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Research Article

Almost Everywhere Convergence of Riesz Means Related to Schrödinger Operator with Constant Magnetic Fields

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We study almost everywhere convergence for Riesz means related to Schrödinger operator with constant magnetic fields. Through researching the weighted norm estimates for the maximal operator with power-weight functions, we obtain the desired result, which is similar to the work given by Anthony Carbery, Jose L. Rubio de Francia, and Luis Vega.

1. Introduction

The magnetic Schrödinger operator (MSO) with constant magnetic fields H_b in $\mathbb{R}^n$ is of the form

$$H_b = -\left(\nabla + \frac{iBz}{2}\right)^2, \quad z \in R^n, \tag{1}$$

where B is a real antisymmetric matrix. If B is degenerated (this requires $n = 2d + l$ to be odd, and $2d$ is the rank of the matrix B), then its eigenvalues have the form $\pm ib_j$, $j = 1, 2, \ldots, d$. In properly chosen coordinates $z = (z_1, z_2, \ldots, z_n) = (x_1, y_1, x_2, y_2, \ldots, x_d, y_d, z_{2d+1}, \ldots, z_{2d+l})$, the operator becomes

$$H_b = -\sum_{j=1}^{d}\left(\left(\partial_{x_j} + i\frac{b_j}{2}y_j\right)^2 + \left(\partial_{y_j} + i\frac{b_j}{2}x_j\right)\right) - \Delta_l, \tag{2}$$

where $\Delta_l = \sum_{j=1}^{l}\partial^2/\partial z_{2d+j}^2$ is the Laplacian in R^l. The spectrum of H_b is in the semiaxis starting from the point $c = \sum b_j$, and its spectral expansion is continuous (see [1]).

Let b_j's be positive. Denote by E_t the spectral function of H_b. It is an integral operator with a kernel $e_t(z, z')$, which is skew-translation invariant; that is, $e_t(z + \omega, z' + \omega) = e_t(z, z')\exp i\langle B(z - z'), \omega\rangle$.

For $\beta > 0$, set the Riesz means of order β as

$$S_\lambda^\beta f(x) = \int_0^\infty \left(1 - \frac{t}{\lambda}\right)_+^\beta dE_t f(x) = \int_c^\infty \left(1 - \frac{t}{\lambda}\right)_+^\beta dE_t f(x), \tag{3}$$

and the kernel of S_λ^β as

$$s_\lambda^\beta(z, z') = s^\beta(\lambda, z, z') = \int_0^\infty \left(1 - \frac{t}{\lambda}\right)_+^\beta de_t(z, z'), \tag{4}$$

with the same skew-translation invariance. We will denote by S_*^β the corresponding maximal operator; that is,

$$S_*^\beta f(x) = \sup_{0<\lambda<\infty}\left|S_\lambda^\beta f(x)\right|. \tag{5}$$

As an indispensable part in harmonic analysis, many people investigate the convergence of Bochner-Riesz means for Fourier transform in norm and almost everywhere, which is defined as

$$\left(T_R^\beta f\right)^\wedge(\xi) = \left(1 - \frac{|\xi|^2}{R}\right)_+^\beta \hat{f}(\xi). \tag{6}$$

Since the convergence of T_R^β in L^p-norm is equivalent to the boundedness of T_1^β in $L^P(\mathbb{R}^n)$, persons look for the L^p

boundedness of it. For $2n/(n+1+2\beta) < p < 2n/(n-1-2\beta)$ and $0 \le \beta \le (n-1/2)$, Carleson, Cordoba, and Fefferman turn out the boundedness in $\mathbb{R}^2$ (see [2–4]). When $n > 2$, it is only proven for $\beta \ge (n-1)/2(n+1)$ (see [4, 5]). Returning the problem about almost everywhere convergence of T_R^{β}, Carbery has finished it for $\beta > 0$ and $2n/(n+1+2\beta) < p < 2n/(n-1-2\beta)$ in two dimensions in 1983 (see [6]). For higher dimensions, it is completed by Christ only for $\beta \ge (n-1)/2(n+1)$ (see [7]). In the special case, $\beta = 0$, Fefferman studies the L^2-boundeness of S_λ^0 for $n \ge 2$ (see [8]). Not until 1988 was it solved by Carbery et al. for $\beta > 0$, $n \ge 2$, and $2 \le p < 2n/(n-1-2\beta)$ (see [9]).

In [1], Rozenblum and Tashchiyan investigated the L^p-norm convergence for Riesz means for Schrödinger operator with constant magnetic fields. They showed that under the restriction theorem similar to one for Fourier transform in [5], it is of the same results as Bochner-Riesz means in $\mathbb{R}^n$. However, very few results are considered for almost everywhere convergence of Riesz means for Schrödinger operator with constant magnetic fields. In the paper, we are interested in it. Through researching the boundedness of the maximal operator in $L(\mathbb{R}^n, |x|^\alpha dx)$, we get the desired result.

