SEMILINEAR ELLIPTIC EQUATIONS HAVING ASYMPTOTIC LIMITS AT ZERO AND INFINITY

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We obtain nontrivial solutions for semilinear elliptic boundary value problems having resonance both at zero and at infinity, when the nonlinear term has asymptotic limits.

1. Introduction

Let Ω be a smooth, bounded domain in $\mathbb{R}^n$, and let A be a selfadjoint operator on $L^2(\Omega)$. We assume that

$$C_0^\infty(\Omega) \subset D := D\big(|A|^{1/2}\big) \subset H^{m,2}(\Omega) \tag{1.1}$$

holds for some $m > 0$, and $\sigma_e(A) = \phi$ with A bounded from below. Let $f(x,t)$ be a Carathéodory function on $\overline{\Omega} \times \mathbb{R}$ satisfying

$$\begin{aligned} f(x,t) &= a_0 t + p_0(x,t), \quad p_0(x,t) = o(t) \quad \text{as } t \longrightarrow 0, \\ f(x,t) &= at + p(x,t), \quad p(x,t) = o(t) \quad \text{as } |t| \longrightarrow \infty. \end{aligned} \tag{1.2}$$

The object of this paper is to prove the following theorem.

Theorem 1.1. *Assume that there is a $\lambda \in \sigma(A)$ such that either*

$$a_0 \le \lambda \le a \tag{1.3}$$

or

$$a \le \lambda \le a_0. \tag{1.4}$$

If $a_0 \in \sigma(A)$, assume also that there is a $\sigma \in (2, 2^*)$, $2^* = 2n/(n-2)$, such that

$$\frac{t p_0(x,t)}{|t|^\sigma} \longrightarrow \alpha_\pm \quad \text{as } t \longrightarrow \pm 0 \tag{1.5}$$

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and

$$\int_{y>0} \alpha_+ |y|^\sigma + \int_{y<0} \alpha_- |y|^\sigma > 0, \quad y \in E(a_0)\backslash\{0\}, \tag{1.6}$$

if $\lambda \le a_0$ *and* < 0 *if* $a_0 \le \lambda$, *where* $E(b) = \{u \in D : (A-b)u = 0\}$. *If* $a \in \sigma(A)$, *assume also that there is a* $\tau \in (1,2)$ *such that*

$$\frac{tp(x,t)}{|t|^\tau} \longrightarrow \beta_\pm \quad \text{as } t \longrightarrow \pm\infty \tag{1.7}$$

and

$$\int_{y>0} \beta_+ |y|^\tau + \int_{y<0} \beta_- |y|^\tau > 0, \quad y \in E(a)\backslash\{0\} \tag{1.8}$$

if $\lambda \le a$ *and* < 0 *if* $a \le \lambda$. *Finally assume that*

$$|f(x,t)| \le C(|t|+1), \quad x \in \Omega,\ t \in \mathbb{R}. \tag{1.9}$$

Then

$$Au = f(x,u) \tag{1.10}$$

has a nontrivial solution.

The proof of Theorem 1.1 will be accomplished by means of a series of lemmas given in the next section.

Many authors have studied special cases of problem (1.10) under hypotheses (1.2) beginning with the work of Amann-Zehnder [1], who considered the Dirichlet problem

$$-\Delta u = f(u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{1.11}$$

They assumed that $f(t) \in C^1(\mathbb{R})$ and that either

$$f'(0) < \lambda < f'(\infty) \tag{1.12}$$

or

$$f'(\infty) < \lambda < f'(0). \tag{1.13}$$

They did not allow $f'(\infty)$ to be in $\sigma(A)$. Since then many authors have weakened some of these requirements (see [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22], and the references therein). In most cases, $f(x,t)$ is required to be continuously differentiable with respect to t, and a and a_0 are not both allowed to be in $\sigma(A)$. In Theorem 1.1, we only require the continuity of $f(x,t)$ with respect to t, allow either or both a_0 and a to be in $\sigma(A)$ and permit $a = a_0 = \lambda$.

2. Lemmas

Theorem 1.1 will be established via a series of lemmas. In describing them, we let Ω be a smooth, bounded domain in $\mathbb{R}^n$, and we let A be a selfadjoint operator on $L^2(\Omega)$.

