UNIFORM STABILIZATION OF A COUPLED STRUCTURAL ACOUSTIC SYSTEM BY BOUNDARY DISSIPATION

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Abstract. We consider a coupled PDE system arising in noise reduction problems. In a two dimensional chamber, the acoustic pressure (unwanted noise) is represented by a hyperbolic wave equation. The floor of the chamber is subject to the action of piezo-ceramic patches (smart materials). The goal is to reduce the acoustic pressure by means of the vibrations of the floor which is modelled by a hyperbolic Kirchoff equation. These two hyperbolic equations are coupled by appropriate trace operators. This overall model differs from those previously studied in the literature in that the elastic chamber floor is here more realistically modeled by a hyperbolic Kirchoff equation, rather than by a parabolic Euler-Bernoulli equation with Kelvin-Voight structural damping, as in past literature. Thus, the hyperbolic/parabolic coupled system of past literature is replaced here by a hyperbolic/hyperbolic coupled model. The main result of this paper is a uniform stabilization of the coupled PDE system by a (physically appealing) boundary dissipation.

1. Introduction

In this paper we study the uniform stabilization of two coupled hyperbolic equations arising in the noise reduction problem for structural acoustic models. The acoustic pressure (unwanted noise) inside a two dimensional chamber is mathematically represented by a hyperbolic wave equation, whereas a hyperbolic Kirchoff equation models the elastic displacements of the one

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dimensional moving floor of the chamber. Such a floor is subject to the action of a piezo-ceramic patch (smart material), which is mathematically modeled as the distributional derivative of a Dirac mass. The interaction between the chamber and the moving floor is represented by appropriate trace operators acting on the interface between the floor and the chamber.

More precisely, let Ω be a two dimensional, open and bounded domain (the chamber) in $\mathbb{R}^{\nvDash}$ with boundary Γ. The boundary is made up of two open, smooth, and disjoint portions Γ_0 and Γ_1. Γ_0, which models the moving floor of the chamber, is assumed to be flat. Two examples of such a domain are given below. Notice that we have scaled the coordinate system for convenience so that $\Gamma_0 = \{(0,1) \times \{0\}\}$.

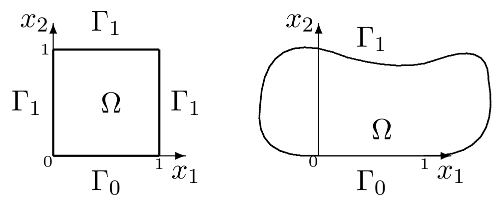

Figure (I) Figure (II)

The acoustic medium within Ω is described by the wave equation in the variable z. The vibrations of the elastic floor Γ_0 are modeled by the variable v. We assume that v will satisfy a Kirchoff equation on Γ_0, coupled with the wave equation satisfied by z in the interior of the domain Ω. Then the PDE model in the variables z and v is as follows:

$$
\begin{cases}
WaveEquation \\
\begin{cases}
z_{tt} = \Delta z & \text{on } Q \\
\frac{\partial z}{\partial \nu}\Big|_{\Gamma_1} = -k_1 z_t & \text{on } \Sigma_1 \\
\frac{\partial z}{\partial \nu}\Big|_{\Gamma_0} = -k_1 z_t - v_t & \text{on } \Sigma_0
\end{cases} \\
\\
Kirchoff Equation \\
\begin{cases}
v_{tt} - \gamma \Delta v_{tt} + \Delta^2 v - z_t = \delta'(x_0) u(t) & \text{on } \Sigma_0 \\
v\big|_{\partial \Sigma_0} = 0; \Delta v\big|_{\partial \Gamma_0} = -k_2 \frac{\partial v_t}{\partial \nu} & \text{on } \partial \Sigma_0
\end{cases} \\
\\
z(0,\cdot) = z_0; z_t(0,\cdot) = z_1; v(0,\cdot) = v_0; v_t(0,\cdot) = v_1 \text{ in } \Omega \times \Omega \times \Gamma_0 \times \Gamma_0
\end{cases}
\tag{1.1}
$$

where $\gamma > 0$, $k_1 \geq 0$, and $k_2 \geq 0$ and

$$(0,T] \times \Omega = Q;\ (0,T] \times \Gamma_1 = \Sigma_1;\ (0,T] \times \Gamma_0 = \Sigma_0;\ (0,T] \times \partial\Gamma_0 = \partial\Sigma_0$$

The control is modeled mathematically by a finite number of distributional derivatives of Dirac masses concentrated at points of the moving floor Γ_0. It is mathematically equivalent to consider only one such distributional derivative (concentrated at the point x_0 on the flat segment $\Gamma_0 \in \mathbb{R}^{\nvDash}$).

The basic structure of acoustic flow models has been known for a long time (see [18]). Some related mathematical questions regarding the spectral properties or the strong stabilization of the model in [18] are studied in [4] and [8]. Smart material technology has suggested the introduction of a dissipation acting at the edge of the floor via moments or shears. This motivates one to consider the damped coupled model (1.1) where the dissipation is exercised through the bending moment of the Kirchoff equation. This model differs in a critical way from other models recently studied in noise reduction problems (see [1], [3]) in that the elastic dynamics of the moving floor is more realistically represented by a hyperbolic Kirchoff equation, rather than by a structurally damped Euler-Bernoulli equation with so-called Kelvin-Voight damping, as in the past models.

The results existing in the literature on the stabilization of structural acoustic models refer to those where the floor is strongly damped by means of structural damping (see [1],[2],[7]). In the case of structural damping present in the model, the component of the uncoupled system corresponding to the Euler-Bernoulli equation represents an analytic semigroup ([22],[6]). This provides, in addition to strong stability properties for the Euler-Bernoulli equation, a lot of regularity properties which facilitate the analysis of stability for the entire structure. The situation is drastically different when the analytic Euler-Bernoulli equation is actually replaced by a more realistic hyperbolic Kirchoff equation.

2. Abstract Models

2.1. **Undamped Problem:** $k_1 = k_2 = 0$.

The case of undamped coupled equations (that is, $k_1 = k_2 = 0$ in (1.1)) is analyzed in a companion paper where a sharp regularity result (to be quoted below) is obtained (see [5]). The following operators and abstract setting for problem (1.1) are also quoted from [5]:

i) Let $\mathcal{A} : L_2(\Gamma_0) \supset \mathcal{D}(\mathcal{A}) \rightarrow L_2(\Gamma_0)$ be the positive self-adjoint operator

$$\mathcal{A}f = \Delta^2 f,\ \mathcal{D}(\mathcal{A}) = \{f \in H^4(\Gamma_0) : f|_{\partial\Gamma_0} = \Delta f|_{\partial\Gamma_0} = 0\} \tag{2.1}$$

- Define the operator $\mathbb{A} : L_2(\Gamma_0) \supset \mathcal{D}(\mathbb{A}) \rightarrow L_2(\Gamma_0)$ as

$$\mathbb{A} = (I + \gamma\mathcal{A}^{\frac{1}{2}})^{-1}\mathcal{A},\ \mathcal{D}(\mathbb{A}) = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \tag{2.2}$$

The operator $\mathbb{A}$ is positive self-adjoint on the space $\mathcal{D}(\mathcal{A}_\gamma^{\frac{1}{4}})$ topologized by the inner product

$$(x, y)_{\mathcal{D}(\mathcal{A}_\gamma^{\frac{1}{4}})} = \Big((I + \gamma\mathcal{A}^{\frac{1}{2}})x, y\Big)_{L_2(\Gamma_0)};\ \forall x, y \in \mathcal{D}(\mathcal{A}_\gamma^{\frac{1}{4}}) \tag{2.3}$$

where

$$\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H^2(\Gamma_0) \cap H_0^1(\Gamma_0) \text{ and } \mathcal{D}(\mathcal{A}_\gamma^{\frac{1}{4}}) = H_0^1(\Gamma_0) \tag{2.4}$$

Let us also note for future reference the following equivalent spaces:

$$\mathcal{D}(\mathcal{A}^{\frac{1}{2}-\varepsilon}) \equiv H^{2-4\varepsilon}(\Gamma_0) \text{ (equivalence in norms for small } \varepsilon > 0) \tag{2.5}$$

- Let $\mathcal{A}_N : L_2^0(\Omega) = L_2(\Omega)\big/\mathcal{N}(\mathcal{A}_N) \to L_2^0(\Omega)$ be the positive self-adjoint operator

$$\mathcal{A}_N f = -\Delta f;\ \mathcal{D}(\mathcal{A}_N) = \{f \in H^2(\Omega) : \frac{\partial f}{\partial \nu}|_\Gamma = 0\} \tag{2.6}$$

where $\mathcal{N}(\mathcal{A}_N)$ is the one-dimensional null space of $\mathcal{A}_N$ in $L_2(\Omega)$.

- Define the Neumann map N for $h \in L_2^0(\Omega)$ as

$$h = Ng \Longleftrightarrow \begin{cases} \Delta h = 0 & \text{on } \Omega \\ \frac{\partial h}{\partial \nu}|_{\Gamma_0} = g & \text{on } \Gamma_0 \end{cases} \tag{2.7}$$

Let us note the following property of the Neumann map for future reference (see [11]):

$$N^* \mathcal{A}_N h = -h\big|_\Gamma \tag{2.8}$$

- Finally, consider the following spaces equivalent in norms:

$$\begin{aligned} Y &\equiv \mathcal{D}(\mathcal{A}_N{}^{\frac{1}{2}}) \times L_2(\Omega) \times \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times \mathcal{D}(\mathcal{A}_\gamma^{\frac{1}{4}}) \\ &\equiv H^1(\Omega) \times L_2(\Omega) \times [H^2(\Gamma_0) \cap H_0^1(\Gamma_0)] \times H_0^1(\Gamma_0) \end{aligned} \tag{2.9}$$

Hence, problem (1.1) with $k_1 = k_2 = 0$ can be written abstractly as

$$\dot{y} = Ay + Bu \text{ on } [\mathcal{D}(A^*)]'; y(0) = y_0 \tag{2.10}$$

where $y(t) = [z(t), z_t(t), v(t), v_t(t)]$

$$A = \begin{bmatrix} 0 & I & 0 & 0 \\ -\mathcal{A}_N & 0 & 0 & \mathcal{A}_N N(\cdot|_{\Gamma_0}) \\ 0 & 0 & 0 & I \\ 0 & -(I + \gamma \mathcal{A}^{\frac{1}{2}})^{-1} N^* \mathcal{A}_N & -\mathbb{A} & 0 \end{bmatrix} = -A^* \tag{2.11}$$

The action of A is described by its domain where, with $y = [y_1, y_2, y_3, y_4]$, $\mathcal{D}(A) = \{y \in Y : y_2 \in \mathcal{D}(\mathcal{A}_N^{\frac{1}{2}}), y_3 \in \mathcal{D}(\mathcal{A}^{\frac{3}{4}}), y_4 \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}}), [y_1 - N(y_4\big|_{\Gamma_0})] \in \mathcal{D}(\mathcal{A}_N)\}$ and A^* is the Y-adjoint of A; while the operator $B :\ U \to [\mathcal{D}(A^*)]'$, $U = \mathbb{R}$ and its adjoint $B^* : \mathcal{D}(A^*) \to U$ are

$$Bu = \begin{bmatrix} 0 \\ 0 \\ 0 \\ (I + \gamma \mathcal{A}^{\frac{1}{2}})^{-1} \delta'(x_0) u \end{bmatrix};\ B^* \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = -\frac{d}{dx} y_4 \Big|_{x=x_0},\ y \in \mathcal{D}(A^*) \tag{2.12}$$

By the skew-adjointness of A on Y (see (2.11)), we see that A generates a s.c. unitary group e^{At} on Y:

$$\begin{aligned} &e^{A^* t} = e^{-At};\ \|e^{A^* t}\|_{\mathcal{L}(Y)} \equiv \|e^{-At}\|_{\mathcal{L}(Y)} \equiv 1 \\ &Re(Ax, x)_Y = Re(A^* x, x)_Y \equiv 0,\ \forall x \in \mathcal{D}(A) = \mathcal{D}(A^*) \end{aligned} \tag{2.13}$$

Also the solution to the undamped problem can be written as:

$$y(t) = e^{At} y_0 + (Lu)(t) \tag{2.14}$$

where

$$(Lu)(t) = \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau \tag{2.15}$$

Now we quote the main theorem, a sharp regularity result, of [5] (Theorem 1.2).

