}, \tag{11}$$

for all n. By $x_{\alpha_n} \in Q_1$, there exists $y_n \in Tx_{\alpha_n}$ such that

$$x_{\alpha_n} \prec y_n, \tag{12}$$

for all n. Since T is increasing on $[x_0, +\infty)$, then by (11) and $\overline{x} \in [x_0, +\infty)$, there exists $z_n \in T\overline{x}$ such that

$$y_n \prec z_n, \tag{13}$$

for all n. This together with (12) implies that

$$x_{\alpha_n} \prec z_n, \tag{14}$$

for all n. Note that $T\overline{x}$ is compact, and there exists a subsequence $\{z_{n_k}\} \subseteq \{z_n\}$ and $z \in T\overline{x}$ such that

$$z_{n_k} \longrightarrow z. \tag{15}$$

From (14) we have $x_{\alpha_{n_k}} \prec z_{n_k}$ for all n_k and hence by (1),

$$d\left(x_{\alpha_{n_k}}, z_{n_k}\right) \preceq \varphi\left(x_{\alpha_{n_k}}\right) - \varphi\left(z_{n_k}\right), \tag{16}$$

for all n_k. Let $n_k \to \infty$, then by (8), (15), and the continuity of φ we have $d(\overline{x}, z) \preceq \varphi(\overline{x}) - \varphi(z)$, that is, $\overline{x} \prec z$. This implies that $T\overline{x} \cap [\overline{x}, +\infty) \neq \emptyset$ and hence $\overline{x} \in Q_1$ by $\overline{x} \in [x_0, +\infty)$.

For all $\alpha \in \Gamma$, if there exists some n_0 such that $x_\alpha \prec x_{\alpha_{n_0}}$, then $\overline{x}$ is an upper bound of $\{x_\alpha\}$ by (11). Otherwise, there exists some $\beta \in \Gamma$ such that $x_{\alpha_n} \prec x_\beta$ for all n. Thus by (1) we have $\varphi(x_{\alpha_n}) - \varphi(x_\beta) \in P$ for all n. Let $n \to \infty$, by (6) we have $\inf_{\alpha\in\Gamma}\varphi(x_\alpha) - \varphi(x_\beta) \in P$; that is, $\varphi(x_\beta) \preceq \inf_{\alpha\in\Gamma}\varphi(x_\alpha)$. So we have $\varphi(x_\beta) = \inf_{\alpha\in\Gamma}\varphi(x_\alpha)$ and hence $\varphi(x_\beta) \preceq \varphi(x_\alpha)$ for all $\alpha \in \Gamma$. Note that $\{\varphi(x_\alpha)\}_{\alpha\in\Gamma}$ is a decreasing chain, then $\beta \geq \alpha$ for all $\alpha \in \Gamma$. Moreover $x_\alpha \prec x_\beta$ for all $\alpha \in \Gamma$ since $\{x_\alpha\}_{\alpha\in\Gamma}$ is an increasing chain. Hence $\{x_\alpha\}_{\alpha\in\Gamma}$ has an upper bound in Q_1. By Zorn's lemma, $(Q_1, \prec)$ has a maximal element x^*; that is, for all $x \in Q_1$, $x^* \prec x$ implies $x = x^*$. By $x^* \in Q_1$, there exists $y^* \in Tx^*$ such that $x^* \prec y^*$. Moreover by the increasing property of T on $[x_0, +\infty)$, there exists $z^* \in Ty^*$ such that $y^* \prec z^*$. Thus we have $x^* \prec z^*$ by $x^* \prec y^*$. This indicates $z^* \in Tx^* \cap [x^*, +\infty)$ and hence $z^* \in Q_1$. Finally the maximality of x^* in Q_1 forces that $x^* = z^* \in Tx^*$; that is, x^* is a maximal fixed point of T in $[x_0, +\infty)$. The proof is complete. ☐

