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Research Article

Saddle-Node Heteroclinic Orbit and Exact Nontraveling Wave Solutions for (2+1)D KdV-Burgers Equation

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We have undertaken the fact that the periodic solution of (2+1)D KdV-Burgers equation does not exist. The Saddle-node heteroclinic orbit has been obtained. Using the Lie group method, we get two-(1+1)-dimensional PDE, through symmetric reduction; and by the direct integral method, spread F-expansion method, and (G'/G)-expansion method, we obtain exact nontraveling wave solutions, for the (2+1)D KdV Burgers equation, and find out some new strange phenomenons of sympathetic vibration to evolution of nontraveling wave.

1. Introduction

We consider the (2+1)-dimensional Korteweg-de Vries Burgers ((2+1)D KdV Burgers) equation

$$(u_t + uu_x - \beta u_{xx} + \alpha u_{xxx})_x + \gamma u_{yy} = 0, \tag{1}$$

where $u : R_x \times R_y \times R_t^+ \to R$, α, β, and γ are real parameters. Equation (1) is model equation for wide class of nonlinear wave models in an elastic tube, liquid with small bubbles, and turbulence [1–3]. Much attention has been put on the study of their exact solutions by some methods [4], such as, a complex line soliton by extended tanh method with symbolic computation [5], exact traveling wave solutions including solitary wave solutions, periodic wave and shock wave solutions by extended mapping method, and homotopy perturbation method [6, 7].

It is well known that the investigation of exact solutions of nonlinear evolution equations plays an important role in the study of nonlinear physical phenomena. Many effective methods have been presented [7–22], such as functional variable separation method [8, 9], homotopy perturbation method [12], F-expansion method [7, 13], Lie group method [14, 15], variational iteration method [16], homoclinic test method [17–19], Exp-function method [20, 21], and homogeneous balance method [22]. Practically, there is no unified method that can be used to handle all types of nonlinearity.

In this paper, we will discuss the existence of periodic traveling wave solution and seek the Saddle-Node heteroclinic orbit, and further use the Lie group method with the aid of the symbolic computation system Maple to construct the non-traveling wave solutions for (1).

2. Existence of Periodic Traveling Wave Solution of (1)

Introducing traveling wave transformation in this form

$$u(x, y, t) = u(\xi), \quad \xi = px + qy - ct \tag{2}$$

permits us to convert (1) into an ODE for $u = u(\xi)$

$$p\left(puu_\xi - \beta p^2 u_{\xi\xi} + \alpha p^3 u_{\xi\xi\xi}\right)_\xi - r u_{\xi\xi} = 0, \tag{3}$$

where $r = pc - q^2\gamma$, Integrating (3) with respect to ξ twice and taking integration constant to A yields

$$2\alpha p^4 u_{\xi\xi} - 2\beta p^3 u_\xi + p^2 u^2 - 2ru = A. \tag{4}$$

Letting $u_\xi = v$, thus nonlinear ordinary differential equation (4) is equivalent to the autonomous dynamic system as follows:

$$\frac{du}{d\xi} = v, \tag{5}$$

$$\frac{dv}{d\xi} = \frac{1}{2\alpha p^4}\left(2\beta p^3 v - p^2 u^2 + 2ru + A\right). \tag{6}$$

The dynamic system (5) has two balance points:

$$P_1(u_1, v_1) = \left(\frac{r + \sqrt{r^2 + p^2 A}}{p^2}, 0 \right),$$
$$P_2(u_2, v_2) = \left(\frac{r - \sqrt{r^2 + p^2 A}}{p^2}, 0 \right). \tag{7}$$

The Jacobi matrixes at the balance points for the right-hand side of (5) are obtained as follows, respectively:

$$J_1 = \begin{pmatrix} 0 & 1 \\ -\frac{\sqrt{r^2 + p^2 A}}{p^4 \alpha} & \frac{\beta}{p\alpha} \end{pmatrix},$$
$$J_2 = \begin{pmatrix} 0 & 1 \\ \frac{\sqrt{r^2 + p^2 A}}{p^4 \alpha} & \frac{\beta}{p\alpha} \end{pmatrix}. \tag{8}$$

Their latent equations are expressed, respectively, as,

$$p^3 \lambda (p\alpha\lambda - \beta) + \sqrt{r^2 + p^2 A} = 0,$$
$$p^3 \lambda (p\alpha\lambda - \beta) - \sqrt{r^2 + p^2 A} = 0. \tag{9}$$