We assume that

$$C_0^\infty(\Omega) \subset D := D\big(|A|^{1/2}\big) \subset H^{m,2}(\Omega) \tag{2.1}$$

holds for some $m > 0$, and $\sigma_e(A) \subset (0,\infty)$. We use the notation

$$a(u,v) = (Au,v), \quad a(u) = a(u,u), \quad u,v \in D. \tag{2.2}$$

D becomes a Hilbert space if we use the scalar product

$$(u,v)_D = \big(|A|u,v\big) + \big(P_0 u,v\big), \quad u,v \in D, \tag{2.3}$$

and its corresponding norm, where P_0 is the projection onto $N(A)$. Let $f(x,t)$ be a Carathéodory function on $\overline{\Omega} \times \mathbb{R}$ satisfying

$$|f(x,t)| \le V(x)^q \big(|t|^{q-1} + 1\big), \quad x \in \Omega,\ t \in \mathbb{R}, \tag{2.4}$$

and

$$\frac{f(x,t)}{V(x)^q} = o\big(|t|^{q-1}\big) \quad \text{as } |t| \longrightarrow \infty, \text{ uniformly}, \tag{2.5}$$

where $q > 2$ satisfies

$$q \le \frac{2n}{n-2m}, \quad 2m < n,\ q < \infty,\ n \le 2m, \tag{2.6}$$

and $V(x) > 0$ is a function in $L^q(\Omega)$ such that

$$\|Vu\|_q \le C\|u\|_D, \quad u \in D. \tag{2.7}$$

(The norm on the left in (2.7) is that of $L^q(\Omega)$.)

Let

$$V = \bigoplus_{\lambda<0} N(A-\lambda). \tag{2.8}$$

By assumption, $p = \dim N(A) + \dim V < \infty$. Let $W = [V \bigoplus N(A)]^\perp$, and let P_-, P_0, P_+ be the projections onto V, $N(A)$, W, respectively. Let $\underline{\lambda}(\overline{\lambda})$ denote the largest (smallest) point in the negative (positive) spectrum of A. Then

$$\begin{aligned} (Av,v) &\le \underline{\lambda}\|v\|^2, \quad v \in V, \\ (Aw,w) &\ge \overline{\lambda}\|w\|^2, \quad w \in W. \end{aligned} \tag{2.9}$$

We let

$$2G(u) = a(u) - 2\int_\Omega F(x,u), \tag{2.10}$$

where

$$F(x,t) = \int_0^t f(x,s)\,ds. \tag{2.11}$$

As is well known, G is in C^1 in D, and

$$\big(G'(u),h\big) = a(u,h) - \big(f(u),h\big), \quad u,h \in D, \tag{2.12}$$

where we write $f(u)$ in place of $f(x,u(x))$. Moreover, u is a solution of

$$Au = f(x,u) \tag{2.13}$$

if and only if it satisfies

$$G'(u) = 0. \tag{2.14}$$

In our first result we make use of the following assumption:
(A) there is a constant $\sigma \in (2,2^*)$ such that

$$\frac{f(x,t)t}{|t|^\sigma} \longrightarrow \alpha_\pm(x) \quad \text{as } t \longrightarrow \pm 0, \text{ uniformly in } x, \tag{2.15}$$

where

$$\int_{y>0} \alpha_+|y|^\sigma + \int_{y<0} \alpha_-|y|^\sigma > 0, \quad y \in N(A)\backslash\{0\}. \tag{2.16}$$

We have the following lemma.

Lemma 2.1. *If* 0 *is an isolated solution of (2.13), and (A) holds, then*

$$C_k(G,0) \cong \delta_{pk}\mathbb{Z} \quad \forall k, \tag{2.17}$$

where $p = \dim V + \dim N(A)$.