Theorem 2.1. *With reference to the coupled P.D.E. system (1.1), we have that*

i) *For each $0 < T < \infty$, the operator L defined in (2.15) satisfies the following property:*

$$L: \; L_2(0,T) \to C([0,T]; Y), \text{ continuously.} \tag{2.16}$$

- *(Abstract Trace Regularity) Equivalently, it follows by duality that the operator $B^* e^{A^* t}$ can be extended continuously from Y to $L_2(0,T)$:*

$$\int_0^T \left| B^* e^{A^* t} y \right|^2 \, dt \leq C_T \|y\|_Y^2, \;\; \forall y \in Y \tag{2.17}$$

Sharp (optimal) regularity results (abstract trace regularity) for the mixed PDE problems have significant implications in the study of associated control theory problems, enabling one to invoke a large body of abstract results on quadratic control theory, min-max game theory, etc. For instance, the abstract results in [11], [15], [17], [24] can now be readily applied to the coupled PDE problem (1.1) over a finite time interval. In the case of infinite time interval, however, in order to invoke the abstract theory as in [11], [15], [17], and [24], additional control theoretic hypotheses such as the Finite Cost Condition and the Detectability Condition are needed. The Finite Cost Condition can be verified by the property of uniform stability in the space Y (see (2.9)) with $L_2(0,\infty;U)$ feedback control. However, this uniform stabilizability property on the space of regularity Y (as given to be $H^1(\Omega) \times L^2(\Omega) \times H^2(\Gamma_0) \times H^1(\Gamma_0)$ for the coupled PDE system (1.1)) fails for the undamped ($k_1 = k_2 = 0$) problem (1.1), which is actually a general pathology of hyperbolic or Petrowski type dynamics with point controls acting through δ or δ' (see [11]). Therefore, the issue of ensuring uniform stabilization is of paramount importance. To remedy this situation, we modify the original conservative dynamics by adding damping terms, as to make it uniformly stable on the regularity space Y, while preserving the same regularity in $C([0,T];Y)$.

2.2. **Damped Problem: $k_1 > 0$, $k_2 > 0$; $u(t) \equiv 0$.**

We are mainly interested in the physically appealing boundary stabilization in which energy decay rates are achieved by introducing some form of dissipation on the boundary. Our main goal is to show that the boundary damping added to the wave equation and the boundary dissipation applied through the bending moments at the edge of Γ_0 are enough to provide the uniform decay rates of the natural energy function associated with the model. For this purpose, we consider the coupled system (1.1) with strictly positive constants k_1 and k_2 which we henceforth take equal to 1 for convenience. Since our main interest is a uniform stability result, we also let $u(t) \equiv 0$.

Now we introduce two more operators, namely, the Dirichlet operator D and the Green's operator G_2 (see [9] and [11]):

$$w = Dh \Leftrightarrow \Big\{\Delta w = 0 \text{ in } \Gamma_0;\ w = h \text{ on } \partial\Gamma_0\Big\} \tag{2.18}$$

$$f = G_2 g \Leftrightarrow \Big\{\Delta^2 f = 0 \text{ in } \Gamma_0;\ f = 0,\ \Delta f = g \text{ on } \partial\Gamma_0\Big\} \tag{2.19}$$

It is easy to show that (see [9])

$$G_2 = -\mathcal{A}^{-\frac{1}{2}} D \tag{2.20}$$

Note that since $\Gamma_0 = (0,1)$ is one dimensional, the Dirichlet operator D in (2.18) has the following form:

$$(Dh)(x_1) = \big[h(1) - h(0)\big]x_1 + h(0),\ 0 \le x_1 \le 1 \tag{2.21}$$

Next notice from (2.20) and [11] that

$$G_2^* \mathcal{A} g = -D^* \mathcal{A}^{\frac{1}{2}} g = \frac{\partial g}{\partial \nu} \tag{2.22}$$

However, again because of $\Gamma_0 = (0,1)$ and $\partial\Gamma_0 = \{0\} \cup \{1\}$, we have that

$$\frac{\partial g}{\partial \nu}(i) = (-1)^{i+1} \frac{\partial g}{\partial x_1}(i),\ i = 0,1 \tag{2.23}$$

Consequently, from (2.21), (2.22), and (2.23), we note the following expressions in $x_1 \in (0,1)$ to be referred to later

$$({G_2}^* \mathcal{A} v_t)(i) = -(D^* \mathcal{A}^{\frac{1}{2}} v_t)(i) = (-1)^{i+1} \frac{\partial v_t}{\partial x_1}(i),\ i = 0,1 \tag{2.24}$$

$$(DD^* \mathcal{A}^{\frac{1}{2}} v_t)(x_1) = -\left(\frac{\partial v_t}{\partial x_1}(t,1) + \frac{\partial v_t}{\partial x_1}(t,0)\right)x_1 + \frac{\partial v_t}{\partial x_1}(t,0) \tag{2.25}$$

$$\nabla DD^* \mathcal{A}^{\frac{1}{2}} v_t = -\left(\frac{\partial}{\partial x_1} DD^* \mathcal{A}^{\frac{1}{2}} v_t = \frac{\partial v_t}{\partial x_1}(t,1) + \frac{\partial v_t}{\partial x_1}(t,0)\right) \tag{2.26}$$

Proceeding as in [10] and [11], the damped problem (1.1) with $k_1 = k_2 = 1$ and $u(t) \equiv 0$ can be written abstractly as (see (2.2), (2.7) and (2.20))

$$z_{tt} = -\mathcal{A}_N z - \mathcal{A}_N N N^* \mathcal{A}_N^* z_t + \mathcal{A}_N N(v_t|_{\Gamma_0}) \tag{2.27}$$

$$\begin{aligned} v_{tt} &= -\mathbb{A} v - \mathbb{A} G_2 G_2^* \mathcal{A} v_t - (I + \gamma \mathcal{A}^{\frac{1}{2}})^{-1} N^* \mathcal{A}_N z_t \\ &= -(I + \gamma \mathcal{A}^{\frac{1}{2}})^{-1} \mathcal{A} v - (I + \gamma \mathcal{A}^{\frac{1}{2}})^{-1} \mathcal{A}^{\frac{1}{2}} DD^* \mathcal{A}^{\frac{1}{2}} v_t \\ &\qquad + (I + \gamma \mathcal{A}^{\frac{1}{2}})^{-1} (z_t|_{\Gamma_0}) \end{aligned} \tag{2.28}$$

At this point, we introduce the following subspace Y_0 of Y (see (2.9)):

$$Y_0 = \Big\{y = [y_1, y_2, y_3, y_4] \in Y : \int_\Gamma y_1\, d\Gamma + \int_\Omega y_2\, d\Omega + \int_{\Gamma_0} y_3\, d\Gamma_0 = 0\Big\}. \tag{2.29}$$

It is easy to show that the coupled problem (1.1) is well-posed in Y_0; that is, the Lumer-Phillips theorem holds on Y_0 with the norm of

$$\mathcal{D}({\mathcal{A}_N}^{\frac{1}{2}}) \times L_2(\Omega) \times \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times \mathcal{D}(\mathcal{A}_\gamma^{\frac{1}{4}}) \tag{2.30}$$

Then the first order equation corresponding to (2.27) and hence to the damped problem (1.1) with $u(t) \equiv 0$ is

$$\dot{y} = A_F y \quad \text{on } [\mathcal{D}(A_F^*)]'; \; y(0) = y_0 \in Y_0 \tag{2.31}$$

where $y(t) = [z(t), z_t(t), v(t), v_t(t)]$ and

$$A_F = \begin{bmatrix} 0 & I & 0 & 0 \\ -\mathcal{A}_N & -\mathcal{A}_N N N^* \mathcal{A}_N & 0 & \mathcal{A}_N N(\cdot|_{\Gamma_0}) \\ 0 & 0 & 0 & I \\ 0 & -(I + \gamma \mathcal{A}^{\frac{1}{2}})^{-1} N^* \mathcal{A}_N & -\mathbb{A} & -\mathbb{A} G_2 G_2^* \mathcal{A} \end{bmatrix} \tag{2.32}$$

with dense domain $\mathcal{D}(A_F) = \{y \in Y_0 : A_F y \in Y_0\}$. The Y_0-adjoint of A_F is

$$A_F^* = \begin{bmatrix} 0 & -I & 0 & 0 \\ \mathcal{A_N} & -\mathcal{A_N N N^* A_N} & 0 & -\mathcal{A_N N}(\cdot|_{\Gamma_\prime}) \\ 0 & 0 & 0 & -I \\ 0 & (I + \gamma \mathcal{A}^{\frac{1}{2}})^{-1} N^* \mathcal{A_N} & \mathbb{A} & -\mathbb{A G_{\not{\in}} G_{\not{\in}}^* A} \end{bmatrix} \tag{2.33}$$

with $\mathcal{D}(A_F{}^*) = \{y \in Y_0 : A_F^* y \in Y_0\}$. The action of A_F and A_F^* is described by their domains.

