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Research Article

On the Dependence of the Limit Functions on the Random Parameters in Random Ergodic Theorems

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We study the structure of the ergodic limit functions determined in random ergodic theorems. When the r random parameters are shifted by the r_0-shift transformation with $r_0 \in \{1, 2, \ldots, r\}$, the major finding is that the (random) ergodic limit functions determined in random ergodic theorems depend essentially only on the r—r_0 random parameters. Some of the results obtained here improve the earlier random ergodic theorems of Ryll-Nardzewski (1954), Gladysz (1956), Cairoli (1964), and Yoshimoto (1977) for positive linear contractions on L_1 and Woś (1982) for sub-Markovian operators. Moreover, applications of these results to nonlinear random ergodic theorems for affine operators are also included. Some examples are given for illustrating the relationship between the ergodic limit functions and the random parameters in random ergodic theorems.

1. A General Argument

The present paper is concerned with the relations between the limit functions in random ergodic theorems and the random parameters concomitant to the limit functions. The first results of the random ergodic theory include Pitt's random ergodic theorem [1] and Ulam-von Neumann's random ergodic theorem [2] concerning a finite number of measure-preserving transformations and Kakutani's random ergodic theorem [3] concerning an infinite number of measure-preserving transformations. Furthermore, Kakutani dealt with the relationship between the random ergodic theorem and the theory of Markov processes with a stable distribution. The random ergodic theorem is usually obtained by using the so-called skew product method as natural extensions of ergodic theorems and has received a great deal of attention from the wider point of view including operator-theoretical treatment. In fact, interesting extensions have been made by many authors.

It was pointed out by Marczewski (see [4]) that the proof of Kakutani's theorem should be found which would not use the hypothesis that the transformations in question are one-to-one. Answering this question, Ryll-Nardzewski [4] improved Kakutani's theorem to the case of random measure-preserving transformations which are not necessarily one-to-one and proved that the limit function is essentially independent of the random parameter. Then, later, Ryll-Nardzewski's theorem was generalized by Gładysz [5] to the case of a finite number of random parameters. The Ryll-Nardzewski theorem was extended by Cairoli [6] to the case of positive linear contractions on L_1 with an additional condition. Yoshimoto [7] extended both Gladysz's theorem and Cairoli's theorem to the case of positive linear contractions on L_1 with a finite number of random parameters. In this paper we inquire further into the problem of the dependence of the limit functions upon the random parameters in random ergodic theorems, and we have an intention of improving the previous random ergodic theorem of Yoshimoto [7].

In what follows, we suppose that there are a given σ-finite measure space (S, β, m) and a probability space $(\Omega, \mathcal{F}, \mu)$. Let $L_p(m) = L_p(S, \beta, m)$, $1 \le p \le \infty$ be the usual Banach spaces of eqivalence classes of β-measurable functions defined on S. From now on, we shall write $f_\omega(s)$ for $f(s, \omega)$ if we wish to regard $f(s, \omega)$ as a function of s defined on S for ω arbitrarily fixed in Ω.

It seems to be worthwhile to include the first random ergodic theorems which may be stated, respectively, as follows.

Theorem 1 (see [1, 2]). *Let $U(s)$, $V(s)$ be two given measure-preserving transformations of S into itself, which generate all the combinations of the transformations: U, V, $U(U)$, $U(V)$, $V(U)$, $V(V)$, $U(U(V))$, $U(V(U))$, ... The ergodic limit exists then for almost every point s of S and almost every choice of the infinite sequence obtained by applying U and V in turn at random, for example, $U(s)$, $V(U(s))$, $V(V(U(s)))$, ...*

Exactly speaking, the first step to the theory of random ergodic theorems was taken by Pitt [1]. The above theorem was stated by Ulam and von Neumann [2] (independently of Pitt), but the essence of the contents is the same as the theorem of Pitt who proved both the pointwise convergence and the $L_p(m)$ ($p \geq 1$, $m(S) < \infty$) mean convergence of random averages in question. Ulam and von Neumann announced the pointwise convergence of random averages in an abstract form, but the proof has never been published. Pitt-Ulam-von Neumann's random ergodic theorem concerning a finite number of measure-preserving transformations was extended by Kakutani [3] to the case of an infinite number of measure-preserving transformations as follows.

Theorem 2 (Kakutani (1948–1950) [3]). *Let $\Phi = \{\varphi_\omega : \omega \in \Omega\}$ be a $\beta \otimes \mathcal{F}$-measurable family of measure-preserving transformations φ_ω defined on S, where $m(S) = \mu(S) = 1$. Let $(\Omega^*_\pm, \mathcal{F}^*_\pm, \mu^*_\pm)$ be the two-sided infinite direct product measure space of $(\Omega, \mathcal{F}, \mu)$. Then, for any $f \in L_p(m)$ ($p \geq 1$), there exists a $(\mathcal{F}^*_\pm, \mu^*_\pm)$-null set N^* such that for any $\omega^* \in \Omega^*_\pm - N^*$, there exists a function $\widetilde{f}_{\omega^*}(s) \in L_p(m)$ such that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f\left(\varphi_{\aleph_k(\omega^*)} \cdots \varphi_{\aleph_0(\omega^*)} s\right) = \widetilde{f}_{\omega^*}(s) \quad m\text{-a.e.}, \tag{1}$$

where $\aleph_n(\omega^*)$ denotes the nth coordinate of ω^*, and this holds also in the norm of $L_p(m)$.

Kakutani's paper on the random ergodic theorem was published in 1950, but the random ergodic theorem had already been dealt with by Kawada ("Random ergodic theorems", Suritokeikenkyu (Japanese) 2, 1948). Later, Kawada reminisced about the circumstances of an affair of his paper. It is the rights of matter that Kawada's result is due to Kakutani's kind suggestion.

Remark 3. In Kakutani's random ergodic theorem (as well as in Pitt-Ulam-von Neumann' theorem), the sequence $(\varphi_{\aleph_n(\omega^*)})_{n\geq 0}$ of measure-preserving transformations on S is chosen at random with the same distribution and independently. In connection with this question, an interesting problem is the following: if we choose a sequence $(\varphi_n)_{n\geq 1}$ at random, not necessarily with the same distribution but independently from a given set Φ of measure-preserving transformations on S, under what condition does the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\varphi_k \cdots \varphi_1 s\right) \tag{2}$$

exist m-a.e. or in $L_2(m)$-mean with probability 1? Revesz made the first step toward the study of this problem (see [8, 9]).

The most general formulation of random ergodic theorems is the following Chacon's type theorem given by Jacobs [10].

Jacobs' General Random Ergodic Theorem [10]. Let σ be an endomorphism of $(\Omega, \mathcal{F}, \mu)$ and let $\{T_\omega : \omega \in \Omega\}$ be a strongly $\mathcal{F}$-measurable family of random linear contractions on $L_1(m)$. Let $\{p_n\}_{n\geq 0}$ be a sequence of $\beta \otimes \mathcal{F}$-measurable functions defined on $S \times \Omega$ which is admissible for $\{T_\omega : \omega \in \Omega\}$. This means that if $h \in L_1(m \otimes \mu)$ and $|h(s,\omega)| \leq p_n(s,\omega)$ $m \otimes \mu$-a.e., then $|T_\omega h_{\sigma\omega}(s)| \leq p_{n+1}(s,\omega)$ $m \otimes \mu$-a.e. Then, for any function $f \in L_1(m)$, there exists a μ-null set $N \in \mathcal{F}$ such that for any $\omega \in \Omega - N$,

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} T_\omega T_{\sigma\omega} \cdots T_{\sigma^k\omega} f(s)}{\sum_{k=0}^{n} p_k(s,\omega)} \left(= \widetilde{f}(s,\omega)\right) \tag{3}$$

exists and is finite m-a.e. on the set $\{s : \sum_{k=0}^{\infty} p_k(s,\omega) > 0\}$ (cf. [11] which includes a further weighted generalization of Jacobs'theorem).

If all T_ω are positive, then Jacobs' theorem yields the following Chacon-Ornstein's type random ergodic theorem (cf. [10, 12]); for $f \in L_1(m)$ and $g \in L_1^+(m)$, there exists a μ-null set N such that for any $\omega \in \Omega - N$,

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} T_\omega T_{\sigma\omega} \cdots T_{\sigma^k\omega} f(s)}{\sum_{k=0}^{n} T_\omega T_{\sigma\omega} \cdots T_{\sigma^k\omega} g(s)} \left(= \widetilde{f_g}(s,\omega)\right) \tag{4}$$

exists and is finite m-a.e. on the set $\{s : \sum_{k=0}^{\infty} T_\omega T_{\sigma\omega} \cdots T_{\sigma^k\omega} g(s) > 0\}$. Moreover, if the family $\{T_\omega, \omega \in \Omega\}$ has a strictly positive invariant function $g \in L_1^+(m)$, then for every $f \in L_1(m)$, there exists a μ-null set N such that for any $\omega \in \Omega - N$

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} T_\omega T_{\sigma\omega} \cdots T_{\sigma^k\omega} f(s) \left(= f^*(s,\omega)\right) \tag{5}$$

exists and is finite m-a.e. Unfortunately, this Cesàro-type result does not hold in general without assuming the existence of a strictly positive invariant function. However, if the family $\{T_\omega : \omega \in \Omega\}$ satisfies the norm conditions $\|T_\omega\|_{L_1(m)} \leq 1$ and $\|T_\omega\|_{L_\infty(m)} \leq 1$ for all $\omega \in \Omega$, then the above Cesàro-type random ergodic theorem holds even without assuming the existence of a strictly positive invariant function in $L_1(m)$. The above limit functions $\widetilde{f}(s,\omega) = [\widetilde{f}]_\omega(s)$, $\widetilde{f_g}(s,\omega) = [\widetilde{f_g}]_\omega(s)$, and $f^*(s,\omega) = [f^*]_\omega(s)$ depend generally on the random parameter ω. Our particular interest is in the relationship between the limit functions and the random parameters in the case that σ is the shift transformation of the random parameter space $(\Omega^*, \mathcal{F}^*, \mu^*)$ being the one-sided infinite product of the same probability space $(\Omega, \mathcal{F}, \mu)$.