Proof. We define

$$2J(u) = \|P_+u\|^2 - \|P_-u\|^2 - \|P_0u\|^2, \tag{2.18}$$

and let

$$\begin{aligned} H_t(u) &= a(u) - 2(1-t)\int_\Omega F(x,u), \\ G_t(u) &= H_t(u) + tJ(u), \quad t \in [0,1]. \end{aligned} \tag{2.19}$$

We note that there is a $\rho > 0$ such that

$$\big(H_t'(u), J'(u)\big) > 0, \quad 0 < \|u\|_D \leq \rho. \tag{2.20}$$

For if (2.20) did not hold, there would be a sequence $\{u_k\} \subset D$ such that

$$\big(H_t'(u_k), J'(u_k)\big) \leq 0, \tag{2.21}$$

and $\rho_k = \|u_k\|_D \to 0$. Let $\tilde{u}_k = u_k/\rho_k$, and write $\tilde{u}_k = \tilde{v}_k + \tilde{y}_k + \tilde{w}_k$, $\tilde{v}_k \in V$, $\tilde{y}_k \in N(A)$, and $\tilde{w}_k \in W$. In particular, we have

$$\big(G'(u_k),h\big) = a(u_k,h) - \big(f(u_k),h\big). \tag{2.22}$$

Thus,

$$\frac{\left(H_t'(u_k), J'(u_k)\right)}{\rho_k^2} = \left\|\tilde{w}_k\right\|_D^2 + \left\|\tilde{v}_k\right\|_D^2 - \frac{(1-t)\left(f(u_k), \hat{u}_k\right)}{\rho_k^2}. \tag{2.23}$$

(Here we take $\hat{u} = w - v - y$.) From this we conclude that (2.21) implies

$$\left\|\tilde{v}_k\right\|_D + \left\|\tilde{w}_k\right\|_D \longrightarrow 0. \tag{2.24}$$

Since $\|\tilde{u}_k\|_D = 1$, we must have a renamed subsequence such that $\tilde{y}_k \to \tilde{y}$ strongly in D with $\|\tilde{y}\|_D = 1$. Consequently,

$$\frac{\left(H_t'(u_k), J'(u_k)\right)}{\rho_k^\sigma} \geq -\frac{(1-t)\left(f(u_k), \hat{u}_k\right)}{\rho_k^\sigma}. \tag{2.25}$$

But

$$\begin{aligned} -\int_\Omega \frac{f(x, u_k)\tilde{y}_k}{\rho_k^{\sigma-1}} &= -\int_\Omega \left[\frac{u_k f(x, u_k)}{|u_k|^\sigma}\right]\left[\left|\tilde{u}_k\right|^{\sigma-2}\tilde{u}_k\tilde{y}_k\right] \\ &\longrightarrow \int_{\tilde{y}>0} \alpha_+\left|\tilde{y}\right|^\sigma + \int_{\tilde{y}<0} \alpha_-\left|\tilde{y}\right|^\sigma > 0 \end{aligned} \tag{2.26}$$

for a subsequence by hypothesis (A), since $\tilde{y} \neq 0$. Moreover,

$$\int_\Omega \frac{f(x, u_k)\tilde{v}_k}{\rho_k^{\sigma-1}} \longrightarrow 0, \qquad \int_\Omega \frac{f(x, u_k)\tilde{w}_k}{\rho_k^{\sigma-1}} \longrightarrow 0. \tag{2.27}$$

This contradicts (2.21) and shows that (2.20) holds for $t < 1$. It is obvious for $t = 1$. Now

$$\left(G_t'(u), J'(u)\right) = \left(H_t'(u), J'(u)\right) + t\left(J'(u), J'(u)\right) \geq t\left\|J'(u)\right\|^2. \tag{2.28}$$

If u is a critical point of G_t, then $J'(u) = 0$, from which it follows that $u = 0$. Thus 0 is an isolated critical point of G_t. Since $2G_1(u) = [a(u) + J(u)]$,

$$G_1''(0) = A + P_+ - P_- - P_0. \tag{2.29}$$

By hypothesis,

$$AP_+ > 0, \qquad A\left(P_- + P_0\right) < 0. \tag{2.30}$$

Consequently, the Morse index of $G_1(0)$ is p. By the homotopy invariance of critical groups, we have

$$C_k(G, 0) \cong C_k\left(G_1, 0\right) \cong \delta_{pk}\mathbb{Z}. \tag{2.31}$$

This gives the desired conclusion. □

In our second result we make use of the following assumption:

(B) there is a constant $\sigma \in (2, 2^*)$ such that

$$\frac{f(x,t)t}{|t|^{\sigma}} \longrightarrow \alpha_{\pm}(x) \quad \text{as } t \longrightarrow \pm 0, \text{ uniformly in } x, \tag{2.32}$$

where

$$\int_{y>0} \alpha_{+}|y|^{\sigma} + \int_{y<0} \alpha_{-}|y|^{\sigma} < 0, \quad y \in N(A)\backslash\{0\}. \tag{2.33}$$

We have the following lemma.