Theorem 2.2. *A_F is a maximal dissipative operator on Y_0 and hence the infinitesimal generator of a s.c. semigroup $e^{A_F t}$ of contractions on Y_0.*

Proof: Since A_F is densely defined, it is enough to show dissipativity of both of A_F and $A_F{}^*$ on Y_0 (see Corollary 4.4 in [20]). Dissipativity follows from the calculations below where we use the skew-adjointness of A (see (2.11)): by (2.13), (2.8), and (2.24), first with $y \in \mathcal{D}(A_F)$ and then with $y \in \mathcal{D}(A_F^*)$

$$\begin{aligned} &Re\big(A_F y, y\big)_{Y_0} \\ &\quad = -\big(\mathcal{A}_N N N^* \mathcal{A}_N y_2, y_2\big)_{L_2(\Omega)} - \big(\mathbb{A} G_2 G_2^* \mathcal{A} y_4, y_4\big)_{\mathcal{D}(\mathcal{A}_\gamma^{\frac{1}{4}})} \\ &\quad = -\big(y_2|_\Gamma, y_2|_\Gamma\big)_{L_2(\Gamma)} - \Big[\frac{\partial y_4}{\partial x_1}^2 (t,0) + \frac{\partial y_4}{\partial x_1}^2 (t,1)\Big] \\ &\quad \le 0, \end{aligned} \tag{2.34}$$

$$\begin{aligned} &Re\big(A_F{}^* y, y\big)_{Y_0} \\ &\quad = -\big(\mathcal{A}_N N N^* \mathcal{A}_N y_2, y_2\big)_{L_2(\Omega)} - \big(\mathbb{A} G_2 G_2^* \mathcal{A} y_4, y_4\big)_{\mathcal{D}(\mathcal{A}_\gamma^{\frac{1}{4}})} \\ &\quad = -\big(y_2|_\Gamma, y_2|_\Gamma\big)_{L_2(\Gamma)} - \Big[\frac{\partial y_4}{\partial x_1}^2 (t,0) + \frac{\partial y_4}{\partial x_1}^2 (t,1)\Big] \\ &\quad \le 0. \; \blacksquare \end{aligned} \tag{2.35}$$

3. Uniform Stability of $e^{A_F t}$

Our main goal is to show the uniform stability of the s.c. contraction semigroup $e^{A_F t}$ in the space Y_0 described in (2.29), corresponding to the coupled damped PDE system with $k_1 = k_2 = 1$ and $u(t) \equiv 0$. Accordingly, the 'energy' of the damped system is identified with the norm of Y_0 where $y_0 = [z_0, z_1, v_0, v_1] \in Y_0$ (see (2.30)):

$$\begin{aligned} E(t) &= \|e^{A_F t} y_0\|_Y^2 = E_z(t) + E_v(t) \qquad (3.1)\\ E_z(t) &= \int_\Omega \Big(|\nabla z(t)|^2 + z_t^2(t) \Big)\, d\Omega \\ &= \|z(t)\|^2_{\mathcal{D}(\mathcal{A}_N{}^{\frac{1}{2}})} + \|z_t(t)\|^2_{L_2(\Omega)}; \qquad (3.2)\\ E_v(t) &= \int_{\Gamma_0} \Big((\Delta v(t))^2 + v_t^2(t) + \gamma |\nabla v_t(t)|^2 \Big)\, d\Gamma_0 \\ &= \|v(t)\|^2_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})} + \|v_t(t)\|^2_{\mathcal{D}(\mathcal{A}_\gamma^{\frac{1}{4}})} \qquad (3.3) \end{aligned}$$

Lemma 3.1. *With respect to the coupled system (1.1) with* $k_1 = k_2 = 1$ *and* $u(t) \equiv 0$, *we have the following expressions:*

$$E(\tilde{t}) + 2\int_0^{\tilde{t}} \int_\Gamma z_t^2 \, d\Gamma\, dt + 2\int_0^{\tilde{t}} \Big(\frac{\partial v_t}{\partial x_1}^2 (t,0) + \frac{\partial v_t}{\partial x_1}^2 (t,1) \Big) dt = E(0) \qquad (3.4)$$

$$\int_0^\infty \Big(\frac{\partial v_t}{\partial x_1}^2 (t,0) + \frac{\partial v_t}{\partial x_1}^2 (t,1) \Big) dt + \|(z_t|_\Gamma)\|^2_{L_2(0,\infty;L_2(\Gamma))} \le \frac{1}{2} E(0) \qquad (3.5)$$

Proof: Initially for $y_0 \in \mathcal{D}(A_F)$, we get from (2.34) in the proof of Theorem 2.2 that $\forall \tilde{t} \in [0,T]$,

$$\begin{aligned} \frac{d}{dt}E(t) &= \frac{d}{dt}\|e^{A_F t} y_o\|_Y^2 = 2\Big(A_F e^{A_F t} y_0, e^{A_F t} y_0 \Big)_Y \qquad (3.6)\\ &= -2\int_\Gamma z_t^2 \, d\Gamma - 2\Big[\frac{\partial v_t}{\partial x_1}^2 (t,0) + \frac{\partial v_t}{\partial x_1}^2 (t,1) \Big] \le 0 \end{aligned}$$

Then (3.4) follows by integrating (3.6), which we then extend to $y_0 \in Y_0$ by density. (3.5) follows immediately from (3.4). ■

From (3.4) in Lemma 3.1, it is clear that $E(t)$ is non-increasing. The main result of this paper is the following theorem which states that $E(t)$ actually decays to zero.

Theorem 3.1. *Let* Ω *be a bounded open domain in* $\mathbb{R}^2$ *of the form in either Fig.(i) or Fig.(ii). Then the contraction semigroup* $e^{A_F t}$ *of Theorem 2.2 describing the damped coupled PDE system (1.1) is uniformly stable on* Y_0; *that is, there exist constants* $\delta > 0$ *and* $M \ge 1$ *such that*

$$\|e^{A_F t}\|_{\mathcal{L}(Y_0)} \le M e^{-\delta t},\ t \ge 0. \text{ Equivalently, } E(t) \le M e^{-\delta t} E(0) \qquad (3.7)$$

Orientation: Our strategy is to study the Kirchoff equation on Γ_0 and the Wave equation on Ω separately and then combine the results. In both cases, we run some multipliers on the corresponding equation. Then, by means of these energy methods, we get an identity (one for Kirchoff equation, another for wave equation) which is commonly met in hyperbolic equations, where the left hand side of the identity contains trace terms and the right hand side consists of interior terms. For each hyperbolic equation, analysis of the equalities obtained by multiplier techniques leads to an estimate of energy plus lower order terms (l.o.t.) terms (with norms topologically weaker than that of the energy) by dissipation terms. Thereon, a standard application of compactness/uniqueness argument gives the result. Let us also note that the analysis of the Kirchoff equation follows closely the technique of [10]. However, since the domain of the Kirchoff equation that we are interested in is only one dimensional ($\Gamma_0 = (0,1)$), the analysis here is easier.

3.1. **Kirchoff Equation Part of the Damped System.** In this section, we will consider the following uncoupled Kirchoff equation:

$$\begin{cases} v_{tt} - \gamma\Delta v_{tt} + \Delta^2 v = f & \text{on } \Sigma_0 \\ v\big|_{\partial\Gamma_0} = 0 & \text{on } \partial\Sigma_0 \\ \Delta v\big|_{\partial\Gamma_0} = -\frac{\partial v_t}{\partial\nu} & \text{on } \partial\Sigma_0 \\ v(0,\cdot) = v_0; v_t(0,\cdot) = v_1 & \text{in } \Gamma_0 \end{cases} \tag{3.8}$$

$$\text{where } f \in L_2(\Sigma_0) \tag{3.9}$$

The Kirchoff equation (3.8) with $f \equiv 0$ is studied in [10], where it is shown that there exists an operator $\mathcal{A}_F$ generating a s. c. semigroup $e^{\mathcal{A}_F t}$ on the space $\{\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times \mathcal{D}(\mathcal{A}_\gamma^{\frac{1}{4}})\}$. Therefore, the solution to (3.8) is the following:

$$\begin{bmatrix} v \\ v_t \end{bmatrix} = e^{\mathcal{A}_F t} \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} + \int_0^t e^{\mathcal{A}_F (t-s)} \begin{bmatrix} 0 \\ f(s) \end{bmatrix} ds \tag{3.10}$$

Hence, it follows a fortiori from (3.10) and (3.9) that

$$\begin{bmatrix} v \\ v_t \end{bmatrix} \in C\Big([0,T]; \begin{bmatrix} \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \\ \mathcal{D}(\mathcal{A}_\gamma^{\frac{1}{4}}) \end{bmatrix}\Big) \tag{3.11}$$

The main result of this section is the following theorem:

Theorem 3.2. *With respect to the Kirchoff equation in (3.8), we have the following inequality:*

$$\begin{aligned} &\int_0^T \left[\left(\frac{\partial v_t}{\partial x_1}(t,0)\right)^2 + \left(\frac{\partial v_t}{\partial x_1}(t,1)\right)^2\right] dt + \int_{\Sigma_0} f^2 \, d\Sigma_0 \\ &\geq C_{h_0\gamma}\Big(\int_0^T E_v(t)dt - \big[E_v(T) + E_v(0)\big] - \|v\|^2_{L_2(0,T;\mathcal{D}(\mathcal{A}^{\frac{1}{2}-\varepsilon}))}\Big) \end{aligned} \tag{3.12}$$

The proof of Theorem 3.2 will be given in the subsequent steps.

Corollary 3.2. *With respect to the Kirchoff equation part of the coupled PDE's with* $k_1 = k_2 = 1$ *and* $u(t) \equiv 0$ *in (1.1), we have the following* *inequality:*

$$\int_0^T \left[\left(\frac{\partial v_t}{\partial x_1}(t,0)\right)^2 + \left(\frac{\partial v_t}{\partial x_1}(t,1)\right)^2\right]dt + \int_{\Sigma_0} z_t^2 \, d\Sigma_0 \geq C_{h_0\gamma}\left(\int_0^T E_v(t)dt - [E_v(T)+E_v(0)] - \|v\|^2_{L_2(0,T;\mathcal{D}(\mathcal{A}^{\frac{1}{2}-\varepsilon}))}\right) \tag{3.13}$$

Proof: Notice that the Kirchoff equation part of (1.1) is the same as (3.8) with $f = z_t|_{\Gamma_0} \in L_2(0,\infty; L_2(\Gamma_0))$, (see Lemma 3.1). Hence, the result follows from Theorem 3.2. ∎

Let us now note the following abstract version of (3.8) (see (2.28)):

$$(I + \gamma\mathcal{A}^{\frac{1}{2}})v_{tt} = -\mathcal{A}v - \mathcal{A}^{\frac{1}{2}}DD^*\mathcal{A}^{\frac{1}{2}}v_t + f \tag{3.14}$$

Remark 3.1. *The analysis of the Kirchoff equation above will follow closely section 6 in* [10]*. However, in our case we have an extra term introduced by the coupling with a wave equation. Here, we will emphasize the influence of the coupling term represented by f in (3.8), which was absent in the problem considered in* [10]*. Whenever convenient, we quote results directly from* [10]*. However, since our case is only one dimensional, some parts of the analysis here is easier than in* [10]*. Compare (3.8) with (1.24) in* [10]*.*

Step 1: A New Variable. Define a new variable r as:

$$r = \mathcal{A}^{-\frac{1}{2}}v_t \in C([0,T]; \mathcal{D}(\mathcal{A}^{\frac{3}{4}})) \tag{3.15}$$

where the regularity follows from (3.11).