Relevant latent roots are as follows respectively:

$$\lambda_1 = \frac{p\beta \pm \sqrt{p^2\beta^2 - 4\alpha\sqrt{r^2 + p^2 A}}}{2p^2\alpha},$$
$$\lambda_2 = \frac{p\beta \pm \sqrt{p^2\beta^2 + 4\alpha\sqrt{r^2 + p^2 A}}}{2p^2\alpha}. \tag{10}$$

Obviously, if $p^2\beta^2 > 4\alpha\sqrt{r^2 + p^2 A}$, then λ_1 are two positive real roots, therefore P_1 is a nonsteady node point. If $0 < p^2\beta^2 < 4\alpha\sqrt{r^2 + p^2 A}$, then λ_1 are conjugate complex roots and real part is positive, so P_1 is a nonsteady focus point. And λ_2 is a positive and minus real root, thus P_2 is a saddle point. From (5), we know the phase trajectory on the phase plane satisfies

$$\frac{dv}{du} = \frac{2\beta p^3 v - p^2 u^2 + 2ru + A}{2\alpha p^4 v}. \tag{11}$$

Integrating (11), we can obtain

$$H(u, v) = Au + ru^2 - \frac{1}{3} p^2 u^3 + 2\beta p^3 uv - \alpha p^4 v^2, \tag{12}$$

where $H(u, v)$ is a total energy or Hamilton function of system (4). Apparently

$$u_\xi \neq -\frac{\partial H}{\partial v}, \qquad v_\xi \neq \frac{\partial H}{\partial u}. \tag{13}$$

Consequently, the system expressed in (12) is not a conservative one, then periodic traveling wave solution of (1) does not exist.

We conclude the above analysis in the following theorem.

Theorem 1. *Under the traveling wave transformation, the periodic solution of (2+1)-dimensional KdV-Burgers equation does not exist.*

But, saddle-node heteroclinic orbits and nontraveling periodic solution do exist, which will be discussed later in this paper.

3. Saddle-Node Heteroclinic Orbits of KdV-Burgers Equation

First, we assume the solutions of (4) in the form

$$u(\xi) = \frac{r + \sqrt{r^2 + p^2 A}}{p^2} + \frac{b}{(1 + e^{a\xi})^2}. \tag{14}$$

Substituting (14) into (4) yields

$$2\left(4\alpha p^4 a^2 + \sqrt{r^2 + p^2 A} + 2\beta p^3 a\right) e^{2a\xi}$$
$$- 4\left(\alpha p^4 a^2 - \sqrt{r^2 + p^2 A} - \beta p^3 a\right) e^{a\xi}$$
$$+ 2\sqrt{r^2 + p^2 A} + p^2 b = 0. \tag{15}$$

Then we get

$$4\alpha p^4 a^2 + \sqrt{r^2 + p^2 A} + 2\beta p^3 a = 0,$$
$$\alpha p^4 a^2 - \sqrt{r^2 + p^2 A} - \beta p^3 a = 0, \tag{16}$$
$$2\sqrt{r^2 + p^2 A} + p^2 b = 0.$$

Solving the system (16) gets

$$a = -\frac{\beta}{5\alpha p}, \qquad b = -\frac{12\beta^2}{25\alpha}, \qquad \sqrt{r^2 + p^2 A} = \frac{6p^2\beta^2}{25\alpha}. \tag{17}$$

Substituting (17) into (14) obtains

$$u(\xi) = \frac{r + \sqrt{r^2 + p^2 A}}{p^2} - \frac{12\beta^2}{25\alpha} \frac{1}{\left(1 + e^{-(\beta/5\alpha p)\xi}\right)^2}$$
$$= u_1 - \frac{3\beta^2}{25\alpha}\left(1 + \tanh\frac{\beta}{20\alpha}\xi\right)^2. \tag{18}$$

Evidently, $\xi \to -\infty \Rightarrow u(\xi) \to u_1$, $\xi \to +\infty \Rightarrow u(\xi) \to u_1 - (6\beta^2/25\alpha) = u_2$. Thus (18) is a saddle-node heteroclinic orbit through nonsteady node point P_1 and saddle point P_2 [23].