Lemma 2.2. *If* 0 *is an isolated solution of (2.13), and (B) holds, then*

$$C_k(G,0) \cong \delta_{p_1 k}\mathbb{Z} \quad \forall k, \tag{2.34}$$

where $p_1 = \dim V$.

Proof. Now we define

$$2J(u) = \|P_{+}u\|^2 - \|P_{-}u\|^2 + \|P_0 u\|^2, \tag{2.35}$$

and let

$$H_t(u) = a(u) - 2(1-t)\int_{\Omega} F(x,u), \quad G_t(u) = H_t(u) + tJ(u), \quad t \in [0,1]. \tag{2.36}$$

We note that there is a $\rho > 0$ such that

$$\big(H_t'(u), J'(u)\big) > 0, \quad 0 < \|u\|_D \le \rho. \tag{2.37}$$

For if (2.37) did not hold, there would be a sequence $\{u_k\} \subset D$ such that

$$\big(H_t'(u_k), J'(u_k)\big) \le 0, \tag{2.38}$$

and $\rho_k = \|u_k\|_D \to 0$. Let $\tilde{u}_k = u_k/\rho_k$, and write $\tilde{u}_k = \tilde{v}_k + \tilde{y}_k + \tilde{w}_k$, $\tilde{v}_k \in V$, $\tilde{y}_k \in N(A)$, and $\tilde{w}_k \in W$. In particular, we have

$$\big(G'(u_k), h\big) = a(u_k, h) - \big(f(u_k), h\big). \tag{2.39}$$

Thus,

$$\frac{\big(H_t'(u_k), J'(u_k)\big)}{\rho_k^2} = \|\tilde{w}_k\|_D^2 + \|\tilde{v}_k\|_D^2 - \frac{(1-t)\big(f(u_k), \hat{u}_k\big)}{\rho_k^2}. \tag{2.40}$$

(Here we take $\hat{u} = w - v + y$.) From this we conclude that (2.38) implies

$$\|\tilde{v}_k\|_D + \|\tilde{w}_k\|_D \longrightarrow 0. \tag{2.41}$$

Since $\|\tilde{u}_k\|_D = 1$, we must have a renamed subsequence such that $\tilde{y}_k \to \tilde{y}$ strongly in D with $\|\tilde{y}\|_D = 1$. Consequently,

$$\frac{\big(H_t'(u_k), J'(u_k)\big)}{\rho_k^{\sigma}} \ge -\frac{(1-t)\big(f(u_k), \hat{u}_k\big)}{\rho_k^{\sigma}}. \tag{2.42}$$

But

$$\int_\Omega \frac{f(x,u_k)\tilde{y}_k}{\rho_k^{\sigma-1}} = \int_\Omega \left[\frac{u_k f(x,u_k)}{|u_k|^\sigma}\right]\left[|\tilde{u}_k|^{\sigma-2}\tilde{u}_k\tilde{y}_k\right] \longrightarrow \int_{\tilde{y}>0} \alpha_+|\tilde{y}|^\sigma + \int_{\tilde{y}<0} \alpha_-|\tilde{y}|^\sigma < 0 \tag{2.43}$$

for a subsequence by hypothesis (B), since $\tilde{y} \neq 0$. Moreover,

$$\int_\Omega \frac{f(x,u_k)\tilde{v}_k}{\rho_k^{\sigma-1}} \longrightarrow 0, \qquad \int_\Omega \frac{f(x,u_k)\tilde{w}_k}{\rho_k^{\sigma-1}} \longrightarrow 0. \tag{2.44}$$

This contradicts (2.21) and shows that (2.37) holds. Now

$$\left(G_t'(u), J'(u)\right) = \left(H_t'(u), J'(u)\right) + t\left(J'(u), J'(u)\right) \geq t\left\|J'(u)\right\|^2. \tag{2.45}$$