Proposition 3.3. *The variable r satisfies the following Kirchoff equation:*

$$\begin{cases} r_{tt} - \gamma\Delta r_{tt} + \Delta^2 r = -DD^*\mathcal{A}^{\frac{1}{2}}v_{tt} + \mathcal{A}^{-\frac{1}{2}}f_t & \text{on } \Sigma_0 \\ r|_{\partial\Gamma_0} = \Delta r|_{\partial\Gamma_0} = 0 & \text{on } \partial\Sigma_0 \\ r(0,\cdot) = \mathcal{A}^{-\frac{1}{2}}v_1; & \\ r_t(0,\cdot) = -(I+\gamma\mathcal{A}^{\frac{1}{2}})^{-1}[\mathcal{A}^{\frac{1}{2}}v_0 + DD^*\mathcal{A}^{\frac{1}{2}}v_1 - \mathcal{A}^{-\frac{1}{2}}f(0,\cdot)] & \end{cases} \tag{3.16}$$

Remark 3.2. *Compare (3.16) with (6.9) in* [10]*. Also notice the extra term $\mathcal{A}^{-\frac{1}{2}}f_t$ in (3.16) as a result of coupling.*

Proof: By the definition (3.15) and the abstract equation (3.14), we get that

$$r_t = \mathcal{A}^{-\frac{1}{2}}v_{tt} = -(I+\gamma\mathcal{A}^{\frac{1}{2}})^{-1}[\mathcal{A}^{\frac{1}{2}}v + DD^*\mathcal{A}^{\frac{1}{2}}v_t - \mathcal{A}^{-\frac{1}{2}}f] \tag{3.17}$$

$$(I+\gamma\mathcal{A}^{\frac{1}{2}})r_t = -\mathcal{A}^{\frac{1}{2}}v - DD^*\mathcal{A}^{\frac{1}{2}}v_t + \mathcal{A}^{-\frac{1}{2}}f \tag{3.18}$$

$$(I+\gamma\mathcal{A}^{\frac{1}{2}})r_{tt} = -\mathcal{A}r - DD^*\mathcal{A}^{\frac{1}{2}}v_{tt} + \mathcal{A}^{-\frac{1}{2}}f_t \tag{3.19}$$

Since $r \in C([0,T];\mathcal{D}(\mathcal{A}^{\frac{3}{4}}))$ (see (3.15)), the boundary conditions stated in (3.16) hold true. As for the initial conditions, they follow from (3.15) and (3.18). ∎

In the rest of the paper, however, we can assume without loss of generality that

$$f \in H_0^1(0,T;L_2(\Gamma_0)) \text{ which is dense in } L_2(0,T;L_2(\Gamma_0)) \tag{3.20}$$

so that $f(0,\cdot) = 0$. Similarly, we will assume that $v_1 \in H_0^2(\Gamma_0)$, which is dense in $H_0^1(\Gamma_0) = \mathcal{D}(\mathcal{A}^{\frac{1}{4}})$ so that $D^*\mathcal{A}^{\frac{1}{2}}v_1 = \frac{\partial v_1}{\partial \nu}|_{\partial\Gamma_0} = 0$. As a result, the initial conditions in (3.16) become

$$r_t(0,\cdot) = -(I+\gamma\mathcal{A}^{\frac{1}{2}})^{-1}\mathcal{A}^{\frac{1}{2}}v_0 \tag{3.21}$$

Therefore, it suffices to prove the desired estimates with these smoother data, and then extend by density.

Step 2: Equivalence of Some Norms Between Old Variable v and New Variable r

Let us note the following norms for later reference:

$$\text{(i)} \quad \left\{\int_{\Gamma_0} |\nabla(\Delta r)|^2 \, d\Gamma_0\right\}^{\frac{1}{2}} = \|\mathcal{A}^{\frac{3}{4}}r\|_{L_2(\Gamma_0)} = \|\mathcal{A}^{\frac{1}{4}}v_t\|_{L_2(\Gamma_0)}$$
$$\text{which is equivalent to } \|v_t\|_{\mathcal{D}(\mathcal{A}_\gamma^{\frac{1}{4}})} \text{ (see (3.15))}. \tag{3.22}$$

$$\text{(ii)} \quad \left\{\int_{\Gamma_0} (|\nabla r_t|^2 + \gamma|\Delta r_t|^2)\, d\Gamma_0\right\}^{\frac{1}{2}} \text{ which is equivalent to}$$
$$\|(I+\gamma\mathcal{A}^{\frac{1}{2}})r_t\|_{L_2(\Gamma_0)} = \|\mathcal{A}^{\frac{1}{2}}v\|_{L_2(\Gamma_0)} \tag{3.23}$$
$$+\mathcal{O}\left(\|D^*\mathcal{A}^{\frac{1}{2}}v_t\|_{L_2(\partial\Gamma_0)} + \|\mathcal{A}^{-\frac{1}{2}}f\|_{L_2(\Gamma_0)}\right) \text{(see (3.18))}$$

Notice from (2.24) and Lemma 3.1 that

$$D^*\mathcal{A}^{\frac{1}{2}}v_t \in L_2(0,T;\mathcal{D}(\mathcal{A}^{\frac{1}{2}})), \text{ hence by (3.23) } r_t \in L_2(0,T;\mathcal{D}(\mathcal{A}^{\frac{1}{2}})) \tag{3.24}$$

Step 3: A Trace Identity

Proposition 3.4. *The solution $r(t, x_1)$ of the equation (3.16) satisfies the following equality where $h_0(x_1) = x_1 - \frac{1}{2}$:*

$$
\begin{aligned}
&\frac{1}{4}\int_0^T \left[\left(\frac{\partial \Delta r}{\partial x_1}(t,0)\right)^2 + \left(\frac{\partial \Delta r}{\partial x_1}(t,1)\right)^2 \right. \\
&\quad \left. + \left(\frac{\partial r_t}{\partial x_1}(t,0)\right)^2 + \left(\frac{\partial r_t}{\partial x_1}(t,1)\right)^2\right] dt \\
&= \frac{1}{2}\int_{\Sigma_0} |\nabla(\Delta r)|^2 d\Sigma_0 + \frac{3}{2}\int_{\Sigma_0} |\nabla r_t|^2 d\Sigma_0 + \frac{\gamma}{2}\int_{\Sigma_0} |\Delta r_t|^2 d\Sigma_0 \qquad (3.25)\\
&\quad + \int_{\Sigma_0} \left(DD^*\mathcal{A}^{\frac{1}{2}} v_{tt} - \mathcal{A}^{-\frac{1}{2}} f_t, h_0 \cdot \nabla(\Delta r)\right) d\Sigma_0 \\
&\quad - \left[\left(r_t + \gamma \Delta r_t, h_0 \cdot \nabla(\Delta r)\right)_{L_2(\Gamma_0)}\right]_0^T
\end{aligned}
$$

Proof: A more general version of the identity for $dim(\Gamma_0) = n \geq 1$ is proved in the appendix A of [10] by using the multiplier $h_0 \nabla(\Delta r)$. Then the LHS of (3.4) is as in (A.7) of [10] and the RHS of (3.4) is stated in (A.8) of [10] after specializing (A.7) and (A.8) from [10] to our one dimensional domain with $h_0 = x_1 - \frac{1}{2}$. ■

Remark 3.3. *The simple one dimensionality of the domain of (3.16) also helps us to avoid running the second multiplier $\Delta r div h$, which is required in the analysis of Kirchoff equations on two or higher dimensional domains. See appendix B in* [10].

<u>Step 4:</u> Analysis of RHS of (3.4)

Proposition 3.5. *With respect to the Kirchoff equation (3.16) and the identity (3.4), we have the following estimate:*

$$
\begin{aligned}
&RHS \text{ of } (3.4) \\
&\geq C_{h_0\gamma}\Bigg(\int_0^T E_v(t)dt - [E_v(0) + E_v(T)] - \int_{\Sigma_0} f^2 \, d\Sigma_0 \qquad (3.26)\\
&\quad - \int_0^T \left[\left(\frac{\partial v_t}{\partial x_1}(t,0)\right)^2 + \left(\frac{\partial v_t}{\partial x_1}(t,1)\right)^2\right] dt - \|r_t\|^2_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}-\varepsilon})}\Bigg)
\end{aligned}
$$

Proof: Let us first notice the following for later reference:

By (3.20), $\mathcal{A}^{-\frac{1}{2}} f \in H_0^1(0,T; H^2(\Gamma_0) \cap H_0^1(\Gamma_0))$ so that

$$\mathcal{A}^{-\frac{1}{2}} f|_{\partial\Gamma_0} \equiv 0 \text{ and } \Delta(I + \gamma\mathcal{A}^{\frac{1}{2}})^{-1}\mathcal{A}^{-\frac{1}{2}} f = -(I + \gamma\mathcal{A}^{\frac{1}{2}})^{-1} f; \quad (3.27)$$

By (3.24), $r_t \in L_2(0,T; \mathcal{D}(\mathcal{A}^{\frac{1}{2}}))$ so that $\Delta r_t = -\mathcal{A}^{\frac{1}{2}} r_t$ (3.28)

With reference to the last two terms on the RHS of (3.4), we have the following equality where the first step below follows from integration by parts and (3.28), whereas the second step follows from (3.18):

$$\int_0^T \left(DD^*\mathcal{A}^{\frac{1}{2}}v_{tt} - \mathcal{A}^{-\frac{1}{2}}f_t, h_0\cdot\nabla(\Delta r)\right)_{L_2(\Gamma_0)} dt$$

$$-\left[\left(r_t+\gamma\Delta r_t, h_0\cdot\nabla(\Delta r)\right)_{L_2(\Gamma_0)}\right]_0^T \tag{3.29}$$

$$=\left[\left(DD^*\mathcal{A}^{\frac{1}{2}}v_t - \mathcal{A}^{-\frac{1}{2}}f - r_t + \gamma\mathcal{A}^{\frac{1}{2}}r_t, h_0\cdot\nabla(\Delta r)\right)_{L_2(\Gamma_0)}\right]_0^T$$

$$+\int_0^T \left(DD^*\mathcal{A}^{\frac{1}{2}}v_t - \mathcal{A}^{-\frac{1}{2}}f, h_0\cdot\nabla(\mathcal{A}^{\frac{1}{2}}r_t)\right)_{L_2(\Gamma_0)} dt$$

$$=\left[\left(-(I+\gamma\mathcal{A}^{\frac{1}{2}})r_t - \mathcal{A}^{\frac{1}{2}}v - r_t + \gamma\mathcal{A}^{\frac{1}{2}}r_t, h_0\cdot\nabla(\Delta r)\right)_{L_2(\Gamma_0)}\right]_0^T$$

$$+\int_0^T \left(DD^*\mathcal{A}^{\frac{1}{2}}v_t - \mathcal{A}^{-\frac{1}{2}}f, h_0\cdot\nabla(\mathcal{A}^{\frac{1}{2}}r_t)\right)_{L_2(\Gamma_0)} dt \tag{3.29}$$

$$=-\left[\left(2r_t + \mathcal{A}^{\frac{1}{2}}v, h_0\cdot\nabla(\Delta r)\right)_{L_2(\Gamma_0)}\right]_0^T$$

$$+\int_0^T \left(DD^*\mathcal{A}^{\frac{1}{2}}v_t - \mathcal{A}^{-\frac{1}{2}}f, h_0\cdot\nabla(\mathcal{A}^{\frac{1}{2}}r_t)\right)_{L_2(\Gamma_0)} dt$$