2. Random Ergodic Theorems

Throughout all that follows, let $(\Omega^*, \mathscr{F}^*, \mu^*)$ be the one-sided infinite product measure space of $(\Omega, \mathscr{F}, \mu)$:

$$(\Omega^*, \mathscr{F}^*, \mu^*) = \left(\prod_{n=1}^{\infty}\Omega_n, \bigotimes_{n=1}^{\infty}\mathscr{F}_n, \bigotimes_{n=1}^{\infty}\mu_n\right),$$
$$(\Omega^*_r, \mathscr{F}^*_r, \mu^*_r) = \left(\prod_{n=1}^{r}\Omega_n, \bigotimes_{n=1}^{r}\mathscr{F}_n, \bigotimes_{n=1}^{r}\mu_n\right), \quad (6)$$
$$(r\text{: a positive integer}),$$
$$(\Omega_n, \mathscr{F}_n, \mu_n) = (\Omega, \mathscr{F}, \mu), \quad n = 1, 2, \ldots.$$

Let φ be the (one-sided) shift transformation defined on Ω^* which means that using the coordinate functions $\aleph_n(\cdot)$,

$$\aleph_n(\varphi\omega^*) = \aleph_{n+1}(\omega^*), \quad n = 1, 2, \ldots. \quad (7)$$

Then, φ is clearly a $\mathscr{F}^*$-measurable and μ^*-measure-preserving transformation defined on Ω^*. Let r be a fixed positive integer. For simplicity, we let $[\omega^*]_r = (\omega_1, \ldots, \omega_r)$ for any $\omega^* = (\omega_1, \ldots, \omega_r, \ldots) \in \Omega^*$. Fix an $r_0 \in \{1, 2, \ldots, r\}$. Suppose that to each $[\omega^*]_r \in \Omega^*_r$, there corresponds a linear contraction operator $T_{[\omega^*]}$ on $L_1(m)$. The family $\{T_{[\omega^*]_r} : \omega^* \in \Omega^*\}$ is said to be strongly $\mathscr{F}^*_r$-measurable if for any $h \in L_1(m)$ the function $T_{[\omega^*]_r}h$ is strongly $\mathscr{F}^*_r$-measurable as an $L_1(m)$-valued function defined on Ω^*_r, namely, for the mapping $\Psi_h : [\omega^*]_r \to T_{[\omega^*]_r}h$ of $\mathscr{F}^*_r$ into $L_1(m)$, $\mu^*_r \circ \Psi_h^{-1}$ has a separable support (cf. [13]).

Our main result is stated as follows.

Theorem 4. *Let r_0 be any fixed integer with $1 \le r_0 \le r$. Let $\{T_{[\omega^*]_r} : \omega^* \in \Omega^*\}$ be a strongly $\mathscr{F}^*_r$-measurable family of positive linear contractions on $L_1(m)$. Suppose that there exists a strictly positive $L_1(m)$-function invariant under $\{T_{[\omega^*]_r} : \omega^* \in \Omega^*\}$. Then, for every $f \in L_1(m)$, there exists a μ^*-null set $N^* \in \mathscr{F}^*$ such that for any $\omega^* \in \Omega^* - N^*$, there exists a function $\tilde{f}_{[\omega^*]_{r-r_0}} \in L_1(m)$ such that*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}T_{[\omega^*]_r}T_{[\varphi^{r_0}\omega^*]_r}\cdots T_{[\varphi^{(k-1)r_0}\omega^*]_r}f_{\varphi^{kr_0}\omega^*}(s) = \tilde{f}_{[\omega^*]_{r-r_0}}(s) \quad m\text{-a.e.} \quad (8)$$

Proof. We need the following lemmas.

Lemma 5 (see [10, 14]). *The strong $\mathscr{F}^*_r$-measurability of the operator family $\{T_{[\omega^*]_r} : \omega^* \in \Omega^*\}$ guarantees that for any $g \in L_1(m \otimes \mu^*)$, there exists a uniquely determined $\beta \otimes \mathscr{F}^*$-measurable version $[T_{[\omega^*]_r}g_{\varphi^{r_0}\omega^*}](s)$ of $T_{[\omega^*]_r}g_{\varphi^{r_0}\omega^*}(s)$ such that excepting a μ^*-null set,*

$$\left[T_{[\omega^*]_r}g_{\varphi^{r_0}\omega^*}\right](s) = T_{[\omega^*]_r}g_{\varphi^{r_0}\omega^*}(s) \quad m\text{-a.e.} \quad (9)$$

Using the measurable version appearing in Lemma 5, we define

$$Tg(s, \omega^*) = \left[T_{[\omega^*]_r}g_{\varphi^{r_0}\omega^*}\right](s). \quad (10)$$

From the norm conditions of $\{T_{[\omega^*]_r} : \omega^* \in \Omega^*\}$ in Theorem 4, it turns out that T is a linear operator on $L_1(m \otimes \mu^*)$ with $\|T\|_{L_1(m\otimes\mu^*)} \le 1$. Moreover, it is easy to check that there exists a strictly positive $L_1(m \otimes \mu^*)$-function invariant under T. Thus, for any $f \in L_1(m\otimes\mu^*)$, we can apply Chacon-Ornstein's ergodic theorem [12] (cf. Hopf's ergodic theorem [15]) to ensure the existence of a function $\tilde{f} \in L_1(m \otimes \mu^*)$ such that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}T^k f(s, \omega^*) = \tilde{f}(s, \omega^*) \quad m \otimes \mu\text{-a.e.} \quad (11)$$

Moreover, as is easily checked, we find that

$$T\tilde{f} = \tilde{f}. \quad (12)$$

One can easily verify that excepting a suitable μ^*-null set $N^*_1 \in \mathscr{F}^*$,

$$T^k f(s, \omega^*) = T_{[\omega^*]_r}T_{[\varphi^{r_0}\omega^*]_r}\cdots T_{[\varphi^{(k-1)r_0}\omega^*]_r}f_{\varphi^{kr_0}\omega^*}(s) \quad m\text{-a.e.}, \quad (13)$$

for all $k = 1, 2, \ldots$. Next we wish to show that $\tilde{f}(s, \omega^*)$ does depend essentially only on $(s, [\omega^*]_{r-r_0})$. To do this, we define sub-σ-fields $\wp_n$, $n = 1, 2, \ldots$, of $\beta \otimes \mathscr{F}^*$ by

$$\wp_n = \beta \otimes \mathscr{F}_1 \otimes \cdots \otimes \beta_n \otimes \mathscr{F}_n \otimes \{\emptyset, \Omega_{n+1}\} \otimes \cdots, \quad n = 1, 2, \ldots, \quad (14)$$
$$(\wp_0 = \beta \otimes \{\emptyset, \Omega_1\} \otimes \{\emptyset, \Omega_2\} \otimes \cdots).$$

It is clear that if $m < n$, then $\wp_m \subset \wp_n$ and that if $f \in L_1(m \otimes \mu^*)$, $f_n = \wp_n f$, then $\wp_m f_n = f_m$ whenever $m \le n$. Therefore, the system $\{f_n, \wp_n : n = 1, 2, \ldots\}$ forms a martingale. For each n, let $E(\cdot \mid \wp_n)$ denote the conditional expectation operator with respect to the sub-σ-field $\wp_n$. Let $h(s, \omega^*)$ be of the form

$$h(s, \omega^*) = \xi(s) \cdot \prod_{i=1}^{\ell}\eta_i(\omega_i), \quad (15)$$

where $\xi \in L_1(m) \cap L_\infty(m)$, $\eta_i \in L_\infty(\Omega_i)$, $i = 1, 2, \ldots, \ell$. Then, the linear combinations of functions of the form (15) are everywhere dense in $L_1(m \otimes \mu^*)$. Thus, for the question confronting us, it suffices to prove the relation $E(\tilde{f} \mid \wp_{r-r_0}) = \tilde{f}$ only for the case when $\tilde{f}$, is of the form (15).

Lemma 6. It holds *that* $E(T^m\cdot \mid \wp_{r-r_0}) = T^m E(\cdot \mid \wp_{(m-1)r_0+r})$, $m = 1, 2, \ldots$.

Proof. It follows that for a sufficiently large ℓ and $n < \ell$, $E(h \mid \wp_n)$ is such that

$$E(h \mid \wp_n)(s, \omega^*) = \xi(s) \cdot \prod_{i=1}^{n}\eta_i(\omega_i) \cdot \prod_{j=n+1}^{\ell}\int_{\Omega}\eta_j(\omega)\,d\mu(\omega) \quad m \otimes \mu^*\text{-a.e.} \quad (16)$$

Thus, excepting a μ^*-null set N_2^*, we get

$$(T^* E(h \mid \wp_n))_{\omega^*}(s) = \left[T^*_{[\omega^*]_r}\xi\right](s)\prod_{i=1}^{n}\eta_i\left(\omega_{i+r_0}\right) \times \prod_{j=n+1}^{\ell}\int_\Omega \eta_j(\omega)\,d\mu(\omega) \quad m\text{-a.e.} \tag{17}$$

On the other hand, since excepting a μ^*-null set

$$(T^* h)_{\omega^*}(s) = \left[T^*_{[\omega^*]_r}\xi\right](s)\prod_{i=1}^{\ell}\eta_i\left(\omega_{i+r_0}\right) \quad m\text{-a.e.}, \tag{18}$$

we have that if $n \geq r - r_0$, then

$$\left(E\left(T^* h \mid \wp_{n+r_0}\right)\right)_{\omega^*}(s) = \left[T_{[\omega^*]_r}\xi\right](s)\prod_{i=1}^{n}\eta_i\left(\omega_{i+r_0}\right) \times \prod_{j=n+1}^{\ell}\eta_j(\omega)\,d\mu(\omega) \quad m\text{-a.e.} \tag{19}$$

Consequently,

$$T^* E(h \mid \wp_n) = E\left(T^* h \mid \wp_{n+r_0}\right) \quad m \otimes \mu^*\text{-a.e.}, \tag{20}$$

and by approximation,

$$T^* E(\cdot \mid \wp_n) = E\left(T^* \cdot \mid \wp_{n+r_0}\right), \tag{21}$$

and so by iteration,

$$T^{*m} E(\cdot \mid \wp_n) = E\left(T^{*m} \cdot \mid \wp_{n+mr_0}\right), \quad m = 0, 1, 2, \ldots,\ n \geq r - r_0. \tag{22}$$

In particular,

$$T^{*m} E\left(\cdot \mid \wp_{r-r_0}\right) = E\left(T^{*m} \cdot \mid \wp_{(m-1)r_0+r}\right), \quad m = 1, 2, \ldots. \tag{23}$$

Since T is the adjoint operator of T^*, we have thus

$$E\left(T^m \cdot \mid \wp_{r-r_0}\right) = T^m E\left(\cdot \mid \wp_{(m-1)r_0+r}\right), \quad m = 1, 2, \ldots. \tag{24}$$

We return to the proof of the theorem. By Lemma 6 and the martingale convergence theorem (cf. [16, 17]), we have

$$\begin{aligned} \left\|E\left(\tilde{f} \mid \wp_{r-r_0}\right) - \tilde{f}\right\|_{L_1(m\otimes\mu^*)} &= \left\|E\left(T^m\tilde{f} \mid \wp_{r-r_0}\right) - T^m\tilde{f}\right\|_{L_1(m\otimes\mu^*)} \\ &\leq \left\|T^m E\left(\tilde{f} \mid \wp_{(m-1)r_0+r}\right) - T^m\tilde{f}\right\|_{L_1(m\otimes\mu^*)} \\ &\leq \left\|E\left(\tilde{f} \mid \wp_{(m-1)r_0+r}\right) - \tilde{f}\right\|_{L_1(m\otimes\mu^*)} \longrightarrow 0 \end{aligned} \tag{25}$$

as $m \to \infty$, and thus

$$E\left(\tilde{f} \mid \wp_{r-r_0}\right) = \tilde{f}, \tag{26}$$

which implies that $\tilde{f}(s, \omega^*)$ depends essentially only on $(s, [\omega^*]_{r-r_0})$. Hence, the theorem follows from (11), (26), and Fubini's theorem. The proof of Theorem 4 has hereby been completed. □

If we take $r_0 = r$ in Theorem 4, we have Cairoli's theorem in which $\tilde{f}(s, \omega^*)$ does not depend essentially on ω^*. The r random parameter generalization (see [7]) of Cairoli's theorem is obtained by taking $r_0 = 1$ in Theorem 4. Adapting the (almost) same argument as used in the proof of Theorem 4, we have the following.