2. Main Results

Theorem 1. *Set $\beta > 0$ and $n \ge 2$. Write $p_\beta = 2n/(n-1-2\beta)$. If $f \in L^p(\mathbb{R}^n)$ with $2 \le p < p_\beta$, then $\lim_{\lambda\to\infty} S_\lambda^\beta f(x) = f(x)$ almost everywhere.*

Usually, we replace the almost everywhere convergence of Riesz means with L^p estimates for the maximal operator. However, we only need to think about weighted L^2 estimates for the maximal operator S_*^β, as follows. In fact, based on the idea in [9], for $2 \le p < p_\beta$, there exists a number α with $0 \le \alpha < 1 + 2\beta$ such that $L^p \subseteq L^2 + L^2(|x|^\alpha)$. We have gotten L^2 boundedness of the maximal operator in another paper.

Theorem 2. *Suppose that $\beta > 0$ and $0 \le \alpha < 1 + 2\beta \le n$. Then,*

$$\int \left|S_*^\beta f(x)\right|^2 |x|^{-\alpha} dx \le C_{\alpha,\beta} \int \left|f(x)\right|^2 |x|^{-\alpha} dx. \tag{7}$$

In order to prove the theorem, we introduce the following lemmas, which are the essential tools. In the following lemmas, we reduce the maximal operator to the one generated by a multiplier with compact support. Since it is easy to see that the boundedness of the maximal operator generated by the multipliers is independent of the index β, the dimension $n = 2d + l$ will not play an important role in the following estimates.

Lemma 3 (see [9]). *If $0 < \varepsilon < 1/2$, then*

$$\int_{||x|-1|\le\varepsilon} \left|f(x)\right|^2 dx \le C_\alpha B_\alpha(\varepsilon) \int \left|f(x)\right|^2 |x|^\alpha dx, \tag{8}$$

where

$$B_\alpha(\varepsilon) = \begin{cases} \varepsilon^\alpha, \varepsilon\left|\log\varepsilon\right|, & 0 \le \alpha < 1, \\ \varepsilon\left|\log\varepsilon\right|, & \alpha = 1, \\ \varepsilon, & 1 < \alpha < n. \end{cases} \tag{9}$$

For a small $\delta > 0$, let $m^\delta(t)$ be a C^∞ function supported in $[1-\delta, 1]$ and satisfy

$$0 \le m^\delta(t) \le 1, \qquad \left|D^k m^\delta(t)\right| \le C\delta^{-k} \quad \forall k \in \mathbb{N}. \tag{10}$$

Define

$$\begin{gathered} K_\lambda^\delta f(x) = \int m^\delta(\lambda t)\, dE_t f(x), \\ K_*^\delta f(x) = \sup_{\lambda>0} \left|K_\lambda^\delta f(x)\right|. \end{gathered} \tag{11}$$

Let $\phi(t) \in C^\infty(\mathbb{R})$ supported in $[-1, 2]$ and $\phi(t) = 1$ for $0 \le t \le 1$. Set

$$h_l(t) = \begin{cases} \phi(t), & l = 0, \\ \phi\left(2^{-l}t\right) - \phi\left(2^{1-l}t\right), & l \ge 1. \end{cases} \tag{12}$$

Denoting by χ_E the character function of the set E, write

$$H_j(x, y) = \begin{cases} \chi_{[0,1)}\left(|x-y|\right), & j = 0, \\ \chi_{[2^{j-1},2^j)}\left(|x-y|\right), & j \ge 1. \end{cases} \tag{13}$$

Let $k^\delta(t)$ be a function with Fourier transform as

$$\widehat{k}_\lambda^\delta(t) = m^\delta(\lambda t). \tag{14}$$

Set

$$\begin{gathered} k_\lambda^{\delta,l,j}(t) = k_\lambda^\delta(t)\, h_l\left(\frac{\delta^2 2^{-j} t}{\lambda}\right), \\ E_t^j f(x) = \int_{\mathbb{R}^n} e_t(x, y) H_j(x, y) f(y)\, dy. \end{gathered} \tag{15}$$

Accordingly, define the operator $K_\lambda^{\delta,l,j}$ as

$$K_\lambda^{\delta,l,j} f(x) = \int \widehat{k}_\lambda^{\delta,l,j}(t)\, dE_t^j f(x), \tag{16}$$

where $\widehat{k}_\lambda^{\delta,l,j}$ is the Fourier transform of $k_\lambda^{\delta,l,j}$. Apparently, since

$$K_\lambda^\delta f(x) = \int \widehat{k}_\lambda^\delta(t)\, dE_t f(x), \tag{17}$$

we decompose

$$\begin{aligned} K_\lambda^\delta f(x) &= \int \widehat{k}_\lambda^\delta(t)\, dE_t f(x) \\ &= \sum_{j=0}^\infty \sum_{l=0}^\infty K_\lambda^{\delta,l,j} f(x) \\ &= \sum_{j=0}^\infty \sum_{l=0}^\infty \int \widehat{k}_\lambda^{\delta,l,j}(t)\, dE_t^j f(x). \end{aligned} \tag{18}$$

Lemma 4. *For $\lambda > 0$, one has*

$$\left\| K_{\lambda}^{\delta,l,j} f(x) \right\|_2^2 \le C 2^{-2M(j+l)} \delta^{2M} \|f\|_2^2, \tag{19}$$

where the constant C is independent of λ and δ.