Therefore, we see by means of (3.4) and (3.29) that

$$\text{RHS of (3.4)} = I + II + III \text{ where}$$

$$I = -\left[\left(2r_t + \mathcal{A}^{\frac{1}{2}}v, h_0\cdot\nabla(\Delta r)\right)_{L_2(\Gamma_0)}\right]_0^T$$

$$II = \frac{1}{2}\int_{\Sigma_0}|\nabla(\Delta r)|^2 d\Sigma_0 + \frac{3}{2}\int_{\Sigma_0}|\nabla r_t|^2 d\Sigma_0 + \frac{\gamma}{2}\int_{\Sigma_0}|\Delta r_t|^2 d\Sigma_0$$

$$III = \int_0^T \left(DD^*\mathcal{A}^{\frac{1}{2}}v_t - \mathcal{A}^{-\frac{1}{2}}f, h_0\cdot\nabla(\mathcal{A}^{\frac{1}{2}}r_t)\right)_{L_2(\Gamma_0)} dt$$

Therefore, Proposition 3.5 will be proved as soon as we show the estimates claimed below:

$$I \le C_{h_0\gamma}[E_v(0)+E_v(T)] + \tag{3.30}$$

$$II \ge C_\gamma \int_0^T E_v(t)dt - C\int_0^T \left[\left(\frac{\partial v_t}{\partial x_1}(t,0)\right)^2 + \left(\frac{\partial v_t}{\partial x_1}(t,1)\right)^2\right] dt$$

$$-C\int_{\Sigma_0} f^2\, d\Sigma_0 \tag{3.31}$$

$$III = \mathcal{O}\Bigg(\int_0^T \left[\left(\frac{\partial v_t}{\partial x_1}(t,0)\right)^2 + \left(\frac{\partial v_t}{\partial x_1}(t,1)\right)^2\right]dt$$

$$+\|f\|^2_{L_2(\Sigma_0)} + \|\mathcal{A}^{\frac{1}{2}}v\|^2_{L_2(\Sigma_0)} + \|r_t\|^2_{L_2(0,T;\mathcal{D}(\mathcal{A}^{\frac{1}{2}-\varepsilon}))}\Bigg) \tag{3.32}$$

Remark 3.4. *Up to this point, there has not been a significant difference between the analysis here and that in* [10]. *The main difference of the analysis here involves the proof of (3.32), which is induced by the coupling of the Kirchoff equation with the wave equation in the PDE system (1.1).*

Proof of (3.30): It follows immediately from the equivalence of norms stated in (3.22) and (3.23) and from the definition of energy $E_v(t)$ in (3.3).

Proof of (3.31): It follows from the equivalence of norms stated in (3.22) and (3.23) that

$$\begin{aligned}
&\frac{1}{2}\int_{\Sigma_0} |\nabla(\Delta r)|^2 d\Sigma_0 + \frac{3}{2}\int_{\Sigma_0} |\nabla r_t|^2 d\Sigma_0 + \frac{\gamma}{2}\int_{\Sigma_0} |\Delta r_t|^2 d\Sigma_0 \\
&\geq \frac{1}{2}\|\mathcal{A}^{\frac{1}{4}} v_t\|^2_{L_2(\Sigma_0)} + \frac{1}{2}\|(I+\gamma\mathcal{A}^{\frac{1}{2}})^{-1} r_t\|^2_{L_2(\Sigma_0)} \\
&= \frac{1}{2}\|\mathcal{A}^{\frac{1}{4}} v_t\|^2_{L_2(\Sigma_0)} + \frac{1}{2}\|\mathcal{A}^{\frac{1}{2}} v\|^2_{L_2(\Sigma_0)} \\
&\qquad + \mathcal{O}\Big(\int_0^T [\|D^*\mathcal{A}^{\frac{1}{2}} v_t\|^2_{L_2(\partial\Gamma_0)} + \|\mathcal{A}^{-\frac{1}{2}} f\|^2_{L_2(\Gamma_0)}] dt\Big)
\end{aligned}$$

To finish the proof (3.31), we first recall that $\|\mathcal{A}^{\frac{1}{4}} v_t\|_{L_2(\Sigma_0)}$ is equivalent to $\|v_t\|_{\mathcal{D}(\mathcal{A}_\gamma^{\frac{1}{4}})}$ and then recall (2.24) for the term $\|D^*\mathcal{A}^{\frac{1}{2}} v_t\|_{L_2(\partial\Gamma_0)}$.

Proof of (3.32): Recall from (3.18), (3.28), and (3.27) that

$$\Delta r_t = -\mathcal{A}^{\frac{1}{2}} r_t \tag{3.33}$$

$$= \mathcal{A}^{\frac{1}{2}}(I+\gamma\mathcal{A}^{\frac{1}{2}})^{-1}[\mathcal{A}^{\frac{1}{2}} v + DD^*\mathcal{A}^{\frac{1}{2}} v_t] - \mathcal{A}^{\frac{1}{2}}(I+\gamma\mathcal{A}^{\frac{1}{2}})^{-1}\mathcal{A}^{-\frac{1}{2}} f$$

$$= F - (I+\gamma\mathcal{A}^{\frac{1}{2}})^{-1} f \tag{3.34}$$

where $F = \mathcal{A}^{\frac{1}{2}}(I+\gamma\mathcal{A}^{\frac{1}{2}})^{-1}[\mathcal{A}^{\frac{1}{2}} v + DD^*\mathcal{A}^{\frac{1}{2}} v_t]$. Therefore, we can write

$$III = III_a + III_b + III_c \tag{3.35}$$

where

$$\begin{aligned}
III_a &= \int_0^T \Big(DD^*\mathcal{A}^{\frac{1}{2}} v_t, h_0\cdot\nabla F\Big)_{L_2(\Gamma_0)} dt \\
III_b &= \int_0^T \Big(DD^*\mathcal{A}^{\frac{1}{2}} v_t, h_0\cdot\nabla(I+\gamma\mathcal{A}^{\frac{1}{2}})^{-1} f\Big)_{L_2(\Gamma_0)} dt \\
III_c &= \int_0^T (\mathcal{A}^{-\frac{1}{2}} f, h_0\cdot\nabla(\mathcal{A}^{\frac{1}{2}} r_t))_{L_2(\Gamma_0)} dt
\end{aligned} \tag{3.36}$$

Since the coupling term does not appear in III_a, we can readily apply Proposition 6.10 from [10] to get that

$$III_a = -\frac{1}{\gamma}\int_0^T \left(\mathcal{A}^{\frac{1}{2}}v, h_0\cdot\nabla(DD^*\mathcal{A}^{\frac{1}{2}}v_t)\right)_{L_2(\Gamma_0)}dt \\ +\mathcal{O}\Big(\int_0^T \|D^*\mathcal{A}^{\frac{1}{2}}v_t\|^2_{L_2(\partial\Gamma_0)}dt\Big) \\ +\mathcal{O}\Big(\int_0^T \|D^*\mathcal{A}^{\frac{1}{2}}v_t\|_{L_2(\partial\Gamma_0)}\cdot\|\mathcal{A}^{\frac{1}{2}}v\|_{L_2(\Sigma_0)}dt\Big) \tag{3.37}$$

Now recalling (2.26), we get

$$\int_0^T \left(\mathcal{A}^{\frac{1}{2}}v, h_0\nabla(DD^*\mathcal{A}^{\frac{1}{2}}v_t)\right)_{L_2(\Gamma_0)}dt \\ = \mathcal{O}\Big(\int_0^T \Big[\Big(\frac{\partial v_t}{\partial x_1}(t,0)\Big)^2 + \Big(\frac{\partial v_t}{\partial x_1}(t,1)\Big)^2\Big]^{\frac{1}{2}}\cdot\|\mathcal{A}^{\frac{1}{2}}v\|_{L_2(\Gamma_0)}dt\Big) \tag{3.38}$$

Therefore, (3.37), (3.1), and (2.25) imply

$$III_a = \mathcal{O}\Big(\int_0^T \Big[\Big(\frac{\partial v_t}{\partial x_1}(t,0)\Big)^2 + \Big(\frac{\partial v_t}{\partial x_1}(t,1)\Big)^2\Big]^{\frac{1}{2}}\cdot\|\mathcal{A}^{\frac{1}{2}}v\|_{L_2(\Gamma_0)}dt \\ + \int_0^T \Big[\Big(\frac{\partial v_t}{\partial x_1}(t,0)\Big)^2 + \Big(\frac{\partial v_t}{\partial x_1}(t,1)\Big)^2\Big]dt\Big) \tag{3.39}$$

As for III_b, we get from (2.25)

$$III_b = \mathcal{O}\Big(\int_0^T \Big[\Big(\frac{\partial v_t}{\partial x_1}(t,0)\Big)^2 + \Big(\frac{\partial v_t}{\partial x_1}(t,1)\Big)^2\Big]^{\frac{1}{2}}\cdot\|f\|_{L_2(\Sigma_0)}dt\Big) \tag{3.40}$$

Hence, to finish the proof of (3.31), we finally consider III_c. First notice that since $dim(\Gamma_0) = 1$,

$$III_c = (\mathcal{A}^{-\frac{1}{2}}f, h_0\cdot\nabla(\mathcal{A}^{\frac{1}{2}}r_t))_{L_2(\Gamma_0)} = \int_{\Gamma_0} h_0(\mathcal{A}^{-\frac{1}{2}}f)\frac{\partial(\mathcal{A}^{\frac{1}{2}}r_t)}{\partial x_1}d\Gamma_0. \tag{3.41}$$

By integration by parts on (3.41)and from the fact that $\mathcal{A}^{-\frac{1}{2}}f|_{\partial\Gamma_0} = 0$ (see (3.27)) we have that

$$III_c = \int_{\Sigma_0} h_0(\mathcal{A}^{-\frac{1}{2}}f)\frac{\partial(\mathcal{A}^{\frac{1}{2}}r_t)}{\partial x_1}d\Sigma_0 = -\int_{\Sigma_0}\frac{\partial}{\partial x_1}(h_0\mathcal{A}^{-\frac{1}{2}}f)\mathcal{A}^{\frac{1}{2}}r_t d\Sigma_0 \\ = -\int_{\Sigma_0}\Big(\mathcal{A}^{\varepsilon}\nabla(h_0\mathcal{A}^{-\frac{1}{2}}f), \mathcal{A}^{\frac{1}{2}-\varepsilon}r_t\Big)d\Sigma_0 \\ = \mathcal{O}\Big(\int_0^T \|f\|_{L_2(\Gamma_0)}\cdot\|r_t\|_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}-\varepsilon})}dt\Big) \tag{3.42}$$