Theorem 4. *Let $(X, d, \prec)$ be a complete partially ordered cone metric space over a total order minihedral and continuous cone P of a normed vector space E. Let $\varphi : X \to E$ be a sequentially continuous vector functional and $T : X \to 2^X$ be a set-valued mapping such that Tx is compact for all $x \in X$. Assume that there exists $y_0 \in X$ such that φ is bounded above on $(-\infty, y_0]$, T is quasi-increasing on $(-\infty, y_0]$, and $Ty_0 \cap (-\infty, y_0] \neq \emptyset$. Then T has a minimal fixed point $x_* \in (-\infty, y_0]$.*

Proof. Set

$$Q_2 = \{x \in (-\infty, y_0] : Tx \cap (-\infty, x] \neq \emptyset\}. \tag{17}$$

Clearly, $Q_2 \neq \emptyset$. By the same method used in the proof of Theorem 3, we can prove that $(Q_2, \prec)$ has a minimal element x_* which is also a minimal fixed point of T in $(-\infty, y_0]$. The proof is complete. ☐

Remark 5. If $T : X \to X$ is a single-valued mapping, then Tx is naturally compact for all $x \in X$. Hence both of Theorems 3 and 4 are still valid for a single-valued mapping.

In particular when T is a single-valued mapping, we have the following further results.

Theorem 6. *Let $(X, d, \prec)$ be a complete partiallly ordered cone metric space over a total order minihedral and continuous cone P of a normed vector space E. Let $\varphi : X \to E$ be a sequentially continuous vector functional and let $T : X \to X$ be a single-valued mapping. Assume that there exists $x_0 \in X$ such that φ is bounded below on $[x_0, +\infty)$, T is increasing on $[x_0, +\infty)$, and $x_0 \prec Tx_0$. Then T has a maximal fixed point x^* and a least fixed point x_* in $[x_0, +\infty)$ such that $x_* \prec x^*$.*

Proof. By Theorem 3 and Remark 5, T has a maximal fixed point $x^* \in [x_0, +\infty)$ and hence $F = \{x \in [x_0, +\infty) : x = Tx\} \neq \emptyset$. Set

$$S = \{I = [x, +\infty) : x \in [x_0, +\infty), x \prec Tx, F \subseteq I\}. \tag{18}$$

Clearly, $[x_0, +\infty) \in S$ and hence $S \neq \emptyset$. Define a relation $\sqsubseteq$ on S by

$$I_1 \sqsubseteq I_2 \Longleftrightarrow I_1 \subseteq I_2, \tag{19}$$

for all $I_1, I_2 \in S$, then it is easy to check that $\sqsubseteq$ is a partial order on S.

Let $\{I_\alpha\}_{\alpha\in\Gamma}$ be a decreasing chain of S, where $I_\alpha = [x_\alpha, +\infty)$. From (1), (18), and (19) we find that $\{x_\alpha\}_{\alpha\in\Gamma}$ is an increasing chain of M, where

$$M = \{x \in [x_0, +\infty) : x \prec Tx, F \subseteq [x, +\infty)\}. \tag{20}$$

Set $\overline{Q}_1 = \{x \in [x_0, +\infty) : x \prec Tx\}$. Clearly, $M \subseteq \overline{Q}_1$. Following the proof of Theorem 3, there exists $\overline{x} \in \overline{Q}_1$ and an increasing sequence $\{x_{\alpha_n}\} \subseteq \{x_\alpha\}$ satisfying (6) such that (8) and (11) are satisfied. From $x_{\alpha_n} \in M$ we have that $x_{\alpha_n} \prec x$ for all $x \in F$ and all n. Thus the increasing property of T on $[x_0, +\infty)$ implies that, for all $x \in F$ and all n,

$$x_{\alpha_n} \prec Tx_{\alpha_n} \prec Tx = x, \tag{21}$$

and hence by (1),

$$d\left(x_{\alpha_n}, x\right) \preceq \varphi\left(x_{\alpha_n}\right) - \varphi(x), \tag{22}$$