Theorem 7. *Let r_0 be any fixed integer with $1 \leq r_0 \leq r$. Let $\{T_{[\omega^*]_r} : \omega^* \in \Omega^*\}$ be a strongly $\mathcal{F}_r^*$-measurable family of linear contractions on $L_1(m)$ with $\|T_{[\omega^*]_r}\|_{L_\infty(m)} \leq 1$ for all $\omega^* \in \Omega^*$. Then, for every $f \in L_p(m \otimes \mu^*)$ with $1 \leq p < \infty$, there exists a μ^*-null set $N^* \in \mathcal{F}^*$ such that for any $\omega^* \in \Omega^* - N^*$, there exists a function $\tilde{f}_{[\omega^*]_{r-r_0}} \in L_p(m)$ such that*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}T_{[\omega^*]_r}T_{[\varphi^{r_0}\omega^*]_r}\cdots T_{[\varphi^{(k-1)r_0}\omega^*]_r}f_{\varphi^{kr_0}\omega^*}(s) = \tilde{f}_{[\omega^*]_{r-r_0}}(s) \quad m\text{-a.e.}, \tag{27}$$

and that if $1 < p < \infty$, then

$$\lim_{n\to\infty}\left\|\frac{1}{n}\sum_{k=1}^{n}T_{[\omega^*]_r}T_{[\varphi^{r_0}\omega^*]_r}\cdots T_{[\varphi^{(k-1)r_0}\omega^*]_r}f_{\varphi^{kr_0}\omega^*}(\cdot) - \tilde{f}_{[\omega^*]_{r-r_0}}(\cdot)\right\|_{L_p(m)} = 0, \tag{28}$$

and that if $m(S) < \infty$ and $f \in L(S \times \Omega^*)\log^+ L(S \times \Omega^*)$, then

$$\lim_{n\to\infty}\left\|\frac{1}{n}\sum_{k=1}^{n}T_{[\omega^*]_r}T_{[\varphi^{r_0}\omega^*]_r}\cdots T_{[\varphi^{(k-1)r_0}\omega^*]_r}f_{\varphi^{kr_0}\omega^*}(\cdot) - \tilde{f}_{[\omega^*]_{r-r_0}}(\cdot)\right\|_{L_1(m)} = 0. \tag{29}$$

Proof. As before, we define $Tg(s,\omega^*) = [T_{[\omega^*]_r} g_{\varphi^{r_0}\omega^*}](s)$ for $g \in L_1(m \otimes \mu^*)$. From the norm conditions of $\{T_{[\omega^*]_r} : \omega^* \in \Omega^*\}$ in Theorem 7, it turns out that T is a linear operator on $L_1(m \otimes \mu^*)$ with $\|T\|_{L_1(m\otimes\mu^*)} \leq 1$ and $\|T\|_{L_\infty(m\otimes\mu^*)} \leq 1$. Then, It follows from the Riesz convexity theorem that $\|T\|_{L_p(m\otimes\mu^*)} \leq 1$ for all p with $1 < p < \infty$. Thus, for any $f \in L_p(m \otimes \mu^*)$ $(1 \leq p < \infty)$, we can apply Dunford and Schwartz's ergodic theorem [18] to ensure the existence of a function $\tilde{f} \in L_p(m \otimes \mu^*)$ such that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} T^k f(s,\omega^*) = \tilde{f}(s,\omega^*) \quad m\otimes\mu\text{-a.e.}, \qquad T\tilde{f} = \tilde{f}, \tag{30}$$

and that if $1 < p < \infty$, then

$$\lim_{n\to\infty} \left\| \frac{1}{n}\sum_{k=1}^{n} T^k f - \tilde{f} \right\|_{L_p(m\otimes\mu^*)} = 0, \tag{31}$$

and that if $m(S) < \infty$ and $f \in L(S\times\Omega^*)\log^+ L(S\times\Omega^*)$, then

$$\lim_{n\to\infty} \left\| \frac{1}{n}\sum_{k=1}^{n} T^k f - \tilde{f} \right\|_{L_1(m\otimes\mu^*)} = 0. \tag{32}$$

Now, adapting the (almost) same argument as used in the proof of Theorem 4, we can find that

$$\tilde{f}(s,\omega^*) = \tilde{f}\left(s,[\omega^*]_{r-r_0}\right) \quad m\otimes\mu^*\text{-a.e.} \tag{33}$$

Note here that if $f \in L_p(m \otimes \mu^*)$, $1 < p < \infty$, (or $f \in L(S\times\Omega^*)\log^+ L(S\times\Omega^*)$), then

$$\sup_{n\geq 1} \frac{1}{n}\sum_{k=1}^{n} |T|^k |f| \in L_p(m\otimes\mu^*) \quad (\text{or}, \in L_1(m\otimes\mu^*)), \tag{34}$$

where $|T|$ denotes the linear modulus of T (see [18, 19]). Therefore, (27) follows from (30), (33), and Fubini's theorem. Equations (28) and (29) follow from (30), (31), (32), (33), (34), and Fubini's theorem. □

In the setting of measure-preserving transformations, Theorem 7 is reduced to the random one-parameter result of Ryll-Nardzewski [4] by taking $r_0 = r = 1$ and to the r random parameter result of Gładysz [5] by taking $r_0 = 1$.

Theorem 8. *Let $r_1, \ldots, r_\nu$ be ν positive integers. Let $\{T^{(i)}_{[\omega^*]_{r_i}} : \omega^* \in \Omega^*\}$, $i = 1, 2, \ldots, \nu$, be strongly $\mathcal{F}^{(i)}_{r_i}$ $(i = 1, 2, \ldots, \nu)$-measurable families of linear contractions on $L_1(m)$ with $\|T^{(i)}_{[\omega^*]_{r_i}}\|_{L_\infty(m)} \leq 1$. For each $i = 1, 2, \ldots, \nu$, we set*

$$T_i\left(k,[\omega^*]_{r_i}\right) = T^{(i)}_{[\omega^*]_{r_i}} T^{(i)}_{[\varphi\omega^*]_{r_i}} \cdots T^{(i)}_{[\varphi^{k-1}\omega^*]_{r_i}}, \quad k = 1, 2, \cdots, \qquad \left(T_i\left(0,[\omega^*]_{r_i}\right) = \text{the identity operator}\right). \tag{35}$$

Then, for every $f \in L_1(m)$, there exists a μ^*-null set N^* such that for every $\omega^* \in \Omega^* - N^*$, there exists a function $\tilde{f}_{[\omega^*]_{q-1}} \in L_1(m)$ with $q = \max(r_1, \ldots, r_\nu)$ such that the multiple averages

$$\frac{1}{n_1\cdots n_\nu}\sum_{k_1=1}^{n_1}\cdots\sum_{k_\nu=1}^{n_\nu} T_1\left(k_1,[\omega^*]_{r_1}\right)\cdots T_\nu \times\left(k_\nu,\left[\varphi^{k_1+\cdots+k_{\nu-1}}\omega^*\right]_{r_\nu}\right) f(s) \tag{36}$$

converge to $\tilde{f}_{[\omega^*]_{q-1}}(s)$ m-a.e. as $n_1 \to \infty, \ldots, n_\nu \to \infty$ independently. Here, $\tilde{f}_{[\omega^*]_{q-1}}(\cdot)$ means that the number of random parameters is at most $q-1$.

Proof. As already seen above, we can define the operators $T_1, \ldots, T_\nu$ on $L_1(m\otimes\mu^*)$ as follows: for $h \in L_1(m\otimes\mu^*)$

$$T_i h(s,\omega^*) = \left[T^{(i)}_{[\omega^*]_{r_i}} h_{\varphi^{r_i}\omega^*}\right](s), \quad i = 1, 2, \ldots, \nu. \tag{37}$$

Each T_i turns out to be a linear contraction on $L_1(m\otimes\mu^*)$ with $\|T_i\|_{L_\infty(m\otimes\mu^*)} \leq 1$. So, it follows from the Riesz convexity theorem that $\|T_i\|_{L_p(m\otimes\mu^*)} \leq 1$ for $1 < p < \infty$. Hence, from Dunford-Schwartz's ergodic theorem [18], we have that for every $\overline{f} \in L_p(m\otimes\mu^*)$ the multiple averages

$$\frac{1}{n_1\cdots n_\nu}\sum_{k_1=1}^{n_1}\cdots\sum_{k_\nu=1}^{n_\nu} T_1^{k_1}\cdots T_\nu^{k_\nu}\overline{f} = E_1\cdots E_\nu\overline{f} \qquad m\otimes\mu^*\text{-a.e.} \tag{38}$$

converge to $E_1\cdots E_\nu\overline{f}$ almost everywhere on $S\times\Omega^*$ as $n_1 \to \infty, \ldots, n_\nu \to \infty$ independently, where each E_i is a projection of $L_p(m\otimes\mu^*)$ onto the manifold $N(I-T_i) = \{g \in L_p(m\otimes\mu^*) : T_i g = g\}$ with

$$\lim_{n_i\to\infty}\left\|\frac{1}{n_i}\sum_{k_i=1}^{n_i} T_i^{k_i}\overline{f} - E_i\overline{f}\right\|_{L_p(m\otimes\mu^*)} = 0, \quad i = 1, 2, \ldots, \nu. \tag{39}$$