(3.35), (3.39), (3.40), and (3.42) finish the proof of (3.32). ∎

Step 5: Analysis of LHS of (3.4)

Proposition 3.6. *With respect to the Kirchoff equation (3.16) and the identity (3.4), we have the following estimate:*

$$\begin{aligned}&\textit{LHS of } (3.4)\\ &\leq C\|r_t\|^2_{L_2(0,T;\mathcal{D}(\mathcal{A}^{\frac{1}{2}-\varepsilon}))} + \int_0^T \left[\left(\frac{\partial v_t}{\partial x_1}(t,0)\right)^2 + \left(\frac{\partial v_t}{\partial x_1}(t,1)\right)^2\right] dt\end{aligned} \tag{3.43}$$

Proof: Since $\Delta r = -\mathcal{A}^{\frac{1}{2}} r$ (see (3.28)), it is true that

$$\frac{\partial \Delta r}{\partial x_1} = -\frac{\partial \mathcal{A}^{\frac{1}{2}} r}{\partial x_1} = -\frac{\partial v_t}{\partial x_1} \tag{3.44}$$

As for the remaining term $\frac{\partial r_t}{\partial x_1}$ in LHS of (3.4), it follows by trace theory that

$$\int_0^T \left[\left(\frac{\partial r_t}{\partial x_1}(t,0)\right)^2 + \left(\frac{\partial r_t}{\partial x_1}(t,1)\right)^2\right] dt \leq C\|r_t\|^2_{L_2(0,T;\mathcal{D}(\mathcal{A}^{\frac{1}{2}-\varepsilon}))} \tag{3.45}$$

Combining (3.44) and (3.45) finishes the proof of Proposition 3.6. ■

<u>Final Step of Proof of Theorem 3.2:</u> Combining Propositions 3.5 and 3.6 almost completes the proof:

$$\begin{aligned}&\int_0^T \left[\left(\frac{\partial v_t}{\partial x_1}(t,0)\right)^2 + \left(\frac{\partial v_t}{\partial x_1}(t,1)\right)^2\right] dt + \int_{\Sigma_0} f^2 \, d\Sigma_0\\ &\geq C_{h_0\gamma}\left(\int_0^T E_v(t)dt - [E_v(T) + E_v(0)] - \|r_t\|^2_{L_2(0,T;\mathcal{D}(\mathcal{A}^{\frac{1}{2}-\varepsilon}))}\right)\end{aligned} \tag{3.46}$$

To complete the proof, note from (3.17) and (2.22) that

$$\begin{aligned}&\|r_t\|^2_{L_2(0,T;\mathcal{D}(\mathcal{A}^{\frac{1}{2}-\varepsilon}))} = C\|v\|^2_{L_2(0,T;\mathcal{D}(\mathcal{A}^{\frac{1}{2}-\varepsilon}))}\\ &\quad + \mathcal{O}\left(\int_0^T \left[\left(\frac{\partial v_t}{\partial x_1}(t,0)\right)^2 + \left(\frac{\partial v_t}{\partial x_1}(t,1)\right)^2\right]^{\frac{1}{2}} \cdot \|f\|_{L_2(\Gamma_0)} dt\right)\end{aligned} \tag{3.47}$$

3.2. **Wave Equation Part of the Damped System.** We will focus on the following Wave equation coming from the coupled P.D.E. system (1.1):

$$\begin{cases} z_{tt} = \Delta z & \text{on } Q,\\ \frac{\partial z}{\partial \nu}\big|_{\Gamma_1} = -z_t & \text{on } \Sigma_1,\\ \frac{\partial z}{\partial \nu}\big|_{\Gamma_0} = -g - z_t & \text{on } \Sigma_0. \end{cases} \tag{3.48}$$

$$\text{where } g \in L_2(\Sigma_0) \tag{3.49}$$

The main result of this section is the following estimate concerning the wave equation (3.48):

Theorem 3.3. *Let $h(x)$ be a smooth (C^2) vector field on Ω, satisfying the following condition:*

$$\int_\Omega H(x)w(x) \cdot w(x) d\Omega \geq \rho \int_\Omega w^2(x) d\Omega, \ \forall w(x) \in L_2(\Omega) \tag{3.50}$$

where $H(x)$ is the transpose of the Jacobian of $h(x)$ and $\rho > 0$ is a strictly positive constant. Then, with respect to the wave equation (3.48), we have the following inequality:

$$\begin{aligned} &C_T\Big\{\|z_t\|^2_{L_2(\Sigma)} + \|g\|^2_{L_2(\Sigma_0)} + \|z\|^2_{L_2(0,T;H^{\frac{1}{2}+\varepsilon_1}(\Omega))} \\ &\quad + \|z\|^2_{H^{\frac{1}{2}+\varepsilon_1}(0,T;L_2(\Omega))}\Big\} \\ &\geq C_{h\alpha\varepsilon_1}\Big(\int_\alpha^{T-\alpha} E_z(t)\,dt - \big[E_z(\alpha) + E_z(T-\alpha)\big]\Big) \end{aligned} \tag{3.51}$$

We have the following corollary of Theorem 3.3:

Corollary 3.7. *Let $h(x)$ be a smooth (C^2) vector field on Ω chosen as in Theorem 3.3. Then, with respect to the wave equation part of (1.1) with $u(t) \equiv 0$, we have the following inequality:*

$$\begin{aligned} &C_T\Big\{\|z_t\|^2_{L_2(\Sigma)} + \|v_t\|^2_{L_2(\Sigma_0)} + \|z\|^2_{L_2(0,T;H^{\frac{1}{2}+\varepsilon_1}(\Omega))} \\ &\quad + \|z\|^2_{H^{\frac{1}{2}+\varepsilon_1}(0,T;L_2(\Omega))}\Big\} \\ &\geq C_{h\alpha\varepsilon_1}\Big(\int_\alpha^{T-\alpha} E_z(t)\,dt - \big[E_z(\alpha) + E_z(T-\alpha)\big]\Big) \end{aligned} \tag{3.52}$$

Proof: Since $v_t \in C\big([0,T]; H^1(\Gamma_0)\big)$ (see (3.11)), Theorem 3.3 holds true for the Kirchoff equation part of the coupled system (1.1) where g is replaced by v_t. ∎

The proof of the Theorem 3.3 will follow from the subsequent lemma and propositions. Before we proceed, let us introduce the following domains where $T > \alpha > 0$ and α is arbitrarily small:

$$Q_\alpha = (\alpha, T-\alpha) \times Q;\ \Gamma_\alpha = (\alpha, T-\alpha) \times \Gamma;\ \Gamma_{0\alpha} = (\alpha, T-\alpha) \times \Gamma_0$$

Proposition 3.8. *Let h be a smooth vector field on Ω. We denote the transpose of its Jacobian as H. Also, ν is the unit normal of the boundary Γ. Then the following (basic trace) identity holds true for the wave equation (3.48):*

$$\begin{aligned} &\int_{\Sigma_\alpha} \frac{\partial z}{\partial \nu} h\cdot\nabla z\, d\Sigma_\alpha + \frac{1}{2}\int_{\Sigma_\alpha} z_t^2 h\cdot\nu\, d\Sigma_\alpha - \frac{1}{2}\int_{\Sigma_\alpha} |\nabla z|^2 h\cdot\nu\, d\Sigma_\alpha \\ &= \int_{Q_\alpha} H\nabla z\cdot\nabla z\, dQ_\alpha + \frac{1}{2}\int_{Q_\alpha}\Big[z_t^2 - |\nabla z|^2\Big] divh\, dQ_\alpha \\ &\quad + \Big[\Big(z_t, h\cdot\nabla z\Big)_{L_2(\Omega)}\Big]_\alpha^{T-\alpha} \end{aligned} \tag{3.53}$$

Proof: We refer to [L-T.4] for the proof of the identity (3.53), where both sides of a Wave equation is multiplied by $h \cdot \nabla z$ and then integrated by parts. Let us note that the (3.53) is true for any wave equation as long as h

is a smooth vector field on a bounded open domain $\Omega \in \mathbb{R}^n$ with a smooth boundary Γ. ▪

Lemma 3.9. *With respect to the wave equation (3.48), the following equalities hold true:*

$$\int_{Q_\alpha} (z_t^2 - |\nabla z|^2)\, divh \; dQ_\alpha = \int_{\Sigma_\alpha} z_t z\, divh \; d\Sigma_\alpha + \int_{\Sigma_{0\alpha}} gz\, divh \; d\Sigma_{0\alpha} + \int_{Q_\alpha} z\nabla(divh) \cdot \nabla z \; dQ_\alpha + \Big[\Big(z_t, z\, divh \Big)_{L_2(\Omega)} \Big]_\alpha^{T-\alpha}, \tag{3.54}$$

$$\int_{Q_\alpha} (z_t^2 - |\nabla z|^2) \; dQ_\alpha = \int_{\Sigma_\alpha} z_t z \; d\Sigma_\alpha + \int_{\Sigma_{0\alpha}} gz \; d\Sigma_{0\alpha} + \Big[\big(z_t, z \big)_{L_2(\Omega)} \Big]_\alpha^{T-\alpha} \tag{3.55}$$

Proof: (3.55) follows from (3.54) with when h is chosen as $h = x$ so $divh = 2$. The proof of (3.54) is achieved through a standard application of energy methods via the multiplier $z\ div(h)$. ▪

We will need the following result which gives a bound for the tangential gradient $\frac{\partial z}{\partial \tau} = \nabla_\tau z$ (see [13]):

Proposition 3.10. *Consider the wave equation (3.48). For arbitrarily small $\alpha > 0$, there exist a constant $C_{T_{\alpha\varepsilon_1}} > 0$ such that*

$$\int_\alpha^{T-\alpha} \int_\Gamma \Big|\frac{\partial z}{\partial \tau}\Big|^2 d\Sigma \le C_{T\alpha\varepsilon_1} \Big\{ \int_0^T \int_\Gamma \Big(\Big|\frac{\partial z}{\partial \nu}\Big|^2 + z_t^2 \Big) d\Sigma + \|z\|^2_{H^{1/2+\varepsilon_1}(Q)} \Big\} \tag{3.56}$$

Let us now analyze the basic trace identity (3.53).