Proof. With the method similar to the proof of Lemma 4 in [9], we write $h(t) = \phi(t) - \phi(2t)$ and expand m^{δ} into a Taylor series around λt. Then,

$$\begin{aligned} \widehat{k}_{\lambda}^{\delta,l,j}(t) &= \int m^{\delta}\left(\lambda\left(t - \frac{2^{-(j+l)}\delta^2 r}{\lambda}\right)\right)\widehat{h}(r)\,dr \\ &= \int m^{\delta}\left(\lambda t - 2^{-(j+l)}\delta^2 r\right)\widehat{h}(r)\,dr \\ &= \int R_M(t,r)\,\widehat{h}(r)\,dr, \end{aligned} \tag{20}$$

where the remainder R_M satisfies

$$\begin{aligned} &\left|R_M(t,r)\right| \\ &\quad \le \left|D^M m^{\delta}\right| \left|2^{-(j+l)}\delta^2 r\right|^M \le 2^{-M(l+j)}\delta^M |r|^M. \end{aligned} \tag{21}$$

But $\widehat{h}$ is a Schwartz function and can be integrated against $|r|^M$. Hence,

$$\left|\widehat{k}_{\lambda}^{\delta,l,j}(t)\right| \le C_M 2^{-M(j+l)}\delta^M. \tag{22}$$

Since $E_{\mathbb{R}}$ is a resolution of the identity, we see

$$E_{\mathbb{R}} f(x) = \int_0^{\infty} dE_t f(x) = f(x). \tag{23}$$

Denote by $e_{\mathbb{R}}(x,y)$ the kernel of $E_{\mathbb{R}}$. For almost all $x_0 \in \mathbb{R}^n$, we let $S = \{y \in \mathbb{R}^n : e_{\mathbb{R}}(x_0,y) f(y) > 0\}$. Decompose

$$\begin{aligned} f(x) &= f(x)\chi_S(x) + f(x)\chi_{\mathbb{R}^n/S}(x) \\ &= f_1(x) + f_2(x). \end{aligned} \tag{24}$$

We have

$$\begin{aligned} \left|E_{\mathbb{R}}^j f(x_0)\right| &= \left|\int_{\mathbb{R}^n} e_{\mathbb{R}}(x_0,y) H_j(x_0,y) f(y)\,dy\right| \\ &= \left|\int_{\mathbb{R}^n} e_{\mathbb{R}}(x_0,y) H_j(x_0,y) f_1(y)\,dy \right. \\ &\qquad \left. + \int_{\mathbb{R}^n} e_{\mathbb{R}}(x_0,y) H_j(x_0,y) f_2(y)\,dy\right| \\ &\le \left|\int_{\mathbb{R}^n} e_{\mathbb{R}}(x_0,y) H_j(x_0,y) f_1(y)\,dy\right| \\ &\quad + \left|\int_{\mathbb{R}^n} e_{\mathbb{R}}(x_0,y) H_j(x_0,y) f_2(y)\,dy\right| \\ &= \int_{\mathbb{R}^n} \left|e_{\mathbb{R}}(x_0,y) H_j(x_0,y) f_1(y)\right| dy \\ &\quad + \int_{\mathbb{R}^n} \left|e_{\mathbb{R}}(x_0,y) H_j(x_0,y) f_2(y)\right| dy \\ &\le \int_{\mathbb{R}^n} \left|e_{\mathbb{R}}(x_0,y) f_1(y)\right| dy \\ &\quad + \int_{\mathbb{R}^n} \left|e_{\mathbb{R}}(x_0,y) f_2(y)\right| dy \\ &= \left|\int_{\mathbb{R}^n} e_{\mathbb{R}}(x_0,y) f_1(y)\,dy\right| \\ &\quad + \left|\int_{\mathbb{R}^n} e_{\mathbb{R}}(x_0,y) f_2(y)\,dy\right| \\ &\le \left|f_1(x_0)\right| + \left|f_2(x_0)\right| \\ &\le C\left|f(x_0)\right|. \end{aligned} \tag{25}$$

It is easy to show

$$\begin{aligned} &\left\|K_{\lambda}^{\delta,l,j} f(x)\right\|_2^2 \\ &\quad = \left\|\int \widehat{k}_{\lambda}^{\delta,l,j}(t)\,dE_t^j f(x)\right\|_2^2 \\ &\quad = \left\|\int dE_t^j\left(\widehat{k}_{\lambda}^{\delta,l,j}(H_b) f(x)\right)\right\|_2^2 \\ &\quad = \left\langle E_{\mathbb{R}}^j\left(\widehat{k}_{\lambda}^{\delta,l,j}(H_b) f\right), E_{\mathbb{R}}^j\left(\widehat{k}_{\lambda}^{\delta,l,j}(H_b) f\right)\right\rangle \\ &\quad \le \left\|\widehat{k}_{\lambda}^{\delta,l,j}(H_b) f\right\|_2^2 \\ &\quad \le \int_0^{\infty} \left|\widehat{k}_{\lambda}^{\delta,l,j}(t)\right|^2 d\left(E_t f, f\right) \\ &\quad \le C_M 2^{-2M(j+l)}\delta^{2M} \int_0^{\infty} d\left(E_t f, f\right) \\ &\quad \le C_M 2^{-2M(j+l)}\delta^{2M} \|f\|_2^2. \end{aligned} \tag{26}$$