Note here that

$$E_\nu\overline{f} \in N(I-T_\nu), E_{\nu-1}E_\nu\overline{f} \in N(I-T_{\nu-1}), \ldots, \qquad E_1\cdots E_\nu\overline{f} \in N(I-T_1). \tag{40}$$

Thus, by Lemma 6 and the (L_p) martingale convergence theorem, we find that excepting a μ^*-null set

$$E_\nu\overline{f}(s,\omega^*) = \left(E_\nu\overline{f}\right)_{[\omega^*]_{r_\nu-1}}(s) \quad m\text{-a.e.},$$
$$E_{\nu-1}E_\nu\overline{f}(s,\omega^*) = \left(E_{\nu-1}E_\nu\overline{f}\right)_{[\omega^*]_{\max(r_{\nu-1},r_\nu)-1}}(s) \quad m\text{-a.e.},$$
$$E_1\cdots E_\nu\overline{f}(s,\omega^*) = \left(E_1\cdots E_\nu\overline{f}\right)_{[\omega^*]_{\max(r_1,\ldots,r_\nu)-1}}(s) \quad m\text{-a.e.} \tag{41}$$

Finally, observe that excepting a μ^*-null set, we get

$$\begin{aligned} T_i^k \overline{f}(s,\omega^*) &= T^{(i)}_{[\omega^*]_{r_i}} T^{(i)}_{[\varphi\omega^*]_{r_i}} \cdots T^{(i)}_{[\varphi^{k-1}\omega^*]_{r_i}} \overline{f}_{\varphi^k\omega^*}(s) \\ &= T_i\left(k,[\omega^*]_{r_i}\right)\overline{f}_{\omega^*}(s) \quad m\text{-a.e.}, \\ & \qquad k=1,2,\ldots, \quad i=1,2,\ldots,\nu, \end{aligned} \tag{42}$$

$$\begin{aligned} T_1^{k_1}\cdots T_\nu^{k_\nu}\overline{f}(s,\omega^*) &= T_1\left(k_1,[\omega^*]_{r_1}\right)\cdots \\ & \quad T_\nu\left(k_\nu,\left[\varphi^{k_1+\cdots+k_{\nu-1}}\omega^*\right]_{r_\nu}\right) \\ & \quad \times \overline{f}_{\omega^*}(s) \quad m\text{-a.e.} \end{aligned} \tag{43}$$

In fact, we have for two operators T_1 and T_2

$$\begin{aligned} T_1^{k_1}T_2^{k_2}\overline{f}(s,\omega^*) &= T^{(1)}_{[\omega^*]_{r_1}}\cdots T^{(1)}_{[\varphi^{k_1-1}\omega^*]_{r_1}}\left[T_2^{k_2}\overline{f}\right]_{\varphi^{k_1}\omega^*}(s) \\ &= T^{(1)}_{[\omega^*]_{r_1}}\cdots T^{(1)}_{[\varphi^{k_1-1}\omega^*]_{r_1}} T^{(2)}_{[\varphi^{k_1}\omega^*]_{r_2}}\cdots \\ & \quad T^{(2)}_{[\varphi^{k_1+k_2-1}\omega^*]_{r_2}}\overline{f}_{\varphi^{k_1+k_2}\omega^*}(s) \\ &= T_1\left(k_1,[\omega^*]_{r_1}\right)T_2\left(k_2,\left[\varphi^{k_1}\omega^*\right]_{r_2}\right) \\ & \quad \times \overline{f}(s,\omega^*) \quad m\text{-a.e.} \end{aligned} \tag{44}$$

To complete the proof of the above equality, assume that (43) has already been established for the $\nu-1$ operators $T_2,\ldots,T_\nu$. Then, it is easily verified by the induction hypothesis that (43) holds for the ν operators $T_1,\ldots,T_\nu$. Hence, taking $\overline{f}(s,\omega^*) = f(s)e(\omega^*)$ $(e(\cdot) = 1)$, the theorem follows from the above arguments. □

In particular, if the operators in question are commutative then we have the following.

Theorem 9. *Let $r_1,\ldots,r_\nu$ be ν positive integers. Let $\{T^{(i)}_{[\omega^*]_{r_i}} : \omega^* \in \Omega^*\}$, $i=1,2,\ldots,\nu$, be a strongly $\mathcal{F}^{(i)}_{r_i}$ $(i=1,2,\ldots,\nu)$-measurable commuting family of linear contractions on $L_1(m)$ with $\|T^{(i)}_{[\omega^*]_{r_i}}\|_{L_\infty(m)} \le 1$. Then, for every $f \in L_1(m)$, there exists a μ^*-null set N^* such that for every $\omega^* \in \Omega^* - N^*$, there exists a function $\widetilde{f}_{[\omega^*]_{q-1}} \in L_1(m)$ with $q=\max(r_1,\ldots,r_\nu)$ such that*

$$\begin{aligned} \lim_{n\to\infty}\frac{1}{n^\nu}\sum_{k_1=0}^{n-1}\cdots\sum_{k_\nu=0}^{n-1} & T_1\left(k_1,[\omega^*]_{r_1}\right)\cdots \\ & T_\nu\left(k_\nu,\left[\varphi^{k_1+\cdots+k_{\nu-1}}\omega^*\right]_{r_\nu}\right)f(s) \\ = \widetilde{f}_{[\omega^*]_{q-1}}(s) \quad & m\text{-a.e.} \end{aligned} \tag{45}$$

In passing, we make mention of the a.e. convergence for sectorial restricted random averages. We say that a sequence $u(n) \subset \mathbf{Z}_+^\nu$ remains in a sector of $\mathbf{Z}_+^\nu$ if there is a constant $C_0 > 0$ such that the ratios n_i/n_j are bounded by C_0 for $1 \le i,\ j \le \nu$ and all $u(n) = (n_1,\ldots,n_\nu)$ (see [20, page 203]). Appealing to Brunel-Dunford-Schwartz' theorem (see [20, Theorem 3.5, page 215]), we find that the multiple averages (38) converge m-a.e. in a sector $\mathbf{Z}_+^\nu$. In addition, in this case, Theorem 8 implies that the limit function depends essentially only on the $q-1$ random parameters. Hence, we have the following.

Theorem 10. *Let $r_1,\ldots,r_\nu$ be ν positive integers. Let $\{T^{(i)}_{[\omega^*]_{r_i}} : \omega^* \in \Omega^*\}$, $i=1,2,\ldots,\nu$, be strongly $\mathcal{F}^{(i)}_{r_i}$ $(i=1,2,\ldots,\nu)$-measurable commuting family of linear contractions on $L_1(m)$ with $\|T^{(i)}_{[\omega^*]_{r_i}}\|_{L_\infty(m)} \le 1$. Then, for every $f \in L_1(m)$, there exists a μ^*-null set N^* such that for every $\omega^* \in \Omega^* - N^*$, there exists a function $\widetilde{f}_{[\omega^*]_{q-1}} \in L_1(m)$ with $q=\max(r_1,\ldots,r_\nu)$ such that the multiple averages*

$$\begin{aligned} \frac{1}{n_1\cdots n_\nu}\sum_{k_1=1}^{n_1}\cdots\sum_{k_\nu=1}^{n_\nu} & T_1\left(k_1,[\omega^*]_{r_1}\right)\cdots \\ & T_\nu\left(k_\nu,\left[\varphi^{k_1+\cdots+k_{\nu-1}}\omega^*\right]_{r_\nu}\right)f(s) \end{aligned} \tag{46}$$

converge to $\widetilde{f}_{[\omega^*]_{q-1}}(s)$ m-a.e. as $u(n)$ $(=(n_1,\ldots,n_\nu) \ge (1,\ldots,1)) \to \infty$ in a sector of $\mathbf{Z}_+^\nu$.

A sub-Markovian operator P^* on $L_\infty(m)$ means that P^* is a positive linear contraction on $L_\infty(m)$ with 1 subinvariant under P^* (i.e., $P^*1 \le 1$) such that $\lim_{n\to\infty}P^*h_n = 0$ m-a.e. for any (decreasing) sequence $(h_n)_{n\ge1} \subset L_\infty(m)$ with $\lim_{n\to\infty}h_n = 0$ m-a.e. Let P denote the positive linear contraction on $L_1(m)$ with P^* as the adjoint operator of P. When $\upsilon = p\mu$ with a P-subinvariant function $p \ge 0$ $(p \ne 0)$, υ is called a P-subinvariant measure. Here, the subinvariant function is not necessarily integrable.

Now, let $\Pi^* = \{P^*_{[\omega^*]_r} : \omega^* \in \Omega^*\}$ be a strongly $\mathcal{F}_r^*$-measurable family of sub-Markovian operators on $L_\infty(m)$. Let $\Pi = \{P_{[\omega^*]_r} : \omega^* \in \Omega^*\}$ be a strongly $\mathcal{F}_r^*$-measurable family of positive linear contractions on $L_1(m)$, where each $P_{[\omega^*]_r}$ is the adjoint operator of the corresponding operator $P^*_{[\omega^*]_r}$. Then, there exists a positive linear contraction U on $L_1(m\otimes\mu^*)$ such that for any $f \in L_1(m\otimes\mu^*)$,

$$Uf(s,\omega^*) = \left[P_{[\omega^*]_r}f_{\varphi^{r_0}\omega^*}\right](s) \quad m\times\mu^*\text{-a.e} \tag{47}$$

In addition, if $\|P_{[\omega^*]_r}\|_{L_\infty(m)} \le 1$ for all $\omega^* \in \Omega^*$, U is also a contraction on $L_\infty(m\otimes\mu^*)$ (cf. [10, 14]). The operator U extends uniquely to a positive linear transformation on the class of nonnegative $\beta\otimes\mathcal{F}^*$-measurable functions defined on $S\times\Omega^*$ (see [20, page 51]).

Lemma 11. *Let the measure m be finite. Assume that m is $E(U\cdot \mid \wp_0)$-subinvariant. Then, U turns out to be a positive Dunford-Schwartz operator on $L_1(m\otimes\mu^*)$.*

Proof. Since m is assumed to be $E(U\cdot \mid \wp_0)$-subinvariant, the function 1 ($\in L_1(m \otimes \mu^*)$) is $E(U\cdot \mid \wp_0)$-subinvariant. According to Lemma 1.5 of Woś [21], we get

$$U1 = E\left(U1 \mid \wp_0\right), \tag{48}$$

so that 1 is also U-subinvariant. Thus, for $f \in L_1(m \otimes \mu^*) \cap L_\infty(m \otimes \mu^*)$,

$$\|Uf\|_\infty \le U\|f\|_\infty = \|f\|_\infty U1 \le \|f\|_\infty. \tag{49}$$

Since U is a positive contraction on $L_1(m \otimes \mu^*)$, and U is a positive Dunford-Schwartz operator on $L_1(m \otimes \mu^*)$. □

Now, the above general theorems can also be applied to sub-Markovian operators. For example, Theorem 7 yields the following theorem which extends both Theorems 3.4 and 3.6 of Woś [21] (In proving the key Lemma 1.5 of [21], Woś showed that if $f \in L_1(m \otimes \mu^*)$ does not depend essentially on parameter ω^*, then $Uf = E(Uf \mid \wp_0)$. Using this equality, he deduced $U1 \le 1$ from $E(U1 \mid \wp_0) \le 1$ in order to prove Theorems 2.8 and 3.4 of [21]. But obviously, 1 does not necessarily belong to $L_1(m \otimes \mu^*)$ in case where m is σ-finite. His arguments are of course correct in the case that m is finite (see, e.g., conditions (i) and (ii) in Theorem 12)).