Proposition 3.11. *With reference to the wave equation (3.48), the following estimate is true:*

$$LHS\ of\ (3.53) \le C_{Th\alpha\varepsilon_1} \Big\{ \int_{\Sigma_\alpha} z_t^2 \; d\Sigma_\alpha + \int_{\Sigma_{0\alpha}} g^2 \; d\Sigma_{0\alpha} + \|z\|^2_{H^{1/2+\varepsilon_1}(Q_\alpha)} \Big\} \tag{3.57}$$

Proof: Since $h = (h \cdot \nu)\nu + (h \cdot \tau)\tau$,

$$h \cdot \nabla z = \frac{\partial z}{\partial \nu} h \cdot \nu + \frac{\partial z}{\partial \tau} h \cdot \tau \text{ on } \Gamma \text{ and } |\nabla z|^2 = \Big(\frac{\partial z}{\partial \nu}\Big)^2 + \Big(\frac{\partial z}{\partial \tau}\Big)^2 \text{ on } \Gamma.$$

Hence, we see from (3.53) that

$$\begin{aligned}\text{LHS of (3.53)} &= \frac{1}{2}\int_{\Sigma_\alpha}\Big(\frac{\partial z}{\partial \nu}\Big)^2 h\cdot\nu\; d\Sigma_\alpha + \int_{\Sigma_\alpha}\Big(\frac{\partial z}{\partial \nu}\frac{\partial z}{\partial \tau}\Big) h\cdot\nu\; d\Sigma_\alpha \\ &\quad + \frac{1}{2}\int_{\Sigma_\alpha} z_t^2 h\cdot\nu\; d\Sigma_\alpha - \frac{1}{2}\int_{\Sigma_\alpha}\Big(\frac{\partial z}{\partial \tau}\Big)^2 h\cdot\tau\; d\Sigma_\alpha \\ &\le C_h'\Big\{\frac{1}{2}\int_{\Sigma_\alpha}\Big(\frac{\partial z}{\partial \nu}\Big)^2 d\Sigma_\alpha + \int_{\Sigma_\alpha}\Big|\frac{\partial z}{\partial \nu}\frac{\partial z}{\partial \tau}\Big| d\Sigma_\alpha + \frac{1}{2}\int_{\Sigma_\alpha} z_t^2\; d\Sigma_\alpha \\ &\quad + \frac{1}{2}\int_{\Sigma_\alpha}\Big(\frac{\partial z}{\partial \tau}\Big)^2 d\Sigma_\alpha\Big\},\end{aligned}$$

where $C_h' = \sup_{x\in\Gamma}|h(x)|$. Next by Proposition 3.10 and the boundary conditions on $\frac{\partial z}{\partial \nu}\big|_\Sigma$ from the equation (3.48), we get that

$$\text{LHS of (3.53)} \le C_{Th\alpha\varepsilon_1}\Big\{\int_{\Sigma_{0\alpha}} g^2\; d\Sigma_{0\alpha} + \int_{\Sigma_\alpha} z_t^2\; d\Sigma_\alpha + \|z\|^2_{H^{1/2+\varepsilon_1}(Q_\alpha)}\Big\},$$

which finishes the proof of Proposition 3.11. ■

Let us now consider the right hand side of (3.53):

Proposition 3.12. *With reference to the wave equation (3.48), the following estimate is true:*

$$\begin{aligned}&\textit{RHS of (3.53)} \\ &\ge \Big(\frac{\rho-\varepsilon}{2}\Big)\int_\alpha^{T-\alpha} E_z(t)\; dt - C_{h,\varepsilon}\big[E_z(\alpha) + E_z(T-\alpha)\big] \\ &\quad - C_{h,\varepsilon}\Big(\int_{Q_\alpha} z^2\; dQ_\alpha + \int_{\Sigma_\alpha} z^2\; d\Sigma_\alpha + \int_{\Sigma_\alpha} z_t^2\; d\Sigma_\alpha + \int_{\Sigma_{0\alpha}} g^2\; d\Sigma_{0\alpha}\Big)\end{aligned} \tag{3.58}$$

Proof: The proof is a standard application of Lemma 3.9 to the identity (3.53). See, for instance, [16]. ■

<u>Final Step of Proof of Theorem 3.3:</u>

Combining Propositions 3.11 and 3.12 we get

$$\begin{aligned}&C_T\Big\{\|z_t\|^2_{L_2(\Sigma)} + \|g\|^2_{L_2(\Sigma_0)} + \|z\|^2_{L_2(\Sigma)} + \|z\|^2_{L_2(Q)} + \|z\|^2_{H^{\frac{1}{2}+\varepsilon_1}(\Omega)}\Big\} \\ &\ge C_{h\alpha\varepsilon_1}\Big(\int_\alpha^{T-\alpha} E_z(t)\; dt - \big[E_z(\alpha) + E_z(T-\alpha)\big]\Big)\end{aligned} \tag{3.59}$$

Next by trace theory we get that

$$\|z\|^2_{L_2(\Gamma)} \le C\|z\|^2_{H^{\frac{1}{2}}(\Omega)} \tag{3.60}$$

Finally, (3.59) and (3.60) finish the proof of Theorem 3.3 since $\|z\|^2_{L_2(0,T;H^{\frac{1}{2}}(\Omega))}$ and $\|z\|^2_{L_2(Q)}$ can be absorbed by $\|z\|^2_{H^{\frac{1}{2}+\varepsilon_1}(Q)}$. ∎

3.3. **Combined Analysis of Coupled P.D.E.'s.** Combining Corollary 3.2 and Corollary 3.7 over the interval $(\alpha, T-\alpha)$ and using the definition of energy (3.1) in associated with the PDE system (1.1), we get the following PDE estimate:

Lemma 3.13. *For T big enough, there exists a constant, $C_T > 0$ such that*

$$E(T) \le C_T\Big\{\Big\|\frac{\partial v_t}{\partial \nu}\Big\|^2_{L_2(\partial\Sigma_0)} + \|z_t\|^2_{L_2(\Sigma_0)} + \|v\|^2_{C([0,T];\,H^{2-\varepsilon}(\Gamma_0))} + \|v_t\|^2_{L_2(\Sigma_0)} + \|z\|^2_{H^{\frac{1}{2}+\varepsilon}(Q)}\Big\} \tag{3.61}$$

Note that we keep track of dependence of the constant C_T only on T, the others being insignificant. Also for convenience, we take $\varepsilon = \varepsilon_1$.

Proof: By Corollaries 3.2 and 3.7, there exists a constant $C_T > 0$ such that

$$\begin{aligned} &C_T\Big\{\Big\|\frac{\partial v_t}{\partial \nu}\Big\|^2_{L_2(\partial\Sigma_0)} + \|z_t\|^2_{L_2(\Sigma_0)} + \|v\|^2_{C([0,T];\mathcal{D}(\mathcal{A}^{\frac{1}{2}-\frac{\varepsilon}{4}}))} + \|v_t\|^2_{L_2(\Sigma_0)} + \|z\|^2_{H^{\frac{1}{2}+\varepsilon}(Q)}\Big\} \\ &\ge \int_\alpha^{T-\alpha} E(t)\,dt - \big[E(0) + E(\alpha) + E(T-\alpha) + E(T)\big] \end{aligned} \tag{3.62}$$

It follows from (3.4) that

$$\int_\alpha^{T-\alpha} E(t)\,dt \ge (T-2\alpha)\Big(E(0) - 2\Big\{\Big\|\frac{\partial v_t}{\partial \nu}\Big\|^2_{L_2(\partial\Sigma_0)} + \|z_t\|^2_{L_2(\Sigma)}\Big\}\Big) \tag{3.63}$$

Since $\mathcal{D}(\mathcal{A}^{\frac{1}{2}-\frac{\varepsilon}{4}}) = H^{2-\varepsilon}(\Gamma_0)$ (see (2.5)), we have that

$$\|v\|^2_{C([0,T];\mathcal{D}(\mathcal{A}^{\frac{1}{2}-\frac{\varepsilon}{4}}))} = \|v\|^2_{C([0,T];\,H^{2-\varepsilon}(\Gamma_0))} \tag{3.64}$$

It then follows from (3.62), (3.63), and (3.64) that there exists a positive constant still denoted as $C_T > 0$ such that

$$\begin{aligned} &C_T\Big\{\Big\|\frac{\partial v_t}{\partial \nu}\Big\|^2_{L_2(\partial\Sigma_0)} + \|z_t\|^2_{L_2(\Sigma_0)} + \|v\|^2_{C([0,T];\,H^{2-\varepsilon}(\Gamma_0))} + \|v_t\|^2_{L_2(\Sigma_0)} + \|z\|^2_{H^{\frac{1}{2}+\varepsilon}(Q)}\Big\} \\ &\ge (T-2\alpha-4)E(0) \ge (T-2\alpha-4)E(T) \end{aligned}$$

Hence Lemma 3.13 is true when $T > 2\alpha + 4$. ∎

In the last lemma, we have an estimate for the energy of the system (1.1) by the dissipative terms plus lower order terms (with respect to the norm of the energy). Our last step is to absorb these lower order terms by means of a standard compactness/uniqueness argument.

Proposition 3.14. *With respect to the coupled P.D.E. system (1.1), for T big enough, there exists a constant $C_T > 0$ such that*

$$C_T\Big(\int_\Sigma z_t^2\, d\Sigma + \int_{\partial\Sigma_0}\Big(\frac{\partial v_t}{\partial\nu}\Big)^2\, d(\partial\Sigma_0)\Big) \geq \|v\|^2_{C([0,T];\,H^{2-\varepsilon}(\Gamma_0))} + \|v_t\|^2_{L_2(\Sigma_0)} + \|z\|^2_{H^{\frac{1}{2}+\varepsilon}(Q)} \tag{3.65}$$

Proof: It follows by a contradiction argument. Assume that there exists a sequence $\{z_n(t), z_n^{'}(t), v_n(t), v_n^{'}(t)\}$ of solutions to the problem (1.1) corresponding to the sequence of initial conditions $\{z_{n0}, z_{n1}, v_{n0}, v_{n1}\}$ such that

$$\|v_n\|^2_{C([0,T];\,H^{2-\varepsilon}(\Gamma_0))} + \|v_n{}'\|^2_{L_2(\Sigma_0)} + \|z_n\|^2_{H^{\frac{1}{2}+\varepsilon,\frac{1}{2}+\varepsilon}(Q)} = 1 \text{ and}$$
$$\lim_{n\to\infty}\left(\|z_n'\|^2_{L_2(\Sigma)} + \left\|\frac{\partial v_n'}{\partial\nu}\right\|^2_{L_2(\partial\Sigma_0)}\right) = 0 \tag{3.66}$$

Let us now define the energy $E_n(t)$ of the system (1.1) corresponding to the initial conditions $\{z_{n0}, z_{n1}, v_{n0}, v_{n1}\}$ as we did in (3.1), (3.2), and (3.3) so that we have (see (3.4)):

$$E_n(T) = E_n(0) - 2\left(\|z_n'\|^2_{L_2(\Sigma)} + \left\|\frac{\partial v_n'}{\partial\nu}\right\|^2_{L_2(\partial\Sigma_0)}\right) \tag{3.67}$$

We claim that $\{z_n(0), z_n^{'}(0), v_n(0), v_n^{'}(0)\}$ is uniformly bounded in Y_0. To see it, we recall the equation (3.61) from Lemma 3.13 and substitute it into (3.67). Therefore, we get first by rearranging the dissipation terms and then by (3.66) that for T large enough (see Lemma 3.13), there exists a constant still denoted by C_T such that $\forall\ n$

$$E_n(0) \leq C_T\Big\{\Big\|\frac{\partial v_t}{\partial\nu}\Big\|^2_{L_2(\partial\Sigma_0)} + \|z_t\|^2_{L_2(\Sigma_0)} + \|v\|^2_{C([0,T];\,H^{2-\varepsilon}(\Gamma_0))} + \|v_t\|^2_{L_2(\Sigma_0)} + \|z\|^2_{H^{\frac{1}{2}+\varepsilon}(Q)}\Big\} \leq \text{ const} \tag{3.68}$$