for all $x \in F$ and all n. Let $n \to \infty$, then by (8) and the continuity of φ we have $d(\overline{x}, x) \preceq \varphi(\overline{x}) - \varphi(x)$; that is,

$$\overline{x} \prec x, \tag{23}$$

for all $x \in F$. This together with $\overline{x} \in \overline{Q}_1$ implies $\overline{x} \in M$. Then in analogy to the proof of Theorem 3, by (6), (8), and $\overline{x} \in M$ we can prove $\{x_\alpha\}_{\alpha\in\Gamma}$ has an upper bound $\widehat{x} \in M$. By (18), we have $[\widehat{x}, +\infty) \in S$. Note that $\widehat{x}$ is an upper bound of $\{x_\alpha\}_{\alpha\in\Gamma}$ in M, then $[\widehat{x}, +\infty) \subseteq I_\alpha$ for all $\alpha \in \Gamma$ and hence by (19),

$$[\widehat{x}, +\infty) \sqsubseteq I_\alpha, \tag{24}$$

for all $\alpha \in \Gamma$. This means $[\widehat{x}, +\infty)$ is a lower bound of $\{I_\alpha\}_{\alpha\in\Gamma}$ in S. By Zorn's lemma, $(S, \sqsubseteq)$ has a minimal element; denote it by $I^* = [x_*, +\infty)$. By (18) we have $x_0 \prec x_* \prec Tx_*$ and

$$x_* \prec x, \tag{25}$$

for all $x \in F$. By the increasing property of T, we have $x_0 \prec x_* \prec Tx_* \prec T(Tx_*)$ and $Tx_* \prec Tx = x$ for all $x \in F$, which implies $[Tx_*, +\infty) \in S$ and $[Tx_*, +\infty) \subseteq I^*$. Moreover by (19), $[Tx_*, +\infty) \sqsubseteq I^*$. The minimality of I^* in S forces that $[Tx_*, +\infty) = I^*$ and so we have $x_* = Tx_*$. Finally by (25), x_* is a least fixed point of T in $[x_0, +\infty)$ and $x_* \prec x^*$. The proof is complete. □

Theorem 7. *Let $(X, d, \prec)$ be a complete partially ordered cone metric space over a total order minihedral and continuous cone P of a normed vector space E. Let $\varphi : X \to E$ be a sequentially continuous vector functional and let $T : X \to X$ be a single-valued mapping. Assume that there exists $x_0 \in X$ such that φ is bounded above on $(-\infty, y_0]$, T is increasing on $(-\infty, y_0]$, and $Ty_0 \prec y_0$. Then T has a minimal fixed point x_* and a largest fixed point in x^* in $(-\infty, x_0]$ such that $x_* \prec x^*$.*

Proof. By Theorem 4 and Remark 5, T has a minimal fixed point in $x_* \in (-\infty, y_0]$. Set

$$\overline{S} = \{J = (-\infty, x] : x \in (-\infty, y_0], Tx \prec x, F \subseteq J\}. \tag{26}$$

Define a relation $\sqsubseteq_{\overline{S}}$ on $\overline{S}$ as follows:

$$J_1 \sqsubseteq_{\overline{S}} J_2 \Longleftrightarrow J_1 \subseteq J_2, \tag{27}$$

for all $J_1, J_2 \in \overline{S}$, then $\sqsubseteq_{\overline{S}}$ is a partial order on $\overline{S}$. In an analogy to the proof of Theorem 4, we can prove $(\overline{S}, \sqsubseteq_{\overline{S}})$ has a minimal element $(-\infty, x^*]$ and x^* is a largest fixed point of T in $(-\infty, y_0]$. The proof is complete. □