□

Lemma 5. *If* $0<\alpha<n$ *and* $\lambda>0$, *then*

$$\int\left|\int \widehat{k}_{\lambda}^{\delta}(t)\,dE_t f(x)\right|^2\frac{dx}{|x|^{\alpha}}\le C_{\alpha}\delta\int|f(x)|^2\frac{dx}{|x|^{\alpha}}, \qquad (27)$$

where C_α *is independent of* δ *and* λ.

Proof. Suppose that f is supported in $\{|x|\le C2^j\}$. Write $f=\chi_{\{0\le|x|\le 1/4\}}(x)f(x)+\chi_{\{C<|x|\le C2^j\}}(x)f(x)+\chi_{\{1/4<|x|\le C\}}(x)f(x)=f_1+f_2+f_3$. If $C\le 1/4$, then $f_3=0$. Since

$$\begin{aligned}&\left(\int\left|K_{\lambda}^{\delta}f(x)\right|^2\frac{dx}{|x|^{\alpha}}\right)^{1/2}\\&\quad=\left(\int\left|K_{\lambda}^{\delta}(f_1+f_2+f_3)(x)\right|^2\frac{dx}{|x|^{\alpha}}\right)^{1/2}\\&\quad\le\left(\int\left|K_{\lambda}^{\delta}f_1\right|^2\frac{dx}{|x|^{\alpha}}\right)^{1/2}+\left(\int\left|K_{\lambda}^{\delta}f_2\right|^2\frac{dx}{|x|^{\alpha}}\right)^{1/2}\\&\qquad+\left(\int\left|K_{\lambda}^{\delta}f_3\right|^2\frac{dx}{|x|^{\alpha}}\right)^{1/2},\end{aligned} \qquad (28)$$

we only need to prove that

$$\begin{aligned}&\int\left|\int\widehat{k}_{\lambda}^{\delta}(t)\,dE_t f_i(x)\right|^2\frac{dx}{|x|^{\alpha}}\\&\quad\le C_{\alpha}\delta\int|f(x)|^2\frac{dx}{|x|^{\alpha}},\quad(i=1,2,3).\end{aligned} \qquad (29)$$

For the case of $i=2$, with Lemma 4, it follows that

$$\begin{aligned}&\int\left|K_{\lambda}^{\delta,j,l}f_2(x)\right|^2dx\\&\quad\le C2^{-l}2^{-jM}\delta^{M}\|f_2\|_2^2\\&\quad\le C2^{-l}2^{-jM}\delta^{M}\int_{C\le|x|\le C2^j}|f(x)|^2dx\\&\quad\le C2^{-l}2^{-jM}\delta^{M}\\&\qquad\times\sum_{k=0}^{j-1}\int_{C2^k\le|x|\le C2^{k+1}}|f(x)|^2dx.\end{aligned} \qquad (30)$$

It is easy to see that

$$\begin{aligned}&\sum_{k=0}^{j-1}\int_{C2^k<|x|\le C2^{k+1}}|f(x)|^2dx\\&\quad\le C\sum_{k=0}^{j-1}2^{-k\alpha}\int_{C2^k<|x|\le C2^{k+1}}|f(x)|^2|x|^{\alpha}dx\\&\quad\le C\int|f(x)|^2|x|^{\alpha}dx.\end{aligned} \qquad (31)$$

Thus, we have

$$\begin{aligned}&\int\left|K_{\lambda}^{\delta,j,l}f_2(x)\right|^2dx\\&\quad\le C2^{-l}2^{-jM}\delta^{M}\int|f(x)|^2|x|^{\alpha}dx.\end{aligned} \qquad (32)$$

Choosing $M>1$, we get

$$\begin{aligned}&\int\left|K_{\lambda}^{\delta,l,j}f_2(x)\right|^2dx\\&\quad\le C_{\alpha}2^{-l}2^{-jM}\delta\int|f(x)|^2|x|^{\alpha}dx.\end{aligned} \qquad (33)$$

On the other hand, E_t is self-adjoint. So,

$$\begin{aligned}&\langle E_t f,g\rangle\\&\quad=\int\int e_t(x,y)f(y)\,dy\,g(x)\,dx\\&\quad=\int f(y)\int e_t(x,y)g(x)\,dx\,dy\\&\quad=\langle f,E_t g\rangle\\&\quad=\int f(y)\int e_t(y,x)g(x)\,dx\,dy.\end{aligned} \qquad (34)$$

