CHARACTERIZATION OF A CLASS OF FUNCTIONS USING DIVIDED DIFFERENCES

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We determine the class of functions, the divided difference of which, at n distinct numbers, is a continuous function of the product of these numbers.

1. Introduction

We first introduce the needed terminology. The divided differences of a function f at distinct points are defined recursively as follows:

$$f[x_1] := f(x_1),$$
$$f[x_1,\dots,x_n] := \frac{f[x_2,\dots,x_n]-f[x_1,\dots,x_{n-1}]}{x_n-x_1}, \quad n\geq 2. \tag{1.1}$$

The following formula is well known, [2],

$$f[x_1,\dots,x_n]=\sum_{j=1}^{n}\frac{f(x_j)}{\omega_n'(x_j)}, \tag{1.2}$$

where $\{x_j\}_{j=1}^n \subset \mathbb{C}$ are distinct numbers and $\omega_n(x) := \prod_{k=1}^n (x-x_k)$.

Furthermore, if $f\in\mathcal{P}_{n-1}$, the set of algebraic polynomials of degree at most $n-1$, then, [2], $f[x_1,\dots,x_{n+1}]=0$. This fact can be proved by induction in n.

The more general definition of divided differences allowing repeated points is that $f[x_1,\dots,x_n] := a_{n-1}$, where $p_{n-1}(x)=\sum_{k=0}^{n-1} a_k x^k$ is the polynomial that interpolates f at $\{x_j\}_{j=1}^n$ in the Hermite sense, [2].

The main results of this paper are the following theorems.

Theorem 1.1. *Let $n\in\mathbb{N}$, $n\geq 2$, be fixed and f be a real-valued function defined on an open set D of the real line $\mathbb{R}$. Assume that the divided difference $f[x_1,\dots,x_n]$ satisfies*

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the functional equation

$$f\left[x_1,\dots,x_n\right]=G\left(\prod_{j=1}^{n}x_j\right) \tag{1.3}$$

for every set of n distinct numbers $\{x_j\}_{j=1}^n\subset D$, *and G is a continuous function on the set of products* $P_n(D):=\{\prod_{j=1}^n x_j : \{x_j\}_{j=1}^n\subset D\}$. *Then*

$$f(x)=\frac{a_{-1}}{x}+\sum_{k=0}^{n-1}a_kx^k. \tag{1.4}$$

Furthermore, $G(t)=(-1)^{n+1}a_{-1}/t+a_{n-1}$.

THEOREM 1.2. *Let f be a complex-valued function defined on an open set D of the complex plane* $\mathbb{C}$. *Assume that f satisfies the conditions of Theorem 1.1 on D. Then f and G have the forms given by Theorem 1.1.*

A similar characterization was obtained in [1] for functions the divided difference of which at n distinct points is a function of the sum of the points.

2. Proofs of the theorems

Proof of Theorem 1.1. First assume that $0\notin D$. We claim that $f\in C^\infty(D)$, that is, f has continuous derivatives of arbitrary order in D. Let $\{x_j\}_{j=1}^n\subset D$ be distinct. From (1.2) and (1.3) for $f[x_1,\dots,x_n]$ we get

$$\begin{aligned}\left(\frac{f\left(x_2\right)-f\left(x_1\right)}{x_2-x_1}\right)&\left(\frac{x_2-x_1}{\omega_n'\left(x_2\right)}\right)\\&+f\left(x_1\right)\left(\frac{1}{\omega_n'\left(x_2\right)}+\frac{1}{\omega_n'\left(x_1\right)}\right)+\sum_{j=3}^{n}\frac{f\left(x_j\right)}{\omega_n'\left(x_j\right)}=G\left(x_1\cdots x_n\right).\end{aligned} \tag{2.1}$$

We let $x_2\to x_1$ in (2.1). All terms on the left-hand side of (2.1) beginning with the second term which tends to $f(x_1)(1/\prod_{j=3}^n(x-x_j))'|_{x=x_1}$, and the right-hand side G have finite limits. Then the first term on the left-hand side of (2.1) has a finite limit, that is, $f'(x_1)$ exists. Since the points $x_1,x_3,\dots,x_n$ are distinct, all terms in the equation obtained from (2.1) after taking the limit $x_2\to x_1$, except possibly the first one, are continuous at x_1 for fixed $\{x_j\}_{j=3}^n$. We obtain $f\in C^1(D)$. Then (1.2) and (1.3) imply $G\in C^1(P_n(D))$. This can be seen by observing that the derivative of the right-hand side of (1.2) with respect to x_1 is continuous at x_1 if the points $\{x_j\}_{j=1}^n$ are distinct and $\{x_j\}_{j=2}^n$ are fixed. Then with $a=\prod_{j=2}^n x_j$, $dG(ax_1)/dx_1$ exists and is continuous at x_1. Therefore, $G'(t)\in C(P_n(D))$ exists because every $t\in P_n(D)$ can be written as a product of n distinct numbers from D and $P_n(D)$ is an open set.