Theorem 12. *Let $\Pi = \{P_{[\omega^*]_r} : \omega^* \in \Omega^*\}$ be a strongly $\mathcal{F}_r^*$-measurable family of positive linear contractions on $L_1(m)$, where each $P_{[\omega^*]_r}$ is the adjoint operator of the corresponding sub-Markovian operator $P^*_{[\omega^*]_r}$. Assume either of the following conditions:*

(i) *m is finite and $E(U\cdot \mid \wp_0)$-subinvariant,*

(ii) *m is σ-finite and $\|P_{[\omega^*]_r}\|_{L_\infty(m)} \le 1$ for all $\omega^* \in \Omega^*$.*

Then, for any $f \in L_p(m \otimes \mu^*)$, $1 \le p < \infty$, there exists a μ^*-null set N^* such that for every $\omega^* \in \Omega^* - N^*$, there exists a function $\widetilde{f}_{[\omega^*]_{r-r_0}} \in L_p(m)$ such that

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} P_{[\omega^*]_r} P_{[\varphi^{r_0}\omega^*]_r} \cdots P_{[\varphi^{(k-1)r_0}\omega^*]_r} f_{\varphi^{kr_0}\omega^*}(s) \\ = \widetilde{f}_{[\omega^*]_{r-r_0}}(s) \quad m\text{-a.e.},\end{aligned} \tag{50}$$

and that if $1 < p < \infty$, then

$$\begin{aligned}\lim_{n\to\infty} \Bigg\| \frac{1}{n}\sum_{k=1}^{n} P_{[\omega^*]_r} P_{[\varphi^{r_0}\omega^*]_r} \cdots P_{[\varphi^{(k-1)r_0}\omega^*]_r} f_{\varphi^{kr_0}\omega^*}(\cdot) \\ -\widetilde{f}_{[\omega^*]_{r-r_0}}(\cdot) \Bigg\|_{L_p(m)} = 0\end{aligned} \tag{51}$$

and that if m is finite and $f \in L(S \times \Omega^*)\log^+ L(S \times \Omega^*)$, then

$$\begin{aligned}\lim_{n\to\infty} \Bigg\| \frac{1}{n}\sum_{k=1}^{n} P_{[\omega^*]_r} P_{[\varphi^{r_0}\omega^*]_r} \cdots P_{[\varphi^{(k-1)r_0}\omega^*]_r} f_{\varphi^{kr_0}\omega^*}(\cdot) \\ -\widetilde{f}_{[\omega^*]_{r-r_0}}(\cdot) \Bigg\|_{L_1(m)} = 0.\end{aligned} \tag{52}$$

Proof of Theorem 1. In view of Lemma 11, either condition (i) or condition (ii) guarantees that U is a positive Dunford-Schwartz operator on $L_1(m \otimes \mu^*)$. Hence, we can apply Theorem 7 to conclude that Theorem 12 follows.

3. (C, α)-Type Random Ergodic Theorems

In this section, we establish a (C,α)-type random ergodic theorem for measure-preserving transformations on S. In this section, we assume that (S, β, m) is a probability space. For real $\alpha > -1$ and $n (= 0, 1, 2, \ldots)$, let A_n^α be the (C, α) coefficient of order α, which is defined by the generating function

$$\frac{1}{(1-t)^{\alpha+1}} = \sum_{n=0}^{\infty} A_n^\alpha t^n, \quad 0 < t < 1. \tag{53}$$

Then, we can easily check that A_n^α is decreasing in n for $-1 < \alpha < 0$ and increasing in n for $\alpha > 0$. For $\alpha > -1$, we have

$$\begin{aligned}&A_n^\alpha > 0, \qquad A_0^\alpha = 1, \qquad A_n^0 = 1, \\ &A_n^{\alpha-1} = A_n^\alpha - A_{n-1}^\alpha, \\ &A_n^\alpha = \sum_{k=0}^{n} A_{n-k}^{\alpha-1} = \binom{n+\alpha}{n}, \quad A_n^\alpha \sim \frac{n^\alpha}{\Gamma(\alpha+1)} \quad (n \longrightarrow \infty),\end{aligned} \tag{54}$$

and moreover,

$$\begin{aligned}\frac{n^\alpha}{\Gamma(\alpha+1)} \le A_n^\alpha \le \frac{(n+1)^\alpha}{\Gamma(\alpha+1)} \quad \text{for } 0 < \alpha \le 1, \\ A_n^{\alpha-1} \le \frac{n^{\alpha-1}}{\Gamma(\alpha)} \quad \text{for } n > 0.\end{aligned} \tag{55}$$

In general, using Hille's theorem (see [22, Theorem 8]), we have the following.

Theorem 13. *Let r_0 be any fixed integer with $1 \le r_0 \le r$. Let $\{T_{[\omega^*]_r} : \omega^* \in \Omega^*\}$ be a strongly $\mathcal{F}_r^*$-measurable family of linear contractions on $L_p(m)$ $(1 \le p < \infty)$. Let $\alpha > 0$ be fixed. If for all $f \in L_p(m \otimes \mu^*)$ there exists a μ^*-null set N^* such that for any $\omega^* \in \Omega^* - N^*$ there exists a function $f^*_{[\omega^*]_{r-r_0}} \in L_p(m)$ such that*

(i) $\lim_{n\to\infty}(1/A_n^\alpha)\sum_{k=0}^{n} A_{n-k}^{\alpha-1} T_{[\omega^*]_r} T_{[\varphi^{r_0}\omega^*]_r} \cdots T_{[\varphi^{kr_0}\omega^*]_r} f_{\varphi^{(k+1)r_0}\omega^*}(s) = \widetilde{f}_{[\omega^*]_{r-r_0}}(s)$ *for m-almost all $s \in S$,*

(ii) $\lim_{\lambda\to 1+0}(\lambda - 1)\sum_{n=0}^{\infty} \lambda^{(n+1)} T_{[\omega^*]_r} T_{[\varphi^{r_0}\omega^*]_r} \cdots T_{[\varphi^{kr_0}\omega^*]_r} f_{\varphi^{(k+1)r_0}\omega^*}(s) = \widetilde{f}_{[\omega^*]_{r-r_0}}(s)$,

(iii) $\lim_{n\to\infty}(1/n^\alpha) T_{[\omega^*]_r} T_{[\varphi^{r_0}\omega^*]_r} \cdots T_{[\varphi^{nr_0}\omega^*]_r} f_{\varphi^{(n+1)r_0}\omega^*}(s) = 0$,

for m-almost all $s \in S$. Conversely, if (ii) holds but (iii) be replaced by the condition that there exists a $\beta \otimes \mathcal{F}^*$-measurable *function $M(s,\omega^*)$ finite almost everywhere such that excepting a μ^*-null set*

$$\left|\frac{1}{A_n^\beta}\sum_{n=0}^{n} A_{n-k}^{\beta-1} T_{[\omega^*]_r} T_{[\varphi^{r_0}\omega^*]_r} \cdots T_{[\varphi^{kr_0}\omega^*]_r} f_{\varphi^{(k+1)r_0}\omega^*}(s)\right| \le M_{\omega^*}(s) \quad m\text{-a.e.}, \tag{56}$$

for all $n \ge 0$, where $\beta \ge 0$, then (i) holds for all $\alpha > \beta$.

Note here that in Theorem 13, (ii) and (iii) do not necessarily imply (i) in general.

Theorem 14. *Let r_0 be any fixed integer with $1 \le r_0 \le r$ and $\{\psi_{[\omega^*]_r} : \omega^* \in \Omega^*\}$ be a $\beta \otimes \mathcal{F}_r^*$-measurable family of measure-preserving transformations on S. Let $0 < \alpha < 1$, $\alpha p > 1$, and $f \in L_p(m)$. Then, there exists a μ^*-null set N^* such that for every $\omega^* \in \Omega^* - N^*$, there exists a function $f^*_{[\omega^*]_{r-r_0}}(\cdot) \in L_p(m)$ such that*

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{A_n^\alpha}\sum_{k=0}^{n} A_{n-k}^{\alpha-1} f\left(\psi_{[\varphi^{kr_0}\omega^*]_r} \cdots \psi_{[\omega^*]_r} s\right) \\
&\quad = \lim_{\lambda\to 1+0} (\lambda-1)\sum_{n=0}^{\infty} \lambda^{-(n+1)} f\left(\psi_{[\varphi^{kr_0}\omega^*]_r} \cdots \psi_{[\omega^*]_r} s\right) \\
&\quad = f^*_{[\omega^*]_{r-r_0}}(s) \quad m\text{-a.e.}, \\
&\lim_{n\to\infty} \left\{\int_S \left|\frac{1}{A_n^\alpha}\sum_{k=0}^{n} A_{n-k}^{\alpha-1} f\left(\psi_{[\varphi^{kr_0}\omega^*]_r} \cdots \psi_{[\omega^*]_r} s\right) - f^*_{[\omega^*]_{r-r_0}}(s)\right|^p dm\right\}^{1/p} = 0.
\end{aligned} \tag{57}$$

Proof. Define the skew product Φ^* of ϕ (the one-sided shift transformation of Ω^*) and $\{\psi_{[\omega^*]_r}\}$ as follows:

$$\Phi^*(s,\omega^*) = \left(\psi_{[\omega^*]_r} s, \varphi^{r_0}\omega^*\right), \quad (s,\omega^*) \in S \times \Omega^*. \tag{58}$$