Hence,

$$e_t(x,y)=e_t(y,x). \qquad (35)$$

With

$$H_j(x,y)=H_j(y,x), \qquad (36)$$

we get

$$\begin{aligned}e_t^j(x,y)&=e_t(x,y)H_j(x,y)\\&=e_t(y,x)H_j(y,x)=e_t^j(y,x),\end{aligned} \qquad (37)$$

and it implies that $K_{\lambda}^{\delta,l,j}$ is also self-adjoint; that is,

$$\langle E_t^j f,g\rangle=\langle f,E_t^j g\rangle. \qquad (38)$$

Therefore, by duality,

$$\begin{aligned}&\int\left|K_{\lambda}^{\delta,l,j}f_2(x)\right|^2\frac{dx}{|x|^{\alpha}}\\&\quad\le C_{\alpha}2^{-l}2^{-jM}\delta\int|f(x)|^2dx\\&\quad\le C_{\alpha}2^{-l}2^{j(\alpha-M)}\delta\int|f(x)|^2\frac{dx}{|x|^{\alpha}}.\end{aligned} \qquad (39)$$

Taking $M>\alpha+1$, we can establish the inequality

$$\begin{aligned}&\int\left|K_{\lambda}^{\delta,l,j}f_2(x)\right|^2\frac{dx}{|x|^{\alpha}}\\&\quad\le C_{\alpha}2^{-l}2^{-j}\delta\int|f(x)|^2\frac{dx}{|x|^{\alpha}}.\end{aligned} \qquad (40)$$

Nextly, we consider $i=1$. By the definition of $K_{\lambda}^{\delta,l,j}$ and f_1, we see that the kernel of $K_{\lambda}^{\delta,l,j}$ is supported in

$$\left\{(x,y):2^{j-1}\le|x-y|\le 2^j\right\} \qquad (41)$$

and $f(y)$ is supported in

$$0 \le |y| \le \frac{1}{4}. \tag{42}$$

So, the support of $K_\lambda^{\delta,l,j} f(x)$ is contained in

$$\left\{x \in \mathbb{R}^n : 2^j + \frac{1}{4} \ge |x| \ge 2^{j-1} - \frac{1}{4} \ge 2^{0-1} - \frac{1}{4} = \frac{1}{4}\right\}. \tag{43}$$

With Lemma 4, we have

$$\begin{aligned}
\int \left|K_\lambda^{\delta,l,j} f_1(x)\right|^2 \frac{dx}{|x|^\alpha}
&\le \int \left|K_\lambda^{\delta,l,j} f_1(x)\right|^2 4^\alpha dx \\
&\le C \int \left|K_\lambda^{\delta,l,j} f_1(x)\right|^2 dx \\
&\le C 2^{-l} 2^{-jM} \delta^M \|f_1\|_2^2 \\
&\le C 2^{-l} 2^{-jM} \delta^M \|f\|_2^2 \\
&\le C 2^{-l} 2^{-jM} \delta^M \left(2^j\right)^\alpha \int |f(x)|^2 \frac{dx}{|x|^\alpha} \\
&\le C 2^{-l} 2^{-j} \delta \int |f(x)|^2 \frac{dx}{|x|^\alpha},
\end{aligned} \tag{44}$$

where $M > \alpha + 1$.

At last, we turn to the case of $i = 3$. Similar to the aforementioned, we have

$$\begin{aligned}
\int \left|K_\lambda^{\delta,l,j} f_3(x)\right|^2 dx
&\le C 2^{-l} 2^{-jM} \delta^M \|f_3\|_2^2 \\
&\le C 2^{-l} 2^{-jM} \delta^M \int_{1/4 \le |x| \le C} |f(x)|^2 dx \\
&\le C 2^{-l} 2^{-jM} \delta^M \left(\frac{1}{4}\right)^{-\alpha} \int_{1/4 \le |x| \le C} |f(x)|^2 \left(\frac{1}{4}\right)^\alpha dx \\
&\le C 2^{-l} 2^{-jM} \delta^M 4^\alpha \int_{1/4 \le |x| \le C} |f(x)|^2 |x|^\alpha dx \\
&\le C 2^{-l} 2^{-jM} \delta^M \int |f(x)|^2 |x|^\alpha dx.
\end{aligned} \tag{45}$$

By duality again, we see

$$\begin{aligned}
\int \left|K_\lambda^{\delta,l,j} f_3(x)\right|^2 \frac{dx}{|x|^\alpha}
&\le C 2^{-l} 2^{-jM} \delta^M \int |f(x)|^2 dx \\
&\le C 2^{-l} 2^{-jM} \delta^M \left(2^j\right)^\alpha \int |f(x)|^2 \frac{dx}{|x|^\alpha} \\
&\le C 2^{-l} 2^{-j(M-\alpha)} \delta^M \int |f(x)|^2 \frac{dx}{|x|^\alpha}.
\end{aligned} \tag{46}$$