Ecumenic, taking the Hamiliton function $H(u,v) = B$, we obtain

$$\frac{du}{d\xi} = v = \frac{3p\beta u \pm \sqrt{3u\left[3A\alpha + 3\left(p^2\beta^2 + r\alpha\right)u - p^2\alpha u^2\right] - 9B\alpha}}{3\alpha p^2}, \tag{19}$$

where B is an arbitrary constant. Integrating (19) with respect to ξ we have

$$\int^{u(\xi)} \frac{3\alpha p^2}{3p\beta s \pm \sqrt{3s\left[3A\alpha + 3\left(p^2\beta^2 + r\alpha\right)s - p^2\alpha s^2\right] - 9B\alpha}} ds = \xi + \xi_0, \tag{20}$$

where ξ_0 is an arbitrary constant. We can see that (4) has the general solution (20) and all partial cases as include above result can be found from the general solution of (20). Example, take $\alpha\sqrt{r^2 + p^2 A} - p^2\beta^2 = 0$, $3B\alpha + A\beta^2 = 0$, $r\alpha + p^2\beta^2 = 0$ in (20), we find a solution of (4) as follows:

$$u(\xi) = -\frac{3\beta^2}{4\alpha}\left[1 + \tanh\left(\frac{\beta}{4p\alpha}\xi + \xi_0\right)\right]^2. \tag{21}$$

It is a heteroclinic orbit too.

4. Li Symmetry of (1)

This section devotes to Li symmetry of (1) [14, 15]. Let

$$\sigma = \sigma\left(x, y, t, u, u_t, u_x, u_y, \ldots\right). \tag{22}$$

be the Li symmetry of (1). From Lie group theory, σ satisfies the following equation

$$\sigma_{xt} + 2u_x\sigma_x + u\sigma_{xx} + \sigma u_{xx} - \beta\sigma_{x^3} + \alpha\sigma_{x^4} + \gamma\sigma_{yy} = 0. \tag{23}$$

We take the function σ in the form

$$\sigma = a_1 u_x + a_2 u_y + a_3 u_t + a_4 u + a_5, \tag{24}$$

where $a_i = a_i(x,y,t) : R_x \times R_y \times R_t^+ \rightarrow R$ $(i = 1, \ldots, 5)$ are functions to be determined later. Substituting (3) into (2) yields

$$a_1 = -\frac{1}{2\gamma}k_2'(t)\,y + k_1(t), \qquad a_2 = k_2(t),$$
$$a_3 = c, \qquad a_4 = 0, \qquad a_5 = \frac{1}{2\gamma}k_2''(t)\,y - k_1'(t), \tag{25}$$

where $k_j(t)$ $(j = 1, 2)$ are arbitrary functions of t, c is an arbitrary constant. Substituting (25) into (24), we obtain the Li symmetries of (1) as follows:

$$\sigma = \left[-\frac{1}{2\gamma}k_2'(t)\,y + k_1(t)\right]u_x + k_2(t)\,u_y + cu_t + \frac{1}{2\gamma}k_2''(t)\,y - k_1'(t). \tag{26}$$

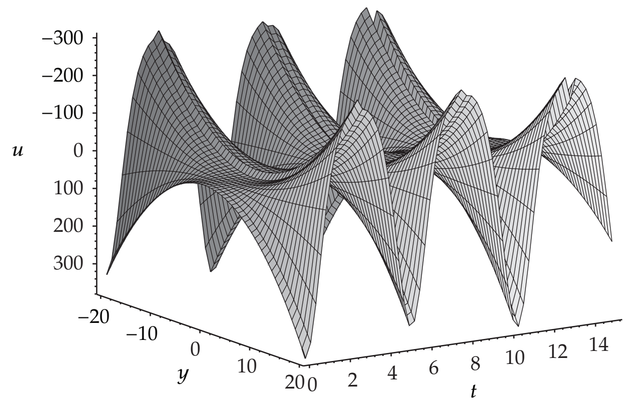

Figure 1: The strange phenomenon which is a sympathetic vibration of periodicity on the t-axis and paraboloid on y-axis for $u_1(x,y,t)$ as $x = 1$.

5. Symmetry Reduction and Solutions of (1)

Based on the integrability of reduced equation of symmetry (26), we are to consider the following three cases.