Let $(f \cdot e)(s,\omega^*) = f(s) \cdot e(\omega^*)$, where $e(\omega^*) = 1$ for all $\omega^* \in \Omega^*$. Then, $(f \cdot e) \in L_p(m \otimes \mu^*)$. Since $L_p(m \otimes \mu^*)$ is reflexive, we see from Yosida-Kakutani's mean ergodic theorem [23] and Déniel's theorem [24] that there exists a function $(f \cdot e)^* \in L_p(m \otimes \mu^*)$ such that

$$\begin{aligned}
&\lim_{n\to\infty} \left\{\iint_{S\times\Omega^*} \left|\frac{1}{A_n^\alpha}\sum_{k=0}^{n} A_{n-k}^{\alpha-1} (f\cdot e)\left(\Phi^{*k}(s,\omega^*)\right) - (f\cdot e)^*(s,\omega^*)\right|^p dm\otimes\mu^*\right\}^{1/p} = 0, \\
&(f\cdot e)^*\left(\Phi^*(s,\omega^*)\right) = (f\cdot e)^*(s,\omega^*) \quad m\otimes\mu^*\text{-a.e.}, \\
&\sup_{n\ge 0}\left|\frac{1}{A_n^\alpha}\sum_{k=0}^{n} A_{n-k}^{\alpha-1} (f\cdot e)(s,\omega^*)\right| \in L_p(m\otimes\mu^*) \left(\subset L_1(m\otimes\mu^*)\right), \\
&\lim_{n\to\infty} \frac{1}{A_n^\alpha}\sum_{k=0}^{n} A_{n-k}^{\alpha-1} (f\cdot e)\left(\Phi^{*k}(s,\omega^*)\right) = (f\cdot e)^*(s,\omega^*) \quad m\otimes\mu^*\text{-a.e.}
\end{aligned} \tag{59}$$

Furthermore, applying Lemma 6 to the operator T^* induced by Φ^*, it follows that

$$(f\cdot e)^*(s,\omega^*) = (f\cdot e)^*\left(s, [\omega^*]_{r-r_0}\right), \tag{60}$$

so that to complete the proof of the theorem, we may take

$$f^*_{[\omega^*]_{r-r_0}}(s) = (f\cdot e)^*_{[\omega^*]_{r-r_0}}(s). \tag{61}$$

□

Remark 15. For example, if $r_0 = r = 1$ in Theorem 14, then $f^*_{[\omega^*]r-r_0}(s) = f^*(s)$. It is worthwhile to note that if $0 < \alpha < 1$ and $\alpha p = 1$ (so, $p = \alpha^{-1} > 1$), then the pointwise (C,α)-convergence for Φ^* does not hold in general (see [24]). For the case of a positive linear contraction on $L_{1/\alpha}(m\otimes\mu^*)$, see Irmisch [25].

In particular, applying Irmisch's theorem to sub-Markovian operators, we have the following.

Theorem 16. *Let $\Pi = \{P_{[\omega^*]_r} : \omega^* \in \Omega^*\}$ be a strongly $\mathcal{F}_r^*$-measurable family of positive linear contractions on $L_1(m)$, where each $P_{[\omega^*]_r}$ is the adjoint operator of the corresponding sub-Markovian operator $P^*_{[\omega^*]_r}$. Assume that m is $E(U\cdot \mid \wp_0)$-subinvariant. Let $0 < \alpha < 1$, $\alpha p > 1$ and $f \in L_p(m)$. Then, there exists a μ^*-null set N^* such that for every $\omega^* \in \Omega^* - N^*$, there exists a function $f^*_{[\omega^*]_{r-r_0}}(\cdot) \in L_p(m)$ such that*

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{A_n^\alpha}\sum_{k=0}^{n} A_{n-k}^{\alpha-1} P_{[\omega^*]_r} P_{[\varphi^{r_0}\omega^*]_r} \cdots P_{[\varphi^{kr_0}\omega^*]_r} \\
&\quad = \lim_{\lambda\to 1+0} (\lambda-1)\sum_{n=0}^{\infty} \lambda^{-(n+1)} P_{[\omega^*]_r} P_{[\varphi^{r_0}\omega^*]_r} \cdots P_{[\varphi^{kr_0}\omega^*]_r} \\
&\quad = f^*_{[\omega^*]_{r-r_0}}(s) \quad m\text{-a.e.}
\end{aligned} \tag{62}$$

Proof. In view of Lemma 11, U is a positive linear contraction on $L_1(m\otimes\mu^*)$ as well as on $L_\infty(m\otimes\mu^*)$. Thus, it follows from the Riesz convexity theorem that $\|U\|_{L_p(m\otimes\mu^*)} \le 1$ for $1 < p < \infty$. Therefore, we reach the assertion of Theorem 16 through Theorems 7 and 13 appealed to Irmisch's theorem [25]. □

Remark 17. The relations between the random ergodic limit functions and the random parameters have been investigated (with satisfactory formulations) only in discrete parameter cases so far. So, it is very interesting to study the continuous analogs of the theorems obtained above. But no continuous results are known from the point of view of the dependence of the limit functions on the random parameters. Here, it is worthwhile to notice that Anzai has obtained a continuous version of Kakutani's random ergodic theorem for Brownian motion in continuous parameter cases (see [26]). Let $\xi = (\xi_t)_{t\geq 0}$ $(\xi_t = \xi_t(\omega), \xi_0 = 0)$ be a Brownian motion (or Wiener process) on a probability space $(\Omega, \mathcal{F}, \mu)$. This process has independent increments; that is, for arbitrary $t_1 < t_2 < \cdots < t_n$, the random variables $\xi_{t_2} - \xi_{t_1}, \ldots, \xi_{t_n} - \xi_{t_{n-1}}$ are independent. In fact, since the process is Gaussian with $E(\xi_t) = 0$ and $E(\xi_t \xi_s) = \min(t, s)$ by definition, it is sufficient to verify only that the increments are uncorrelated. Thus, if $s < t < u < v$, then

$$E((\xi_t - \xi_s)(\xi_v - \xi_u)) = \frac{1}{2}((t-s) + (v-u) - |t-s+v-u|) = 0. \tag{63}$$

Let $\{\psi_t : t \geq 0\}$ be an arbitrary ergodic measurable semiflow on a finite measure space (S, β, m). Then, Anzai's result may be stated as follows: for any $f \in L_1(m)$ and for almost all $\omega \in \Omega$, there exists a null set $N = N_f(\omega) \in \mathcal{F}$ such that

$$\lim_{\alpha\to\infty} \frac{1}{\alpha} \int_0^\alpha f\left(\psi_{\xi_t(\omega)} s\right) dt = \int_S f(s)\, dm \tag{64}$$

holds for any $s \in S - N$. This is an immediate consequence of the ergodicity of the measure-preserving skew product semiflow $\{\Psi_t t \geq 0\}$ defined by $\Psi_t(\omega, s) = (\varphi_t \omega, \psi_{\xi_t(\omega)} s)$, where $\{\varphi_t : t \geq 0\}$ is the ergodic semiflow on Ω given by $\xi_a(\varphi_t \omega) = \xi_{a+t}(\omega)$. It is an interesting and important problem to generalize Anzai's result for Brownian motions to the case of contraction operator quasisemigroups in $L_1(m)$ associated with $\{\varphi_t\}$. We do not discuss it in the present paper.

4. Applications to Nonlinear Random Ergodic Theorems

The random ergodic theorems obtained above can be applied to the nonlinear random ergodic theorems for affine systems (see [27]). An affine operator U on $L_1(m)$ is an operator of the type $Uf = Tf + h$, where T is a linear contraction on $L_1(m)$, and h is a fixed element of $L_1(m)$. Then, U is nonlinear and nonexpansive. The fixed points of U are solutions of Poisson's equation for T, which is $(I-T)f = h$. When we assume T to be mean ergodic, the averages

$$\frac{1}{n}\sum_{k=1}^{n} U^k f = \frac{1}{n}\sum_{k=1}^{n} T^k f + \frac{1}{n}\sum_{k=1}^{n}\sum_{j=0}^{k-1} T^j h \tag{65}$$

converge if and only if $h \in (I-T)L_1(m)$. If $h \in (I-T)L_1(m)$, then there exists a unique $\xi \in \overline{(I-T)L_1(m)}$ such that $(I - T)\xi = h$, and the limit of $(1/n)\sum_{k=1}^n U^k f$ is $Ef + \xi$, where Ef is the limit of $(1/n)\sum_{k=1}^n T^k f$. Therefore, iterating $Uf = Tf + (I - T)\xi$ will yield almost everywhere convergence of the averages of $U^n f$ for any $f \in L_1(m)$. Now, let $\{(T_{[\omega^*]_r}, U_{[\omega^*]_r}, \xi_{\omega^*}, \varphi^{r_0}) : \omega^* \in \Omega^*\}$ be a random affine system on $L_1(m)$ such that $\{T_{[\omega^*]_r} : \omega^* \in \Omega^*\}$ is a strongly measurable family of linear contractions on $L_1(m)$ as well as on $L_\infty(m)$ and such that for some $g \in L_1(m \otimes \mu^*)$,

$$U_{[\omega^*]_r} = T_{[\omega^*]_r} + [(I-T)g]_{\omega^*}, \quad \omega^* \in \Omega^*, \tag{66}$$

where

$$Tf(s, \omega^*) = \left[T_{[\omega^*]_r} f_{\varphi^{r_0}\omega^*}\right](s), \tag{67}$$

for $f \in L_1(m \otimes \mu^*)$ (cf. [11, 27]). For each $f \in L_1(m \otimes \mu^*)$, we define a sequence of random functions $\{V_f(n, \omega^*) : \omega^* \in \Omega^*\}$, $n = 0, 1, \ldots$, in $L_1(m)$ inductively by

$$\begin{aligned} V_f(0, \omega^*) &= f_{\omega^*}, \\ V_f(1, \omega^*) &= f_{\omega^*} + U_{[\omega^*]_r} V_f(0, \varphi^{r_0}\omega^*), \\ &\vdots \\ V_f(n+1, \omega^*) &= f_{\omega^*} + U_{[\omega^*]_r} V_f(n, \varphi^{r_0}\omega^*). \end{aligned} \tag{68}$$

Theorem 18. *Let $\{V_f(n, \omega^*) : \omega^* \in \Omega^*\}$ be the sequence of random functions associated with $f \in L_1(m \otimes \mu^*)$ which is determined by a random affine system $\{(T_{[\omega^*]_r}, U_{[\omega^*]_r}, [(I - T)g]_{\omega^*}, \varphi^{r_0}) : \omega^* \in \Omega^*\}$ given in $L_1(m)$ with $\|T_{[\omega^*]_r}\|_{L_\infty(m)} \leq 1$. Then, there exists a μ^*-null set N^* such that for any $\omega^* \in \Omega^* - N^*$, there exists a function $\widetilde{f}_{[\omega^*]_{r-r_0}} \in L_1(m)$ such that*

$$\lim_{n\to\infty} \frac{V_f(n, \omega^*)(s)}{n+1} = \widetilde{f}_{[\omega^*]_{r-r_0}}(s) \quad m\text{-a.e.} \tag{69}$$