Through we choose $M > \alpha + 1$, it is not difficult to get

$$\begin{aligned}
\int \left|K_\lambda^{\delta,l,j} f_3(x)\right|^2 \frac{dx}{|x|^\alpha}
\le C 2^{-l} 2^{-j} \delta \int |f(x)|^2 \frac{dx}{|x|^\alpha}.
\end{aligned} \tag{47}$$

Combining (28), (40), (44) with (47), when f is supported in $\{|x| \le C2^j\}$, we come to the conclusion

$$\begin{aligned}
\int \left|K_\lambda^{\delta,l,j} f(x)\right|^2 \frac{dx}{|x|^\alpha}
\le C 2^{-l} 2^{-j} \delta \int |f(x)|^2 \frac{dx}{|x|^\alpha}.
\end{aligned} \tag{48}$$

Now, we hope to establish (48) for all $f \in L^2(\mathbb{R}^n)$. Decompose $f = \sum_{i \in \mathbb{Z}^n} \chi_{Q_i} f = \sum_{i \in \mathbb{Z}^n} f_i$, where $\{Q_i\}$ are disjoint cubes of common side $10 \cdot 2^j$ with Q_0 centered at 0. Since $\{K_\lambda^{\delta,l,j} f_i(x)\}$ have essentially disjoint supports, it suffices to prove (48) for every f_i. When $i = 0$, we have proved it. If $i > 1$, then

$$\begin{aligned}
0 &< \left(\frac{1}{2} + l\right) 10 \cdot 2^j \\
&< |x|^{-\alpha} < \left(\frac{1}{2} + l + 1\right) 10 \cdot 2^j \quad (l \ge 0).
\end{aligned} \tag{49}$$

Therefore, we only need to confirm

$$\int \left|K_\lambda^{\delta,l,j} f(x)\right|^2 dx \le C_\alpha 2^{-l} 2^{-j} \delta \int |f(x)|^2 dx. \tag{50}$$

In fact, it follows from Lemma 4 that

$$\begin{aligned}
\int \left|K_\lambda^{\delta,l,j} f(x)\right|^2 dx
&\le C 2^{-2M(j+l)} \delta^{2M} \int |f(x)|^2 dx \\
&\le C 2^{-l} 2^{-j} \delta \int |f(x)|^2 dx.
\end{aligned} \tag{51}$$

At present, we complete the proof of Lemma 5. □

Lemma 6. *For $\delta > 0$, $k \in \mathbb{Z}$, and $0 \le \alpha < n$, one gets*

$$\int_{\mathbb{R}^n} \int_{2^{k-1}}^{2^k} \left|K_\lambda^\delta f(x)\right|^2 \frac{d\lambda}{\lambda} \frac{dx}{|x|^\alpha} \le C_\alpha \delta \int |f(x)|^2 \frac{dx}{|x|^\alpha}. \tag{52}$$

Proof. Applying Minkowski and Cauchy-Schwartz's inequalities into the left hand side of (52), we get

$$\begin{aligned}\int_{\mathbb{R}^n}\left|\int_{2^{k-1}}^{2^k} K_\lambda^\delta f(x)\frac{d\lambda}{\lambda}\right|^2|x|^{-\alpha}dx &\le C\left(\int_{2^{k-1}}^{2^k}\left(\int_{\mathbb{R}^n}\left|K_\lambda^\delta f(x)\right|^2|x|^{-\alpha}dx\right)^{1/2}\frac{d\lambda}{\lambda}\right)^2\\ &\le C\left(\left(\int_{2^{k-1}}^{2^k}\int_{\mathbb{R}^n}\left|K_\lambda^\delta f(x)\right|^2|x|^{-\alpha}dx\frac{d\lambda}{\lambda}\right)^{1/2}\right.\\ &\qquad\left.\times\left(\int_{2^{k-1}}^{2^k}\frac{d\lambda}{\lambda}\right)^{1/2}\right)^2\\ &\le C\int_{2^{k-1}}^{2^k}\int_{\mathbb{R}^n}\left|K_\lambda^\delta f(x)\right|^2|x|^{-\alpha}dx\frac{d\lambda}{\lambda}.\end{aligned}\tag{53}$$

Now, it suffices to prove that

$$\begin{aligned}&\int_{\mathbb{R}^n}\left|K_\lambda^\delta f(x)\right|^2|x|^{-\alpha}dx\\ &\quad\le C_\alpha\delta\int_{\mathbb{R}^n}\left|f(x)\right|^2|x|^{-\alpha}dx,\end{aligned}\tag{54}$$

where $C_\alpha\delta$ is uniform in $2^{k-1}\le\lambda\le 2^k$. For $0\le\alpha<n$, it is equivalent to

$$\int_{\mathbb{R}^n}\left|K_\lambda^\delta g(x)\right|^2\frac{dx}{|x|^\alpha}\le C_\alpha\delta\int_{\mathbb{R}^n}\left|g(x)\right|^2\frac{dx}{|x|^\alpha}.\tag{55}$$

It is just as the result of Lemma 5. □

Now, we come back to the proof of Theorem 2.