If u is a critical point of G_t, then $J'(u) = 0$, from which it follows that $u = 0$. Thus 0 is an isolated critical point of G_t. Since $2G_1(u) = [a(u) + J(u)]$,

$$G_1''(0) = A + P_+ - P_- + P_0. \tag{2.46}$$

By hypothesis,

$$A\left(P_+ + P_0\right) > 0, \qquad AP_- < 0. \tag{2.47}$$

Consequently, the Morse index of $G_1(0)$ is p_1. By the homotopy invariance of critical groups, we have

$$C_k(G,0) \cong C_k\left(G_1,0\right) \cong \delta_{p_1 k}\mathbb{Z}. \tag{2.48}$$

This gives the desired conclusion. □

Lemma 2.3. *If $N(A) = \{0\}$, 0 is an isolated solution of (2.13), and*

$$\frac{f(x,t)}{t} \longrightarrow 0 \quad \text{as } t \longrightarrow 0, \tag{2.49}$$

then (2.34) holds.

Proof. We follow the proof of Lemma 2.2. In this case $P_0 = 0$, and (2.37) holds because (2.38) implies (2.41), which is now the same as $\|u_k\|_D \to 0$. This contradicts the fact that $\|u_k\|_D = 1$. Thus, (2.45) holds. We can now follow the continuation of the proof of Lemma 2.2 keeping in mind that $P_0 = 0$. □

Our next result assumes

(C) there is a constant $\sigma \in (1,2)$ such that

$$\frac{f(x,t)t}{|t|^\sigma} \longrightarrow \alpha_\pm(x) \quad \text{as } t \longrightarrow \pm\infty, \text{ uniformly in } x, \tag{2.50}$$

where

$$\int_{y>0} \alpha_+ |y|^\sigma + \int_{y<0} \alpha_- |y|^\sigma > 0, \quad y \in N(A)\backslash\{0\}. \tag{2.51}$$

We have the following lemma.

LEMMA 2.4. *If (C) holds, then*

$$G(u) \longrightarrow -\infty \quad \text{as } \|u\|_D \longrightarrow \infty,\ u \in V \oplus N(A). \tag{2.52}$$

Proof. Assume that there is a sequence $\{u_k\} \subseteq V \oplus N(A)$ such that $\rho_k = \|u_k\|_D \to \infty$ and $G(u_k)$ is bounded from below. Let $\tilde{u}_k = u_k/\rho_k$, and write $\tilde{u}_k = \tilde{v}_k + \tilde{y}_k$, $\tilde{v}_k \in V$, $\tilde{y}_k \in N(A)$. Since

$$\frac{G(u_k)}{\rho_k^2} = -\|\tilde{v}_k\|_D^2 - 2\int_\Omega \frac{F(x,u_k)}{\rho_k^2}\,dx, \tag{2.53}$$

and $f(x,t)/t \to 0$ as $t \to \infty$, we see that $\|\tilde{v}_k\|_D \to 0$. Thus, there is a renamed subsequence such that $\tilde{u}_k \to \tilde{y}$ in D. Consequently,

$$\frac{G(u_k)}{\rho_k^\sigma} = \frac{-\|v_k\|_D^2}{\rho_k^\sigma} - 2\int_\Omega \frac{F(x,u_k)}{\rho_k^\sigma}\,dx \longrightarrow -\int_{\tilde{y}>0} \alpha_+ |\tilde{y}|^\sigma - \int_{\tilde{y}<0} \alpha_- |\tilde{y}|^\sigma < 0, \tag{2.54}$$

since $\tilde{y} \neq 0$. This contradicts the assumption that $G(u_k)$ is bounded from below. □

Similarly, we have the following lemma.