Case 1. Taking $k_2(t) = 0$ and $c = 0$ in (26) yields

$$\sigma = k_1(t)\,u_x - k_1'(t). \tag{27}$$

The solution of the differential equation $\sigma = 0$ is

$$u = \frac{k_1'(t)}{k_1(t)}x + F(y,t), \quad F(y,t) : R_y \times R_t^+ \rightarrow R. \tag{28}$$

Substituting (28) into (1) yields the function $F(y,t)$ which satisfies the following linear PDE:

$$\frac{k_1''}{k_1} + \gamma\frac{\partial^2 F}{\partial y^2} = 0. \tag{29}$$

By integrating both sides, we find out the following result:

$$F(y,t) = -\frac{k_1''}{2\gamma k_1}y^2 + k_3(t)\,y + k_4(t), \tag{30}$$

where $k_3(t)$, $k_4(t)$ are new arbitrary functions of t. Substituting (30) into (28), we can get the solutions of (1) as follows:

$$u_1(x,y,t) = \frac{k_1'(t)}{k_1(t)}x - \frac{k_1''}{2\gamma k_1}y^2 + k_3(t)\,y + k_4(t). \tag{31}$$

(1) Given $k_i(t) = \mathrm{cn}(t, 0.95)$ $(i = 1, 3, 4)$, $x = 1$, $\gamma = 0.6$ in (31), the local structure of u_1 is obtained (Figure 1). Where $\mathrm{cn}(t, 0.95)$ is an Jacobian elliptic cosine function.

(2) Given $k_1(t) = \mathrm{sech}(t)$, $k_3(t) = \sin(t)$, $k_4(t) = \mathrm{cn}(t, 0.1)$, $y = 1$, $\gamma = 0.6$ in (31), the local structure of u_1 is obtained (Figure 2).

Case 2. Take $k_1(t) = t$, $k_2(t) = 1$ and $c = 0$ in (26), then

$$\sigma = tu_x + u_y - 1. \tag{32}$$

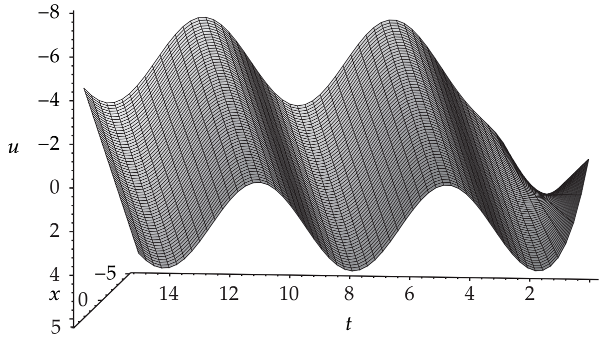

FIGURE 2: The periodic solution which is a periodic nontraveling wave traveling on the t-axis for $u_1(x, y, t)$ as $y = 1$.

Solving the differential equation $\sigma = 0$, we can get

$$u = y + F(t, \xi), \quad \xi = x - ty. \tag{33}$$

Substituting (33) into (1) and integrating once with respect to ξ yield

$$F_t + FF_\xi + \gamma t^2 F_\xi - \beta F_{\xi\xi} + \alpha F_{\xi\xi\xi} = 0. \tag{34}$$

Again, further using the transformation of dependent variable to (34),

$$F(t, \xi) = F(\theta), \quad \theta = k\left(t - \frac{1}{3}\gamma t^3 + \xi\right). \tag{35}$$

Substituting (35) into (34) and integrating once with respect to θ yield

$$2k^2\alpha F'' - 2k\beta F' + F^2 + 2F + A = 0, \tag{36}$$

where A is an integration constant, $F' = dF(\theta)/d\theta$. We assume that the solution of (36) can be expressed in the form

$$F(\theta) = a_0 + a_1 w(\theta) + a_2 w(\theta)^2, \tag{37}$$

where a_i $(i = 0, 1, 2)$ are constants to be determined later, $w(\theta)$ satisfies the following auxiliary equation

$$w' = p + qw^2. \tag{38}$$

Substituting (37) and (38) into (36) and equating the coefficients of all powers of w to zero yield a set of algebra equations for a_0, a_1, a_2, and A as follows.

$$\begin{aligned}
&w^4\colon a_2\left(a_2 + 12\alpha k^2 q^2\right) = 0, \\
&w^3\colon -4\beta k a_2 q + 2a_1 a_2 + 4\alpha k^2 a_1 q^2 = 0, \\
&w^2\colon a_1^2 + 16\alpha k^2 a_2 q p - 2\beta k a_1 q + 2a_2 + 2a_2 a_0 = 0, \\
&w^1\colon 2a_1 a_0 - 4\beta k a_2 p + 2a_1 + 4\alpha k^2 a_1 q p = 0, \\
&w^0\colon 2a_0 + A + 4\alpha k^2 a_2 p^2 + a_0^2 - 2\beta k a_1 p = 0.
\end{aligned} \tag{39}$$