Theorem 8. *Let $(X, d, \prec)$ be a complete partially ordered cone metric space over a total order minihedral and continuous cone P of a normed vector space E. Let $\varphi : X \to E$ be a sequentially continuous mapping and let $T : X \to X$ be a single-valued mapping. Assume that there exists $x_0, y_0 \in X$ with $x_0 \prec y_0$ such that T is increasing on $[x_0, y_0]$ and $x_0 \prec Tx_0, Ty_0 \prec y_0$. Then T has a largest fixed point x^* and a least fixed point x_* in $[x_0, y_0]$ such that $x_* \prec x^*$.*

Proof. For all $x \in [x_0, y_0]$, by (1) we have $\varphi(y_0) \preceq \varphi(x) \prec \varphi(x_0)$; that is, φ is bounded on $[x_0, y_0]$. In an analogy to the proof of Theorem 3, we can prove T has a maximal fixed point and a minimal fixed point in $[x_0, y_0]$ by investigating the existence of maximal element and minimal element, respectively, in $D_1 = \{x \in [x_0, y_0] : x \prec Tx\}$ and $D_2 = \{x \in [x_0, y_0] : Tx \prec x\}$. Let

$$\begin{aligned} S_1 &= \{I = [x, y_0] : x \in [x_0, y_0], x \prec Tx, G \subseteq I\}, \\ S_2 &= \{J = [x_0, x] : x \in [x_0, y_0], Tx \prec x, G \subseteq J\}, \end{aligned} \tag{28}$$

where $G = \{x \in [x_0, y_0] : Tx = x\}$ is nonempty. Define $\sqsubseteq_1$ on S_1 and $\sqsubseteq_2$ on S_2, respectively, by

$$\begin{aligned} I_1 \sqsubseteq_1 I_2 &\Longleftrightarrow I_1 \subseteq I_2, \quad \forall I_1, I_2 \in S_1, \\ J_1 \sqsubseteq_2 J_2 &\Longleftrightarrow J_1 \subseteq J_2, \quad \forall J_1, J_2 \in S_2, \end{aligned} \tag{29}$$

then it is easy to check that $\sqsubseteq_1$ and $\sqsubseteq_2$ are partial orders on S_1 and S_2, respectively. In an analogy to the proof of Theorem 4, we can prove $(S_1, \sqsubseteq_1)$ has a minimal element $I_* = [x_*, y_0]$ and $(S_2, \sqsubseteq_2)$ has a minimal element $J_* = [x_0, y^*]$. By the definitions of S_1 and S_2, we have $x_*, y^* \in [x_0, y_0]$,

$$x_* \prec x \prec y^*, \tag{30}$$

$$x_* \prec Tx_* \prec Ty^* \prec y^*. \tag{31}$$

Moreover by (30) and the increasing property of T on $[x_0, y_0]$, for all $x \in G$, we have

$$x_0 \prec Tx_0 \prec Tx_* \prec x \prec Ty^* \prec Ty_0 \prec y_0, \tag{32}$$

and so by (31),

$$x_* \prec Tx_* \prec T\left(Tx_*\right) \prec T\left(Ty^*\right) \prec Ty^* \prec y^*. \tag{33}$$

From (32) and (33) we have that $[Tx_*, y_0] \in S_1$, $[x_0, Ty^*] \in S_2$, and

$$\left[Tx_*, y_0\right] \sqsubseteq_1 I_*, \qquad \left[x_0, Ty^*\right] \sqsubseteq_2 J_*, \tag{34}$$

which implies $[Tx_*, y_0] = I_*$ and $[x_0, Ty^*] = J_*$ by the minimality of I_* and J_*. This means that $Tx_* = x_*$ and $Ty^* = y^*$. Hence x_* is the least fixed point and y^* is the largest fixed point of T in $[x_0, y_0]$ by (31). The proof is complete. □

Remark 9. Theorems 3–8 are extensions of [4, Theorems 3 and 4] and [2, Theorems 3, 4, and 5] to the case of cone metric spaces. It is worth mentioning that in Theorems 4, 7, and 8, not only the existence of maximal and minimal fixed points but also the existence of largest and least fixed points is obtained. Therefore Theorems 4, 7, and 8 are still new even in the case of metric space and hence they indeed improve [2, Theorems 3, 4, and 6].