LEMMA 2.5. *Assume*

(D) *there is a constant $\sigma \in (1,2)$ such that*

$$\frac{f(x,t)t}{|t|^\sigma} \longrightarrow \alpha_\pm(x) \quad \text{as } t \longrightarrow \pm\infty, \text{ uniformly in } x, \tag{2.55}$$

where

$$\int_{y>0} \alpha_+ |y|^\sigma + \int_{y<0} \alpha_- |y|^\sigma < 0, \quad y \in N(A)\backslash\{0\}. \tag{2.56}$$

Then

$$G(u) \longrightarrow \infty \quad \text{as } \|u\|_D \longrightarrow \infty,\ u \in W \oplus N(A). \tag{2.57}$$

LEMMA 2.6. *If*

$$\frac{f(x,t)}{t} \longrightarrow 0 \quad \text{as } |t| \longrightarrow \infty, \tag{2.58}$$

then

$$G(u) \longrightarrow -\infty \quad \text{as } \|u\|_D \longrightarrow \infty,\ u \in V, \tag{2.59}$$

$$G(u) \longrightarrow \infty \quad \text{as } \|u\|_D \longrightarrow \infty,\ u \in W. \tag{2.60}$$

Proof. Assume $\{v_k\} \subset V$, $\rho_k = \|v_k\|_D \to \infty$, and $G(v_k) \to m > -\infty$. Let $\tilde{v}_k = v_k/\rho_k$. Then $\|\tilde{v}_k\| = 1$, and there is a renamed subsequence such that $\tilde{v}_k \to \tilde{v}$ in D and a.e. in Ω. Thus

$$\frac{2G(v_k)}{\rho_k^2} = -\|\tilde{v}_k\|_D^2 - \frac{2\int_\Omega F(x, v_k)\,dx}{\rho_k^2} \longrightarrow -\|\tilde{v}\|_D < 0. \tag{2.61}$$

This proves (2.59). Similarly, if $\{w_k\} \subset W$, and $\rho_k = \|w_k\|_D \to \infty$, let $\tilde{w}_k = w_k/\rho_k$. Then $\|\tilde{w}_k\| = 1$, and there is a renamed subsequence such that $\tilde{w}_k \to \tilde{w}$ weakly in D, strongly in $L^2(\Omega)$, and a.e. in Ω. Then,

$$\frac{2G(w_k)}{\rho_k^2} = \|\tilde{w}_k\|_D^2 - \frac{2\int_\Omega F(x, w_k)\,dx}{\rho_k^2} \geq 1 - \frac{2\int_\Omega F(x, w_k)\,dx}{\rho_k^2} \longrightarrow 1. \tag{2.62}$$

This proves (2.60). □

LEMMA 2.7. *Assume (2.58). If $N(A) \neq \{0\}$, assume also that there is a constant $\sigma \in (1, 2)$ such that*

$$\frac{f(x,t)t}{|t|^\sigma} \longrightarrow \alpha_\pm(x) \quad \text{as } t \longrightarrow \pm\infty, \text{ uniformly in } x, \tag{2.63}$$

where

$$\int_{y>0} \alpha_+ |y|^\sigma + \int_{y<0} \alpha_- |y|^\sigma \neq 0, \quad y \in N(A)\backslash\{0\}. \tag{2.64}$$

Then G satisfies the PS condition.

Proof. If

$$G(u_k) \longrightarrow c, \qquad G'(u_k) \longrightarrow 0, \tag{2.65}$$

assume that $\rho_k = \|u_k\|_D \to \infty$. Let $\tilde{u}_k = u_k/\rho_k$, and write $\tilde{u}_k = \tilde{v}_k + \tilde{y}_k + \tilde{w}_k$, $\tilde{v}_k \in V$, $\tilde{y}_k \in N(A)$, and $\tilde{w}_k \in W$. In particular, we have

$$\big(G'(u_k), h\big) = a\big(u_k, h\big) - \big(f(u_k), h\big) = o\big(\|h\|_D\big). \tag{2.66}$$

Setting $h = \tilde{w}_k, -\tilde{v}_k$, respectively, and dividing by ρ_k, we conclude that

$$\|\tilde{v}_k\|_D + \|\tilde{w}_k\|_D \longrightarrow 0. \tag{2.67}$$

Since $\|\tilde{u}_k\|_D = 1$, we must have a renamed subsequence such that $\tilde{y}_k \to \tilde{y}$ strongly in D with $\|\tilde{y}\|_D = 1$. Consequently,

$$\left(\frac{G'(u_k)}{\rho_k^{\sigma-1}}, \tilde{y}_k\right) = -\left(\frac{f(u_k)}{\rho^{\sigma-1}}, \tilde{y}_k\right) \longrightarrow 0. \tag{2.68}$$