Solving the system of function equations with the aid of Maple, we obtain

$$a_0 = \frac{3\beta^2 - 25\alpha}{25\alpha}, \qquad a_1 = \frac{6\beta^2 q}{25s\alpha}, \qquad a_2 = \frac{3\beta^2 q}{25\alpha p}. \tag{40}$$

when $k = \beta/10s\alpha$, $pq < 0$, $A = (625\alpha^2 - 36\beta^4)/625\alpha^2$, where $s = \sqrt{-pq}$.

It is known that solutions of (38) are as follows [24]:

$$w(\theta) = -s\tanh(s\theta), \qquad w(\theta) = -s\coth(s\theta). \tag{41}$$

Substituting (41), (40), (37), and (35) into (33), we obtain solutions of (1) as follows:

$$\begin{aligned}
&u_2(x, y, t) \\
&\quad = \frac{1}{25\alpha}\left\{3\beta^2 - 25\alpha - 3q\beta^2\right. \\
&\qquad \times\left[\tanh\left(\frac{\beta}{10s\alpha}\left(x - ty + t - \frac{\gamma}{3}t^3\right)\right)\right. \\
&\qquad\quad \left.\left. - 2pq\tanh^2\left(\frac{\beta}{10s\alpha}\left(x - ty + t - \frac{\gamma}{3}t^3\right)\right)\right]\right\} \\
&\qquad + y, \\
&u_3(x, y, t) \\
&\quad = \frac{1}{25\alpha}\left\{3\beta^2 - 25\alpha - 3q\beta^2\right. \\
&\qquad \times\left[\coth\left(\frac{\beta}{10s\alpha}\left(x - ty + t - \frac{\gamma}{3}t^3\right)\right)\right. \\
&\qquad\quad \left.\left. - 2pq\coth^2\left(\frac{\beta}{10s\alpha}\left(x - ty + t - \frac{\gamma}{3}t^3\right)\right)\right]\right\} \\
&\qquad + y.
\end{aligned} \tag{42}$$

(see Figures 3 and 4).

Remark 2. If we direct assume that the solution of (34) can be expressed in the form

$$F(t, \xi) = a_0(t) + a_1(t)\, w(\theta) + a_2(t)\, w(\theta)^2, \tag{43}$$

where $\theta = f(t)\xi + g(t)$, $f(t)$, and $g(t)$ are continuous functions of t to be determined later. $w(\theta)$ satisfies the auxiliary equation (38). Substituting (43) and (38) into (34), equating the coefficients of all powers of w to zero yields a set of

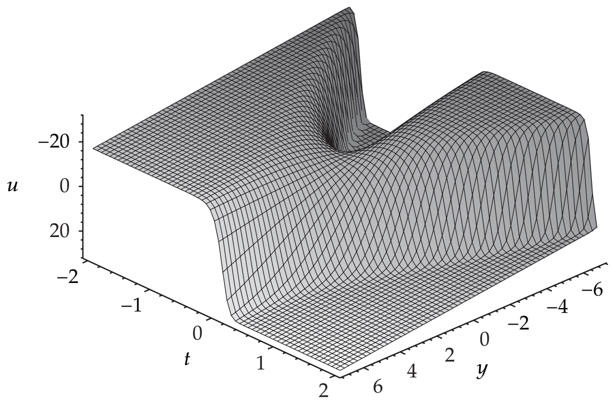

FIGURE 3: Local structure of $u_2(x, y, t)$ is shown as $x = 1$, $\alpha = 1$, $\beta = 10$, $p = -1$, $q = 1$, and $\gamma = 6$.