Proof. As in [3], we can decompose

$$\left(1-\frac{t}{\lambda}\right)_+^\beta=\sum_{k=0}^{\infty}2^{-k\beta}m^{2^{-k}}\left(\frac{t}{\lambda}\right).\tag{56}$$

Thus,

$$S_*^\beta f(x)\le\sum_{k=0}^{\infty}2^{-k\beta}K_*^{2^{-k}}f(x).\tag{57}$$

Consequently, we consider

$$\int_{\mathbb{R}^n}\left|K_*^\delta f(x)\right|^2\frac{dx}{|x|^\alpha}\le C_\alpha\delta\int_{\mathbb{R}^n}\left|f(x)\right|^2\frac{dx}{|x|^\alpha}.\tag{58}$$

Let

$$G^\delta f(x)=\left(\int_0^\infty\left|K_\lambda^\delta f(x)\right|^2\frac{d\lambda}{\lambda}\right)^{1/2}\tag{59}$$

and $\overline{G}^\delta$ be defined in the same way but using instead of m^δ the function

$$\overline{m}^\delta(\lambda)=\delta\lambda\frac{dm^\delta(\lambda)}{\lambda},\tag{60}$$

which satisfies the same estimates as m^δ. Then, by the fundamental theorem in calculus and Hölder's inequality, we have

$$\begin{aligned}\left|K_*^\delta f(x)\right|^2&\le\int_0^\infty 2\left|K_\lambda^\delta f(x)\frac{dK_\lambda^\delta f(x)}{d\lambda}\right|d\lambda\\ &\le 2\delta^{-1}\int_0^\infty\frac{\left|K_\lambda^\delta f(x)\right|}{\lambda^{1/2}}\lambda\delta\frac{\left|dK_\lambda^\delta f(x)/d\lambda\right|}{\lambda^{1/2}}d\lambda\\ &\le 2\delta^{-1}\left(\int_0^\infty\left|K_\lambda^\delta f(x)\right|^2\frac{d\lambda}{\lambda}\right)^{1/2}\\ &\quad\times\left(\int_0^\infty\left|\lambda\delta\frac{dK_\lambda^\delta f(x)}{d\lambda}\right|^2\frac{d\lambda}{\lambda}\right)^{1/2}\\ &=2\delta^{-1}G^\delta f(x)\overline{G}f(x).\end{aligned}\tag{61}$$

Take a Schwartz function ψ such that $\psi(0)=0$ and $\psi(t)=1$ if $1/2\le t\le 2$ and $\psi_k(t)=\psi(2^k t)$. Then, when $2^{k-1}\le\lambda\le 2^k$, we have

$$K_\lambda^\delta f(x)=\int m^\delta(\lambda t)\psi_k(t)\,dE_t f(x).\tag{62}$$

Using Lemma 6, we get

$$\begin{aligned}&\int_{\mathbb{R}^n}\int_{2^{k-1}}^{2^k}\left|K_\lambda^\delta f(x)\right|^2\frac{d\lambda}{\lambda}\frac{dx}{|x|^\alpha}\\ &\quad=\int_{\mathbb{R}^n}\int_{2^{k-1}}^{2^k}\left|\int m^\delta(\lambda t)\psi_k(t)\,dE_t f(x)\right|^2\frac{d\lambda}{\lambda}\frac{dx}{|x|^\alpha}\\ &\quad=\int_{\mathbb{R}^n}\int_{2^{k-1}}^{2^k}\left|\int m^\delta(\lambda t)\,dE_t\left(\psi_k(H_b)f\right)(x)\right|^2\frac{d\lambda}{\lambda}\frac{dx}{|x|^\alpha}\\ &\quad\le C_\alpha\delta\int\left|\psi_k(H_b)f(x)\right|^2\frac{dx}{|x|^\alpha}\\ &\quad=C_\alpha\delta\int\left|\int\psi_k(t)\,dE_t f(x)\right|^2\frac{dx}{|x|^\alpha}.\end{aligned}\tag{63}$$

From Theorem 3.1 in page 411 of [10] and the density of L_c^∞ in $L^p(w)$, we induce that

$$\left(\sum_{k=-\infty}^{\infty}\left|\int\psi_k(t)\,dE_t f(x)\right|^2\right)^{1/2}\tag{64}$$

is bounded in $L^2(dx/|x|^\alpha)$. As a result,

$$\begin{aligned}\int \left(G^\delta f(x)\right)^2 \frac{dx}{|x|^\alpha} &= \int \int_0^\infty \left|K_\lambda^\delta f(x)\right|^2 \frac{d\lambda}{\lambda} \frac{dx}{|x|^\alpha} \\ &= \int \sum_{k=-\infty}^{\infty} \int_{2^{k-1}}^{2^k} \left|K_\lambda^\delta f(x)\right|^2 \frac{d\lambda}{\lambda} \frac{dx}{|x|^\alpha} \\ &= \sum_{k=-\infty}^{\infty} \int \int_{2^{k-1}}^{2^k} \left|K_\lambda^\delta f(x)\right|^2 \frac{d\lambda}{\lambda} \frac{dx}{|x|^\alpha} \\ &\leq \sum_{k=-\infty}^{\infty} C_\alpha \delta \int \left| \int \psi_k(t)\, dE_t f(x) \right|^2 \frac{dx}{|x|^\alpha} \\ &\leq C_\alpha \delta \int \sum_{k=-\infty}^{\infty} \left| \int \psi_k(t)\, dE_t f(x) \right|^2 \frac{dx}{|x|^\alpha} \\ &\leq C_\alpha \delta \int |f(x)|^2 \frac{dx}{|x|^\alpha}.\end{aligned} \tag{65}$$