Now we give an example to demonstrate Theorem 3.

Example 10. Let $X = \{1, 2, 3, 4\}$, $E = \mathbb{R}^2$ with the norm $\|u\| = \sqrt{u_1^2 + u_2^2}$ for all $u = (u_1, u_2) \in \mathbb{R}^2$ and $P = \mathbb{R}_+^2$. Clearly, P is a strongly minihedral and continuous cone of E. Define a mapping $d : \mathbb{R} \times \mathbb{R} \to P$ by

$$d(x, y) = \left(|x - y|, |x - y|^{1/2}\right), \quad \forall x, y \in \mathbb{R}, \tag{35}$$

then $(\mathbb{R}, d)$ is a complete cone metric space over P and hence (X, d) is a complete cone metric subspace of $(\mathbb{R}, d)$. Define a vector functional $\varphi : [1, +\infty) \to E$ by

$$\varphi(x) = \left(\frac{6}{x}, \frac{3\sqrt{2} + 2\sqrt{3}}{\sqrt{x}}\right), \tag{36}$$

for all $x \in [1, +\infty)$. For arbitrary $x \in [1, +\infty)$, let $\{x_n\} \subseteq [1, +\infty)$ be a sequence such that $x_n \xrightarrow{d} x$, then $x_n \xrightarrow{|\cdot|} x$ and hence $\|\varphi(x_n) - \varphi(x)\| \to 0$, that is, $\lim_{n \to \infty} \varphi(x_n) = \varphi(x)$. This means that $\varphi : [1, +\infty) \to E$ is sequentially continuous; in particular, $\varphi : X \to E$ is sequentially continuous. From (35) and (36) it is easy to check that

$$\begin{gathered} 1 \prec 1, \quad 1 \prec 2, \quad 1 \prec 3, \quad 1 \prec 4, \\ 2 \prec 2, \quad 2 \prec 3, \quad 2 \prec 4, \\ 3 \prec 3, \quad 3 \nprec 4, \quad 4 \prec 4, \quad 4 \nprec 3, \end{gathered} \tag{37}$$

where $\prec$ is the partial order defined by (1). Let $T : X \to 2^X$ be a set-valued mapping such that

$$\begin{gathered} T1 = \{3, 4\}, \qquad T2 = \{1, 3\}, \\ T3 = \{1, 2, 3, 4\}, \qquad T4 = \{1, 2, 3\}. \end{gathered} \tag{38}$$

Fix $x_0 = 2$, then $[x_0, +\infty) = \{x \in X : 2 \prec x\} = \{2, 3, 4\}$ by (37), and so $Tx_0 \cap [x_0, +\infty) = \{3\} \neq \emptyset$. For $x, y \in [x_0, +\infty)$, if $x \prec y$ and $x \neq y$, then we have only two cases: $x = 2 \prec 3 = y$ and $x = 2 \prec 4 = y$ by (37). Fix $x = 2$ and $y = 3$, for all $u \in Tx$, there exists $v = 3, 4 \in Ty$ such that $u \prec v$. Fix $x = 2$ and $y = 4$, for all $u \in Tx$, there exists $v = 3 \in Ty$ such that $u \prec v$. This means that $T : X \to 2^X$ is increasing on $[x_0, +\infty)$. Therefore all the conditions of Theorem 3 are satisfied and hence T has a fixed point $3 \in [x_0, +\infty)$.

Fix $x = 4$; for all $y \in T4$, we have $x = 4 \nprec y$ by (37); that is, (2) is not satisfied. Therefore the existence of fixed points could not be obtained by generalized Caristi's fixed point theorems in cone metric spaces of [1, 3].

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