Hence, $E_n(0)$ is uniformly bounded and the claim is proved. By (3.3), there exists a subsequence still ordered by n such that

$$\{z_{n0}, z_{n1}, v_{n0}, v_{n1}\} \longrightarrow \{\tilde{z}_0, \tilde{z}_1, \tilde{v}_0, \tilde{v}_1\} \text{ weakly in } Y_0.$$

Now denote by $\{\tilde{z}(t), \tilde{z}_t(t), \tilde{v}(t), \tilde{v}_t(t)\}$ the solution to the problem (1.1) corresponding to the initial conditions $\{\tilde{z}_0, \tilde{z}_1, \tilde{v}_0, \tilde{v}_1\}$. Since $e^{A_F t}$ is a s.c. semigroup of contractions on Y_0, $\{z_n(t), z_n^{'}(t), v_n(t), v_n^{'}(t)\}$ is uniformly bounded on $C([0,T];Y_0)$; therefore, there exists a subsequence still ordered by n such that

$\{z_n(t), z_n^{'}(t), v_n(t), v_n^{'}(t)\} \longrightarrow \{\tilde{z}(t), \tilde{z}^{'}(t), \tilde{v}(t), \tilde{v}^{'}(t)\}$ weakly in $C([0,T];Y_0)$. It then follows by compact imbeddings below

$$\begin{cases} H^2(\Gamma_0) \hookrightarrow H^{2-\varepsilon}(\Gamma_0), \\ H_0^1(\Gamma_0) \hookrightarrow L_2(\Gamma_0), \\ H^1(\Omega) \hookrightarrow H^{1-\varepsilon}(\Omega), \text{ and} \\ H^1(0,T;L_2(\Omega)) \hookrightarrow H^{1-\varepsilon}(0,T;L_2(\Omega)) \end{cases} \tag{3.69}$$

that

$$\begin{cases} v_n \longrightarrow \tilde{v} \text{ strongly in } C([0,T];H^{2-\varepsilon}(\Gamma_0)), \\ v_n' \longrightarrow \tilde{v}_t \text{ strongly in } L_2(\Sigma_0) \\ z_n \longrightarrow \tilde{z} \text{ strongly in } H^{1-\varepsilon}(Q). \end{cases} \tag{3.70}$$

Therefore, by (3.70) and the limit in the assumptions noted in (3.66), we get that

$$\int_\Sigma \tilde{z}_t{}^2 \, d\Sigma_0 = \int_{\partial\Sigma_0} \left(\frac{\partial \tilde{v}_t}{\partial \nu}\right)^2 \, d(\partial\Sigma_0) = 0 \tag{3.71}$$

Hence, by use of (3.71) in the equation (1.1), $\{\tilde{z}, \tilde{v}\}$ satisfies the following equations:

$$\begin{cases} \tilde{z}_{tt} = \Delta \tilde{z} & \text{on } Q \\ \frac{\partial \tilde{z}}{\partial \nu}\big|_{\Gamma_1} = 0 & \text{on } \Sigma_1 \\ \frac{\partial \tilde{z}}{\partial \nu}\big|_{\Gamma_0} = -\tilde{v}_t & \text{on } \Sigma_0 \\ \tilde{z}_t\big|_\Gamma = 0 & \text{on } \Sigma \end{cases} \tag{3.72}$$

$$\begin{cases} \tilde{v}_{tt} - \gamma\Delta\tilde{v}_{tt} + \Delta^2\tilde{v} = 0 & \text{on } \Sigma_0 \\ \tilde{v}\big|_{\partial\Sigma_0} = \Delta\tilde{v}\big|_{\partial\Gamma_0} = \frac{\partial \tilde{v}_t}{\partial \nu} = 0 & \text{on } \partial\Sigma_0 \end{cases} \tag{3.73}$$

where $\{\tilde{z}(0,\cdot), \tilde{z}_t(0,\cdot), \tilde{v}(0,\cdot), \tilde{v}_t(0,\cdot)\} = \{\tilde{z_0}, \tilde{z_1}, \tilde{v_0}, \tilde{v_1}\} \in Y_0$

It is known that the solution of the Kirchoff problem (3.73) with three zero boundary conditions is the trivial solution (see [19]):

$$\tilde{v} \equiv \tilde{v}_t \equiv 0 \tag{3.74}$$

Thus, using (3.74) in (3.72), we see that $\tilde{z}$ satisfies the following:

$$\begin{cases} \tilde{z}_{tt} = \Delta \tilde{z} & \text{on } Q \\ \frac{\partial \tilde{z}}{\partial \nu}\big|_\Gamma \equiv \tilde{z}_t\big|_\Gamma \equiv 0 & \text{on } \Sigma \end{cases} \tag{3.75}$$

where $\{\tilde{z_0}, \tilde{z_1}\} \in H^1(\Omega) \times L_2(\Omega)$ such that by (2.29)

$$\int_\Gamma \tilde{z_0} \, d\Gamma + \int_\Omega \tilde{z_1} \, d\Omega = 0 \tag{3.76}$$

Lumer-Phillips theorem holds true for the uncoupled $\tilde{z}$ problem (3.75) in $H^1(\Omega) \times L_2(\Omega)$ subject to (3.76). Hence, we also get that

$$\tilde{z} \equiv \tilde{z}_t \equiv 0 \tag{3.77}$$

However, (3.74) and (3.77) contradict with the assumptions in (3.66) and the proof of Proposition 3.14 is finished. ∎

Final Step of the Proof of Theorem 3.1:

By Lemma 3.13 and Proposition 3.14, we see that when T is large enough, there exists a positive constant $C_T > 0$ such that

$$C_T E(T) \leq \|z_t\|^2_{L_2(\Sigma)} + \left\| \frac{\partial v_t}{\partial \nu} \right\|^2_{L_2(\partial \Sigma_0)}$$

Now recall the equality (3.4) rewritten below as:

$$E(T) + 2\|z_t\|^2_{L_2(\Sigma)} + 2\left\| \frac{\partial v_t}{\partial \nu} \right\|^2_{L_2(\partial \Sigma_0)} = E(0)$$

Consequently, we get that $(2C_T + 1)E(T) \leq E(0)$ and hence $E(T) < E(0)$, implying that $\|e^{A_F t}\|_{\mathcal{L}(Y)} < 1$. Therefore, A_F is the infinitesimal generator of a uniformly stable semigroup on Y and the proof of Theorem 3.1 is finished. ∎

References

[1] G. Avalos, The exponential stability of a coupled hyperbolic/parabolic system arising in structural acoustics, *Abstr. Appl. Anal.* **1** (1996), 203–219.

[2] G. Avalos and I. Lasiecka, Differential Riccati equation for the active control of a problem in structural acoustics, *J. Optim. Theory Appl.* **91** (1996), 695–728.

[3] G. Avalos and I. Lasiecka, Uniform decay rates of solutions to a structural acoustic model with nonlinear dissipation, *Appl. Math. Comput. Sci.* **8** (1998), 287–312.

[4] J. T. Beale, Spectral properties of an acoustic boundary condition, *Indiana Univ. Math. J.* **25** (1976), 895–917.

[5] M. and R. Triggiani, Sharp regularity of a coupled system of a wave and a Kirchoff equation with point control arising in noise reduction, *Differential Integral Equations*, to appear.

[6] S. Chen and R. Triggiani, Proof of extensions of two conjectures on structural damping for elastic systems, *Pacific J. Math.* **136** (1989), 15–55.

[7] F. Fahroo and C. Wang, A new model for acoustic interaction and its exponential stability, *Manuscript*, 1996.

[8] W. Littman and B. Liu, On the spectral properties and stabilization of acoustic flow, *Univ. Minnesota, IMA Preprint Series*, **#1436**, 1996.

[9] I. Lasiecka and R. Triggiani, Regularity theory for a class of Euler-Bernoulli equations: A cosine operator approach, *Boll. Un. Mat. Ital.* **7** (1989), 199–228.

[10] I. Lasiecka and R. Triggiani, Exact controllability and uniform stabilization of Kirchoff plates with boundary control only on $\Delta w|_\Sigma$ and homogeneous boundary displacement, *J. Differential Equations*, **93** (1991), 62–101.

[11] I. Lasiecka and R. Triggiani, Differential and algebraic Riccati equations with application to boundary/point control problems: continuous theory and approximation theory, *Lecture Notes in Control and Information Sciences*, **164**, Springer-Verlag, New York, 1991.

[12] I. Lasiecka and R. Triggiani, Regularity theory of hyperbolic equations under boundary terms, *Appl. Math. Optim.* **10** (1983), 275–285.

[13] I. Lasiecka and R. Triggiani, Uniform Stabilization of the Wave Equation with Dirichlet or Neumann Feedback Control Without Geometrical Conditions, *Appl. Math. Optim.* **25** (1992), 189–224.

[14] I. Lasiecka and R. Triggiani, Recent advances in regularity of second-order hyperbolic mixed problems, and applications, invited paper for *Book Series, Dynamics Reported*, Springer-Verlag, New York.

[15] I. Lasiecka and R. Triggiani, Deterministic optimal control theory for PDE's: an abstract approach, *Encyclopedia of Mathematics and its Applications*, Cambridge Univ. Press, 1998.
[16] I. Lasiecka and R. Triggiani, Exact Controllability for the Wave Equation with Neumann Boundary Control, *Appl. Math. Optim.* **19** (1989), 243–290.
[17] C. McMillan and R. Triggiani, Min-max game theory and algebraic Riccati equations for boundary control problems with continuous input-solution map, Part 2: the general case, *Appl. Math. Optim.* **29** (1994), 1–65.
[18] P. M. Mores and K. U. Ingard, *Theoretical Acoustics*, McGraw-Hill, New York, 1968.
[19] N. Ourada and R. Triggiani, Uniform stabilization of the Euler-Bernoulli Equation with feedback operator only in the Neumann boundary condition, *Differential Integral Equations*, **4** (1991), 277–292.
[20] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differnetial Equations*, Springer-Verlag, New York, 1983.
[21] D. L. Russell, Controllability and stabilizability theory for linear partial differential equations, recent progress and open questions, *SIAM Rev.* **20** (1978), 639–739.
[22] D. L. Russell, Mathematical models for elastic beam and their control-theoretic properties, *Semigroup Theory and Applications, Pitman Research Notes*, **152** (1986), 177–217.
[23] R. Triggiani, Regularity with interior control, Part 2:Kirchoff equations, *J. Differential Equations*, **103** (1993), 394–420.
[24] R. Triggiani, Min-max game theory for partial differential equations with boundary/point control and disturbance. An abstract approach., *Lecture Notes in Control and Inform. Sci.* **197**, Springer-Verlag, London, 1994.

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