But

$$\int_\Omega \frac{f(x,u_k)\tilde{y}_k}{\rho_k^{\sigma-1}} = \int_\Omega \left[\frac{u_k f(x,u_k)}{|u_k|^\sigma}\right]\left[\frac{|\tilde{u}_k|^\sigma}{\tilde{u}_k}\right]\tilde{y}_k \longrightarrow \int_{\tilde{y}>0} \alpha_+|\tilde{y}|^\sigma + \int_{\tilde{y}<0} \alpha_-|\tilde{y}|^\sigma \tag{2.69}$$

for any subsequence. By hypothesis, this cannot vanish, since $\tilde{y} \neq 0$. This contradiction shows that $\rho_k \leq C$, and the usual methods obtain a convergent subsequence of $\{u_k\}$ (cf. [20]). □

The following lemma is well known (cf. [2]).

LEMMA 2.8. *If $E = V \oplus W$, $k = \dim V < \infty$, $G \in C^1(E, \mathbb{R})$ satisfies the PS condition, u_0 is the only critical point of G, (2.59) holds, and*

$$\inf_W G > -\infty, \tag{2.70}$$

then

$$C_k(G, u_0) \neq 0. \tag{2.71}$$

3. The final proof

We can now give the proof of Theorem 1.1.

Proof. Assume that 0 is the only solution of (1.10) and that (1.3) holds. Let

$$V_0 = \bigoplus_{\mu < a_0} N(A - \mu), \qquad W_0 = V_0^{\perp}. \tag{3.1}$$

If $a_0 \notin \sigma(A)$, then

$$C_k(G, 0) \cong \delta_{p_0 k}\mathbb{Z} \quad \forall k, \tag{3.2}$$

where $p_0 = \dim V_0$ by Lemma 2.3. If $a_0 \in \sigma(A)$, then (3.2) holds by Lemma 2.2. On the other hand, if $a \notin \sigma(A)$, then (2.59) and (2.60) hold by Lemma 2.6, where

$$V = \bigoplus_{\mu \leq a} N(A - \mu), \qquad W = V^{\perp}. \tag{3.3}$$

This implies (2.70). For if

$$G(w_k) \longrightarrow -\infty, \tag{3.4}$$

then we must have $\|w_k\|_D \leq C$ by (2.60). Then there is a renamed subsequence such that $w_k \to w$ weakly in D, strongly in $L^2(\Omega)$ and a.e. in Ω. It then follows that

$$G(w_k) \geq -\int_{\Omega} F(x, w_k)\, dx \longrightarrow -\int_{\Omega} F(x, w)\, dx > -\infty \tag{3.5}$$

(cf. [20]). Therefore,

$$C_p(G, 0) \neq 0, \quad p = \dim V. \tag{3.6}$$

If $a \in \sigma(A)$, then (2.59) holds by Lemma 2.4, while (2.60) holds as before. Thus, (3.6) holds in this case as well. Now we note that $p_0 < p$, since $E(\lambda) \subset V$, while $E(\lambda) \not\subset V_0$.

This contradiction proves the theorem when (1.3) holds. Assume next that (1.4) holds. Let

$$V = \bigoplus_{\mu < a} N(A-\mu), \qquad W = V^{\perp},$$
$$V_0 = \bigoplus_{\mu \le a_0} N(A-\mu), \qquad W_0 = V_0^{\perp}. \tag{3.7}$$

If $a_0 \notin \sigma(A)$, then (3.2) holds by Lemma 2.3. If $a_0 \in \sigma(A)$, then (3.2) holds by Lemma 2.4. However, if $a \notin \sigma(A)$, then (2.59) and (2.60) hold by Lemma 2.6. Hence, (2.70) holds as before. This implies (3.6). If $a \in \sigma(A)$, then (2.59) and (2.60) hold again by Lemma 2.6, implying (3.6) in this case as well. Since $E(\lambda) \subset V_0$, $E(\lambda) \not\subset V$, we have $p < p_0$, providing the necessary contradiction. This completes the proof. □

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