Proof. It follows that there exists a μ^*-null set N_1^* such that for any $\omega^* \in \Omega^* - N_1^*$,

$$\begin{aligned} V_f(n, \omega^*)(s) = f_{\omega^*}(s) &+ \sum_{k=1}^{n} T_{[\omega^*]_r} T_{[\varphi^{r_0}\omega^*]_r} \cdots T_{[\varphi^{(k-1)r_0}\omega^*]_r} f_{\varphi^{kr_0}\omega^*}(s) \\ &- T_{[\omega^*]_r} T_{[\varphi^{r_0}\omega^*]_r} \cdots T_{[\varphi^{(n-1)r_0}\omega^*]_r} g_{\varphi^{nr_0}\omega^*}(s) \\ &+ \sum_{k=1}^{n-1} T_{[\omega^*]_r} T_{[\varphi^{r_0}\omega^*]_r} \cdots T_{[\varphi^{(k-1)r_0}\omega^*]_r} g_{\varphi^{kr_0}\omega^*}(s) \\ &+ g_{\omega^*}(s). \end{aligned} \tag{70}$$

Moreover, we have

$$\lim_{n\to\infty}\frac{f(s,\omega^*)}{n+1}=\lim_{n\to\infty}\frac{g(s,\omega^*)}{n+1}=0\quad m\text{-a.e.},$$
$$\lim_{n\to\infty}\frac{1}{n+1}T_{[\omega^*]_r}T_{[\varphi^{r_0}\omega^*]_r}\cdots T_{[\varphi^{(n-1)r_0}\omega^*]_r}g_{\varphi^{nr_0}\omega^*}(s)=0\quad m\text{-a.e.} \tag{71}$$

Therefore, by Theorem 7, there exists a μ^*-null set N_2^* such that for every $\omega^*\in\Omega^*-N_2^*$, there exist functions $f^*_{[\omega^*]_{r-r_0}}$, $g^*_{[\omega^*]_{r-r_0}}\in L_1(m)$ such that

$$\lim_{n\to\infty}\frac{V_f(n,\omega^*)(s)}{n+1}=f^*_{[\omega^*]_{r-r_0}}(s)+g^*_{[\omega^*]_{r-r_0}}(s)\quad m\text{-a.e.} \tag{72}$$

Hence, the theorem follows by putting $N^*=N_1^*\cup N_2^*$ and

$$\widetilde{f}\left(s,[\omega^*]_{r-r_0}\right)=f^*\left(s,[\omega^*]_{r-r_0}\right)+g^*\left(s,[\omega^*]_{r-r_0}\right). \tag{73}$$

□

As far as we are concerned with the ergodic behaviors of Cesàro-type processes for nonexpansive operators on L_p, one can only expect weak convergence in general. In fact, the pointwise convergence of the $(C,1)$ averages of nonlinear and nonexpansive operators on L_p may fail to hold. In addition, these $(C,1)$ averages do not need converge in the strong operator topology of L_p (see [27, 28]). The so-called nonlinear sums introduced by Wittmann [29] make it possible to consider the pointwise convergence and the strong convergence under some additional conditions. To make the most of advantageous results in the theory of linear ergodic theorems, it is very rational to consider a class of affine operators as a model case (cf. [27]). Under the above setting, observe that

$$\begin{aligned}&\left[U_{[\omega^*]_r}U_{[\varphi^{r_0}\omega^*]_r}\cdots U_{[\varphi^{kr_0}\omega^*]_r}f_{\varphi^{(k+1)r_0}\omega^*}\right](s)\\&=\left[T_{[\omega^*]_r}T_{[\varphi^{r_0}\omega^*]_r}\cdots T_{[\varphi^{kr_0}\omega^*]_r}f_{\varphi^{(k+1)r_0}\omega^*}\right](s)\\&\quad+\sum_{j=1}^{k}\left[T_{[\omega^*]_r}T_{[\varphi^{r_0}\omega^*]_r}\cdots T_{[\varphi^{(j-1)r_0}\omega^*]_r}g_{\varphi^{jr_0}\omega^*}\right](s)\\&\quad+g_{[\omega^*]_r}(s),\quad k=1,2,\ldots.\end{aligned} \tag{74}$$

Then, Theorem 7, together with Theorems 2.1 and 2.2 of [27], yields the following theorem.

Theorem 19. *Let $\{(T_{[\omega^*]_r},U_{[\omega^*]_r},[(I-T)g]_{\omega^*},\varphi^{r_0}):\omega^*\in\Omega^*\}$ be a random affine system given in $L_1(m)$ with $\|T_{[\omega^*]_r}\|_{L_\infty(m)}\le 1$. If $f\in L_1(m\otimes\mu^*)$, then there exists a μ^*-null set N^* such that for any $\omega^*\in\Omega^*-N^*$, there exists a function $f^*_{[\omega^*]_{r-r_0}}\in L_1(m)$ such that*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}U_{[\omega^*]_r}U_{[\varphi^{r_0}\omega^*]_r}\cdots U_{[\varphi^{(k-1)r_0}\omega^*]_r}f_{\varphi^{kr_0}\omega^*}(s)=f^*_{[\omega^*]_{r-r_0}}(s)\quad m\text{-a.e.} \tag{75}$$

5. Examples

Example 1. Let $\{\tau_\omega:\omega\in\Omega\}$ be a $\beta\otimes\mathcal{F}$-measurable family of m-measure-preserving transformations on S, and let ϕ be μ-measure-preserving transformation on Ω. Then, for any $f\in L_1(m)$, there exists a μ-null set $N^*\in\beta$ such that for each $\omega\in\Omega-N^*$, there exists a function $f^*_\omega\in L_1(m)$ such that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}f\left(\tau_{\phi^{k-1}\omega}\cdots\tau_\omega s\right)=f^*_\omega(s)\quad m\text{-a.e.}, \tag{76}$$

which follows from Birkhoff's ergodic theorem applied to the so-called skew product Φ of ϕ and $\{\tau_\omega\}$ defined by

$$\Phi(s,\omega)=\left(\tau_\omega s,\phi\omega\right),\quad (s,\omega)\in S\times\Omega. \tag{77}$$

If the skew product transformation Φ is ergodic, then the function $f^*_\omega(s)$ is constant almost everywhere on $S\times\Omega$. It is worthwhile to notice that, in general, the limit function $f^*_\omega(s)$ depends on the two variables s and ω. To illustrate this, we consider the measure spaces (S,β,m) and $(\Omega,\mathcal{F},\mu)$ and transformations given by

$$\begin{aligned}&S=\left\{s_1,\ldots,s_p\right\},\\&m\{s_1\}=\cdots=m\left\{s_p\right\}=\frac{1}{p}\quad(p\text{: integer},\ p\ge 2),\\&\Omega=\left\{\omega_1,\ldots,\omega_{2q}\right\},\\&\mu\{\omega_1\}=\cdots=\mu\left\{\omega_{2q}\right\}=\frac{1}{2q}\quad(q\text{: integer},\ 2q\ge p),\\&\tau_{\omega_{2j-1}}s_i=\begin{cases}s_{i+1}, & \text{if } 1\le i\le p-1,\\ s_1, & \text{if } i=p,\end{cases}\quad j=1,2,\ldots,q,\\&\tau_{\omega_{2j}}=\tau^{-1}_{\omega_{2j-1}},\quad j=1,2,\ldots,q,\\&\phi\omega_j=\begin{cases}\omega_{j+1}, & \text{if } 1\le j\le 2q-1,\\ \omega_1, & \text{if } j=2q.\end{cases}\end{aligned} \tag{78}$$

Thus, for instance, if we consider a function $f(s,\omega)$ defined by

$$\begin{aligned}f_{\omega_j}(s_i)&=i\cdot\delta_{ji},\quad\text{if } j=i\ (i,j=1,2,\ldots,p),\\&=0,\quad\text{if } i=1,2,\ldots,p;\ \text{and } j=p+1,\ldots,2q\end{aligned} \tag{79}$$

where δ_{ji} denotes the Kronecker delta, then the limit function $f^*_\omega(s)$ determined by (76) depends essentially on the two variables s and ω (therefore, the limit function $f^*_\omega(s)$ is not necessarily independent of the random variable ω.) In fact, one can easily find that under the above setting

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{2n}\sum_{k=1}^{2n}f_{\omega_1}\left(\tau_{\phi^{2k-1}\omega_1}\cdots\tau_{\omega_1}s_p\right)=f^*_{\omega_1}\left(s_p\right)=\frac{1}{2},\\&\lim_{n\to\infty}\frac{1}{2n}\sum_{k=1}^{2n}f_{\omega_p}\left(\tau_{\phi^{2k-1}\omega_1}\cdots\tau_{\omega_1}s_p\right)=f^*_{\omega_p}\left(s_p\right)=\frac{p}{2}.\end{aligned} \tag{80}$$

See also Example 3 below.

The following example given by Gładysz [5] will be a great help to understand the subject of this paper.

Example 2 (see Gladysz [5]). In this example, we consider the measure spaces (S, β, m) and $(\Omega, \mathscr{F}, \mu)$ taken to be $S = \Omega = [0, 1)$, $\beta = \mathscr{F}$ = the σ-field of Borel sets, and $m = \mu$ = the Lebesgue measure. Let r be a fixed integer with $r \geq 2$, and let β_j, $j = 1, 2, \ldots, r$, be real constants such that

$$\beta_1 = \cdots = \beta_{r-1} = \frac{1}{r}, \qquad \beta_r = -1 + \frac{1}{r}. \tag{81}$$

Define a $\beta \otimes \mathscr{F}_r^*$-measurable family $\{\tau_{[\omega^*]_r} : \omega^* \in \Omega^*\}$ of measure-preserving transformations on S by

$$\tau_{[\omega^*]_r} s = s + \sum_{i=1}^{r} \beta_i \omega_i \quad (\text{mod} 1), \ s \in S. \tag{82}$$

Take $r_0 = 1$. Then, for a function $f \in L_1(m)$ given by $f(s) = e^{2\pi i s} = \exp(s)$, there exists a μ^*-null set N^* such that for each $\omega^* \in \Omega^* - N^*$, there exists a function $f^*_{[\omega^*]_{r-1}}(\cdot) \in L_1(m)$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\tau_{[\varphi^{k-1}\omega^*]_r} \cdots \tau_{[\omega^*]_r} s\right) = f^*_{[\omega^*]_{r-1}}(s) \quad m\text{-a.e.}, \tag{83}$$

where

$$\begin{aligned} f^*&\left(s, [\omega^*]_{r-1}\right) \\ &= K \cdot \exp(s) \cdot \exp\left[\beta_1 \omega_1 + \cdots + (\beta_1 + \cdots + \beta_{r-1}) \omega_{r-1}\right], \\ &\quad H(\omega^*) = \exp\left[(\beta_2 + \cdots + \beta_r) \omega_1 + \cdots + \beta_r \omega_{r-1}\right], \\ K &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} H\left(\varphi^k \omega^*\right) \\ &= \int_{\Omega^*} H(\omega^*)\, d\mu^*(\omega^*) \quad (\varphi \text{ is ergodic}) \\ &= \exp\left[\frac{1}{r}\left(\omega_1 + 2\omega_2 + \cdots + (r-1)\omega_{r-1}\right)\right] \quad (\neq 0). \end{aligned} \tag{84}$$