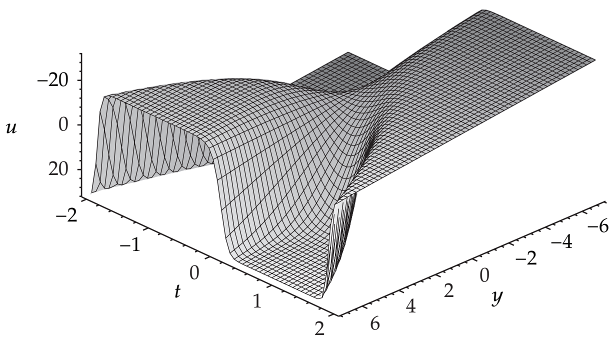

FIGURE 4: Local structure of $u_3(x, y, t)$ is shown as $x = 1$, $\alpha = 1$, $\beta = 10$, $p = -1$, $q = 1$, $\gamma = 6$.

function equations for $a_0(t)$, $a_1(t)$, $a_2(t)$, $f(t)$, and $g(t)$ as follows:

$$\begin{aligned}
w^5&: 2fa_2q\left(12f^2q^2\alpha + a_2\right) = 0,\\
w^4&: -3fq\left(-2a_1q^2f^2\alpha + 2qfa_2\beta - a_1a_2\right) = 0,\\
w^3&: -2\beta a_1f^2q^2 + 2a_2^2fp + 40\alpha a_2f^3q^2p + 2a_2g'q\\
&\quad + 2a_0a_2fq + a_1^2fq + 2a_2f'\xi q + 2\gamma t^2a_2fq = 0,\\
w^2&: -8\beta a_2f^2pq + a_1g'q + a_2' + a_0a_1fq + \gamma t^2a_1fq\\
&\quad + 8\alpha a_1f^3pq^2 + a_1f'\xi q + 3a_1a_2fq = 0,\\
w^1&: a_1^2fp + 16\alpha a_2f^3p^2q + a_1' + 2\gamma t^2a_2fp + 2a_0a_2fp\\
&\quad + 2a_2g'p + 2a_2f'\xi p - 2\beta a_1f^2pq = 0,\\
w^0&: a_1g'p + a_1f'\xi p + a_0a_1fp - 2\beta a_2f^2p^2 + a_0'\\
&\quad + 2\alpha a_1f^3p^2q + \gamma t^2a_1fp = 0.
\end{aligned} \tag{44}$$

Solving the system of function equations, we obtain

$$a_0(t) = \frac{3\beta^2}{25\alpha}, \qquad a_1(t) = \pm\frac{6\beta^2q}{25s\alpha},$$
$$a_2(t) = \frac{3\beta^2q}{25\alpha p}, \qquad f(t) = \pm\frac{\beta}{10s\alpha}, \qquad g(t) = \mp\frac{\beta\gamma}{30s\alpha}t^3. \tag{45}$$

This result indicate the idea is equivalent to idea of Case 2 above.

Case 3. Take $k_2(t) = 0$ and $c = 1$ in (26), then

$$\sigma = k_1(t)u_x + u_t - k_1(t). \tag{46}$$

Solving the differential equation $\sigma = 0$, we obtain

$$u = k_1(t) + F(\xi, y), \quad \xi = x - \int k_1(t)\,dt. \tag{47}$$

Substituting (47) into (1) yield

$$\alpha F_{\xi\xi\xi\xi} - \beta F_{\xi\xi\xi} + FF_{\xi\xi} + F_\xi^2 + \gamma F_{yy} = 0. \tag{48}$$

Using the transformation $F(\xi, y) = F(\eta)$, $\eta = k\xi - cy$ and integrating the resulting equation with respect to η we have

$$k^2F^2 + 2\gamma c^2F + 2k^4\alpha F'' - 2k^3\beta F' + A = 0, \tag{49}$$

where A is an arbitrary constant, $F' = dF/d\eta$. Suppose that the solution of ODE (49) can be expressed by a polynomial in (G'/G) as follows:

$$F(\eta) = b_n\left(\frac{G'}{G}\right)^n + \cdots, \tag{50}$$

where $G = G(\eta)$ satisfies the second-order LODE in the form [25]

$$G'' + \lambda G' + \mu G = 0. \tag{51}$$

Balancing F'' with F^2 in (49) gives $n = 2$. So that

$$F(\eta) = b_2\left(\frac{G'}{G}\right)^2 + b_1\left(\frac{G'}{G}\right) + b_0, \quad b_2 \neq 0, \tag{52}$$

where b_i $(i = 0, 1, 2)$ and μ are constants to be determined later. Substituting (52) and (51) into (49). Setting these coefficients of the G'/G to zero, yields a set of algebraic equations as follows:

$$\begin{aligned}
&k^2b_2\left(12\alpha k^2 + b_2\right) = 0,\\
&2k^2\left(10\alpha k^2b_2\lambda + b_1b_2 + 2\alpha k^2b_1 + 2\beta kb_2\right) = 0,\\
&8\alpha k^4b_2\lambda^2 + 2\beta k^3b_1 + k^2b_1^2 + 16\alpha k^4b_2\mu + 2k^2b_2b_0\\
&\qquad + 6\alpha k^4b_1\lambda + 4\beta k^3b_2\lambda + 2\gamma c^2b_2 = 0,\\
&2k^2b_1b_0 + 4\beta k^3b_2\mu + 4\alpha k^4b_1\mu + 2\gamma c^2b_1 + 2\alpha k^4b_1\lambda^2\\
&\qquad + 2\beta k^3b_1\lambda + 12\alpha k^4b_2\lambda\mu = 0,\\
&2\gamma c^2b_0 + 2\alpha k^4b_1\lambda\mu + A + 2\beta k^3b_1\mu + 4\alpha k^4b_2\mu^2 + k^2b_0^2 = 0.
\end{aligned} \tag{53}$$

Solving the algebraic equations above yields

$$b_0 = \frac{15k^3\lambda\alpha\left(5k\lambda\alpha + 2\beta\right) - 3k^2\beta^2 + 25c^2\alpha\gamma}{25k^2\alpha},$$
$$b_1 = -\frac{12k\left(5k\alpha\lambda + \beta\right)}{5}, \qquad b_2 = -12k^2\alpha. \tag{54}$$

when $25k^2\alpha^2(4\mu - \lambda^2) + \beta^2 = 0$ and $625\alpha^2(Ak^2 - c^2\gamma^2) + 36k^4\beta^4 = 0$. Consequently, we obtain the following solution of (1) for $\lambda^2 - 4\mu > 0$:

$$\begin{aligned} u_4(x,y,t) &= -12k^2\alpha\tau^2 \\ &\quad \times \Bigg[\left(C_1 \sinh\tau\left(k\left(x - \int k_1(t)\,dt\right) - cy\right)\right. \\ &\qquad \left. + C_2 \cosh\tau\left(k\left(x - \int k_1(t)\,dt\right) - cy\right)\right) \\ &\qquad \times \left(C_1 \cosh\tau\left(k\left(x - \int k_1(t)\,dt\right) - cy\right)\right. \\ &\qquad + C_2 \sinh\tau \\ &\qquad \left.\times\left(k\left(x - \int k_1(t)\,dt\right) - cy\right)\right)^{-1}\Bigg]^2 \\ &\quad + \left(12k^2\lambda\alpha\tau - \frac{12k\left(5k\alpha\lambda + \beta\right)}{5}\right) \\ &\quad \times \Bigg[\left(C_1 \sinh\tau\left(k\left(x - \int k_1(t)\,dt\right) - cy\right)\right. \\ &\qquad \left. + C_2 \cosh\tau\left(k\left(x - \int k_1(t)\,dt\right) - cy\right)\right) \\ &\quad \times \left(C_1 \cosh\tau\left(k\left(x - \int k_1(t)\,dt\right) - cy\right)\right. \\ &\qquad + C_2 \sinh\tau \\ &\qquad \left.\times\left(k\left(x - \int k_1(t)\,dt\right) - cy\right)\right)^{-1}\Bigg]^2 \\ &\quad + \frac{15k^3\lambda\alpha\left(5k\lambda\alpha + 2\beta\right) - 3k^2\beta^2 + 25c^2\alpha\gamma}{25k^2\alpha} \\ &\quad + \frac{\lambda^2}{4}, \end{aligned} \tag{55}$$

where $\tau = (1/2)\sqrt{\lambda^2 - 4\mu}$.

6. Conclusions

Based on the fact that the periodic solution of (2+1)D KdV-Burgers equation does not exist, we have obtained Saddle-node Heteroclinic Orbits. By applying the Lie group method, we reduce the (2+1)D KdV Burgers equation to (1+1)-dimensional equations including the (1+1)-dimensional linear partial differential equation with constants coefficients (29), (48) and (1+1)-dimensional nonlinear partial differential equation with variable coefficients (34). By solving the equations (29), (34), and (48), we obtain some new exact solutions and discover the strange phenomenon of sympathetic vibration to evolution of nontraveling wave soliton for the (2+1)D KdV Burgers equation. Our results show that the unite of Lie group method with others is effective to search simultaneously exact solutions for nonlinear evolution equations. Other structures of solutions with symmetry (26) are to be further studied.

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