At last, with (61) and Hölder's inequality, we come to the result that

$$\int \left(K_*^\delta f(x)\right)^2 \frac{dx}{|x|^\alpha} \leq C_\alpha \int |f(x)|^2 \frac{dx}{|x|^\alpha}. \tag{66}$$

□

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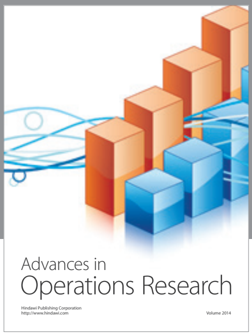
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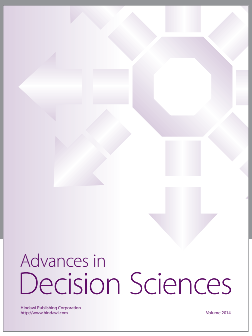
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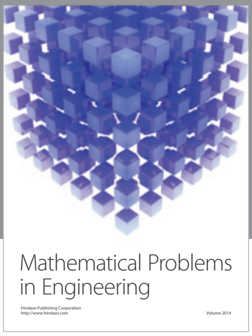
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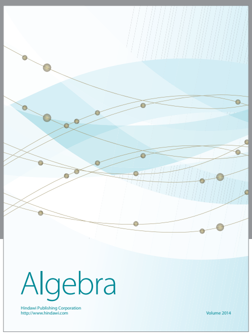
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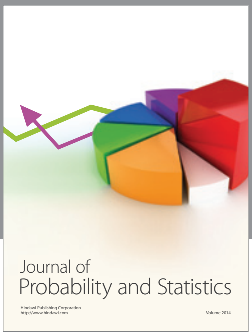
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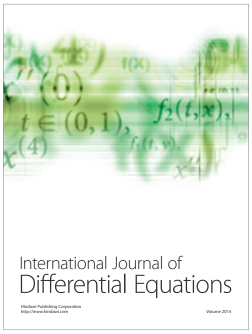
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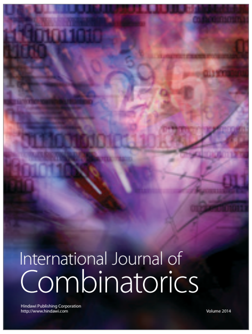
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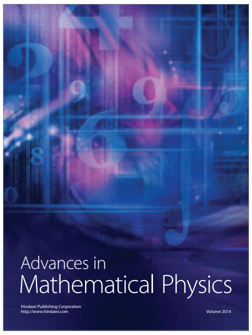
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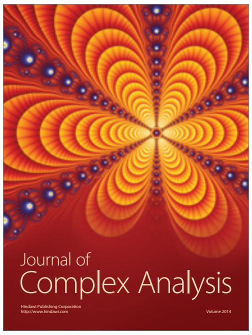
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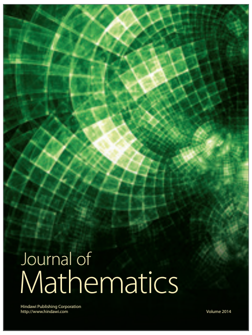
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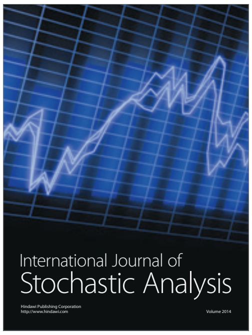
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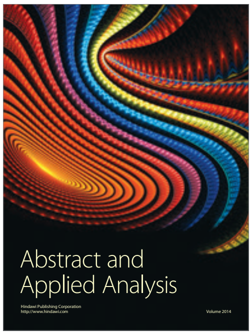
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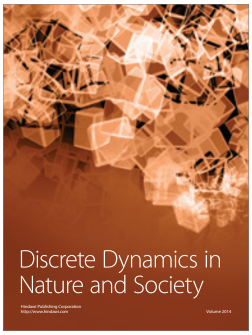
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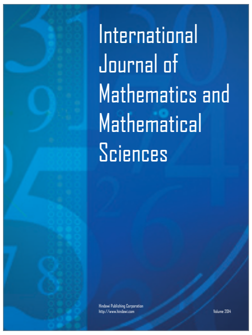
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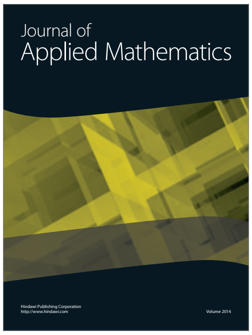
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