Supplement. Let $0 < \alpha < 1$, $\alpha p > 1$, and $r_0 = 1$. Then, for the function $f(s) = \exp(s)$ ($\in L_p(m)$), there exists a μ^*-null set N^* such that for any $\omega^* \in \Omega^* - N^*$, there exists a function $f^* \in L_p(m \otimes \mu^*)$ such that (if necessary, apply Hille's theorem [22])

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} f\left(\tau_{[\varphi^k\omega^*]_r} \cdots \tau_{[\varphi\omega^*]_r} \tau_{[\omega^*]_r} s\right) \\ &\quad = \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} f\left(\tau_{[\varphi^k\omega^*]_r} \cdots \tau_{[\varphi\omega^*]_r} \tau_{[\omega^*]_r} s\right) \\ &\quad = \lim_{\lambda\to 1+0} (\lambda - 1) \sum_{n=1}^{\infty} \lambda^{-(n+1)} f\left(\tau_{[\varphi^{n-1}\omega^*]_r} \cdots \tau_{[\varphi\omega^*]_r} \tau_{[\omega^*]_r} s\right) \\ &\quad = f^*_{[\omega^*]_{r-1}}(s) \quad m\text{-a.e.} \end{aligned} \tag{85}$$

Next, if (for example) $r = 5$ and $r_0 = 3$, we let $\beta_1, \ldots, \beta_5$ be real constants such that

$$\begin{gathered} \beta_1 \neq 0, \qquad \beta_2 \neq 0, \qquad \beta_3 = 0, \\ \beta_1 + \beta_2 \neq 0, \qquad \beta_4 = -\beta_1, \qquad \beta_5 = -\beta_2. \end{gathered} \tag{86}$$

Using the skew product transformation Θ defined by

$$\Theta(s, \omega^*) = \left(\tau_{[\omega^*]_5} s, \varphi^3 \omega^*\right), \tag{87}$$

we have for the function $f(s) = \exp(s)$,

$$\begin{aligned} f&\left(\Theta^k(s, \omega^*)\right) \\ &= \exp\left[s + \sum_{j=1}^{5} \beta_j \omega_j + \sum_{j=1}^{5} \beta_j \omega_{3+j} + \cdots + \sum_{j=1}^{5} \beta_j \omega_{3k+j}\right] \\ &= \exp\left[s + \beta_1 \omega_1 + \beta_2 \omega_2\right], \quad k = 1, 2, \ldots. \end{aligned} \tag{88}$$

Hence it follows at once that for almost all $\omega^* \in \Omega^*$, there exists a function $f^* \in L_1(m \otimes \mu^*)$ such that

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} f\left(\tau_{[\varphi^{3k}\omega^*]_5} \cdots \tau_{[\varphi^3\omega^*]_5} \tau_{[\omega^*]_r} s\right) \\ &\quad = \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} f\left(\tau_{[\varphi^{3k}\omega^*]_5} \cdots \tau_{[\varphi^3\omega^*]_5} \tau_{[\omega^*]_r} s\right) \\ &\quad = \lim_{\lambda\to 1+0} (\lambda - 1) \sum_{n=0}^{\infty} \lambda^{-(n+1)} f\left(\tau_{[\varphi^{3k}\omega^*]_5} \cdots \tau_{[\varphi^3\omega^*]_5} \tau_{[\omega^*]_r} s\right) \\ &\quad = \exp\left[s + \beta_1 \omega_1 + \beta_2 \omega_2\right] = f^*_{[\omega^*]_{5-3}}(s) \\ &\quad = f^*_{[\omega^*]_2}(s) \quad m\text{-a.e.} \end{aligned} \tag{89}$$

Example 3. In the setting of Example 2, let q be an β-measurable function with $|q(s)| = 1$. Then, for $f \in L_p(m \otimes \mu^*)$, $1 \leq p < \infty$, there exists a μ^*-null set N^* such that for any $\omega^* \in \Omega^* - N^*$, there exists a function $H_{\omega^*} \in L_p(m)$ such that

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} q\left(\tau_{[\omega^*]_r} s\right) q\left(\tau_{[\varphi\omega^*]_r} \tau_{[\omega^*]} s\right) \cdots \\ &\quad q\left(\tau_{[\varphi^{k-1}\omega^*]_r} \cdots \tau_{[\omega^*]_r} s\right) \\ &\quad \times f\left(\tau_{[\varphi^k\omega^*]_r} \cdots \tau_{[\omega^*]_r} s\right) = H_{\omega^*}(s), \\ &\quad H_{\omega^*}(s) = q\left(\tau_{[\omega^*]_r} s\right) H_{\varphi\omega^*}\left(\tau_{[\omega^*]_r} s\right), \end{aligned} \tag{90}$$

for m-almost all $s \in S$. In fact, letting $Q(s,\omega^*) = q(s)\cdot e(\omega^*)$ with $e = 1 \in L_p(\mu^*)$ and

$$\Phi^*(s,\omega^*) = \left(\tau_{[\omega^*]_r} s, \varphi\omega^*\right), \tag{91}$$

we have by Gladysz's theorem ([5], Satz 5) that there exists a function $F \in L_p(m\otimes\mu^*)$ such that

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} &Q(\Phi^*(s,\omega^*))\,Q\left(\Phi^{*2}(s,\omega^*)\right)\cdots\\ &Q\left(\Phi^{*k}(s,\omega^*)\right) f\left(\Phi^{*k+1}(s,\omega^*)\right)\\ &= [Q(s,\omega^*)]^{-1}\cdot F(s,\omega^*)\left(= q(s)^{-1}F(s,\omega^*)\right),\\ F(s,\omega^*) &= Q(s,\omega^*)\,F(\Phi^*(s,\omega^*))\\ &= q(s)\,F(\Phi^*(s,\omega^*)),\end{aligned} \tag{92}$$

almost everywhere on $S\times\Omega^*$. Hence, we may take

$$H(s,\omega^*) = q(s)^{-1}F(s,\omega^*), \tag{93}$$

and then

$$H(s,\omega^*) = q\left(\tau_{[\omega^*]_r} s\right) H(\Phi^*(s,\omega^*)) \quad m\otimes\mu^*\text{-a.e.} \tag{94}$$

Moreover, since q is also $\beta\otimes\mathcal{F}_r^*$-measurable and $H \in L_1(m\otimes\mu^*)$ if we consider the function

$$\overline{q}(s,\omega_1,\dots,\omega_r) = q\left(s + \sum_{i=1}^{r}\beta_i\omega_i\right) \tag{95}$$

Then, by Gladysz's lemma ([5], Hilfssatz 3) there exists a $\beta\otimes\mathcal{F}_{r-1}^*$-measurable function $g = g(s,\omega_1,\dots,\omega_{r-1})$ such that

$$H(s,\omega^*) = g(s,\omega_1,\dots,\omega_{r-1}) \quad m\otimes\mu^*\text{-a.e.} \tag{96}$$

Consequently, we have for any $\omega^* \in \Omega^* - N^*$,

$$H_{\omega^*}(s) = g_{[\omega^*]_{r-1}}(s) \quad m\text{-a.e.} \tag{97}$$

Remark 20. It is an interesting problem to ask what happens if we transform a function $f \in L_p$ with a random sequence $T_1, T_2, \dots, T_n, \dots$, of operators chosen at random from some stock of linear operators on L_p given in advance. What can we say about the limit $\lim_{n\to\infty}(1/n)\sum_{k=1}^{n} T_1T_2\cdots T_k f$? Unfortunately, we cannot expect any convergence for every random sequence chosen from the stock. Therefore, it is desirable to consider how to choose almost every (not every) random sequence from the stock (cf. Revesz [30] and Yoshimoto [31]). In Pitt [1] and Ulam and von Neumann [2], the random ergodic theorem for two-measure-preserving transformations U, V (cited in Section 1) means the existence of the a.e. limit of the form $\lim_{n\to\infty}(1/n)\sum_{k=1}^{n} f(T_1T_2\cdots T_k s)$ for almost every sequence $(T_n)_{n\ge1}$ of the infinite sequences obtained by applying U and V in turn at random. This is just the case that the transformations T_n, $n \ge 1$, are chosen at random with the same distribution and independently. See also Remark 3 (the case that the transformations T_n, $n \ge 1$, are chosen at random, not necessarily with the same distribution but independently). In general the random system $\{(T_{[\omega^*]_r}, \varphi) : \omega^* \in \Omega^*\}$ of linear contractions on $L_p(m)$ as given in Theorem 4 plays a role of such an advance stock of linear contractions on $L_1(m)$. To illustrate this, we let $S = X = [0,1)$ and consider the $\beta\otimes\mathcal{F}_r^*$-measurable, m-measure-preserving transformations $\pi_{[\omega^*]_r}$, $\omega^* \in \Omega^*$, defined by

$$\pi_{[\omega^*]_r} s = s + \sum_{j=1}^{r}\delta_j\omega_j \quad (\mathrm{mod}\,1), \quad \left(s\in S, \delta_j \neq 0\right). \tag{98}$$

In this case, the random system $\{(\pi_{[\omega^*]_r}, \varphi) : \omega^* \in \Omega^*\}$ is taken as a stock of measure-preserving transformations on S. If $\omega_1, \dots, \omega_r$ are linearly independent irrational numbers, then $\{\pi_{[\omega^*]_r}\}$ is ergodic. Thus, for any $f \in L_1(m)$ with period 1,

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f\left(s + k\sum_{j=1}^{r}\delta_j\omega_j\right) &= \lim_{\lambda\to 1+0}(\lambda-1)\sum_{n=1}^{\infty}\lambda^{-(n+1)} f\left(s + n\sum_{j=1}^{r}\delta_j\omega_j\right)\\ &= \int_0^1 f(s)\,dm \quad m\text{-a.e.}\end{aligned} \tag{99}$$

This is an immediate consequence of the ergodicity of the family $\{\pi_{[\omega^*]_r}\}$. We can state this fact in terms of stochastic processes. For example, see Gładysz [5], Satz 3.

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