Next, from (1.2) we have

$$\lambda^{n-1} f\big[\lambda x_1,\dots,\lambda x_n\big] = \sum_{j=1}^{n} \frac{f\big(\lambda x_j\big)}{\omega_n'\big(x_j\big)} = \lambda^{n-1} G\big(\lambda^n x_1\cdots x_n\big). \tag{2.2}$$

Differentiating (2.2) with respect to λ and setting $\lambda = 1$, we obtain

$$\begin{aligned}\big(xf'(x)\big)\big[x_1,\dots,x_n\big] &= \sum_{j=0}^{n} \frac{x_j f'\big(x_j\big)}{\omega_n'\big(x_j\big)} \\ &= (n-1)G\big(x_1\cdots x_n\big) + nx_1\cdots x_n G'\big(x_1\cdots x_n\big).\end{aligned} \tag{2.3}$$

Equation (2.3) for $f_1(x) = xf'(x)$ has the same form as (1.3) for $f(x)$, and $G_1(t) = (n-1)G(t) + ntG'(t) \in C(P_n(D))$. Using the same argument and induction, we can show that for every $k \in \mathbb{N}$, $f_k(x) \in C^1(D)$ and $G_k(t) \in C^1(P_n(D))$, where the functions f_k and G_k are defined recursively by $f_{k+1}(x) = xf_k'(x)$, $k \geq 0$, $f_0(x) = f(x)$, and $G_{k+1}(t) = (n-1)G_k(t) + ntG_k'(t)$, $k \geq 0$, $G_0(t) = G(t)$. From the definition of f_k, we get

$$f_k(x) = \sum_{j=1}^{k} a_{k,j} x^j f^{(j)}(x), \tag{2.4}$$

where $a_{k,1} = a_{k,k} = 1$ and $a_{k+1,j} = a_{k,j-1} + j a_{k,j}$, $j = 2,\dots,k$. Since $0 \notin D$, we get $f^{(k)}(x) \in C(D)$ for every $k \geq 0$, that is, $f \in C^\infty(D)$.

We proceed by induction with respect to $n \in \mathbb{N}$, $n \geq 2$. For $n = 2$ we have

$$\frac{f\big(x_2\big) - f\big(x_1\big)}{x_2 - x_1} = G\big(x_1x_2\big), \quad x_1, x_2 \in D,\ x_1 \neq x_2, \tag{2.5}$$

and $f \in C^\infty(D)$. Set $x_1 = x$ and let $x_2 \to x$. We get

$$f'(x) = G\big(x^2\big) = \frac{f\big(x^2\big) - f(1)}{x^2 - 1}, \quad x \in D \setminus \{1\}. \tag{2.6}$$

We may assume that $1 \in D$, otherwise we consider $f(ax)$ instead of $f(x)$ for some $a \in D$. Set $g(t) =: f(1+t)$. Since $0 \in D(g)$, the domain of g, and $g \in C^\infty(D(g))$, g has a power series representation

$$g(t) = \sum_{k=0}^{\infty} g_k t^k, \quad t \in (-r,r), \tag{2.7}$$

for small $r > 0$. Relations (2.6), with $x = t+1$, and (2.7) yield

$$\sum_{k=1}^{\infty} g_k k t^{k-1} = g'(t) = \frac{\big(g\big(t^2+2t\big) - g(0)\big)}{t^2+2t} = \sum_{s=0}^{\infty} g_{s+1} t^s (t+2)^s. \tag{2.8}$$

Comparing the coefficients of t^j in (2.8) we obtain

$$(j+1)g_{j+1} = \sum_{s=0}^{j} \binom{s}{j-s} 2^{2s-j} g_{s+1}, \quad j \geq 0, \tag{2.9}$$

or, equivalently, (with $\nu = s+1$)

$$\left(j+1-2^j\right)g_{j+1} = \sum_{\nu=2}^{j} \binom{\nu-1}{j-\nu+1} 2^{2\nu-j-2} g_\nu, \quad j \geq 2. \tag{2.10}$$

Equation (2.10) is nontrivial only if $j \geq 2$. This is so because adding to g (or f) a linear function and multiplying g (or f) by a constant does not change its properties. The binomial coefficients in (2.10) are not zero only if $\nu - 1 \geq j - \nu + 1$, that is, if $\nu \geq j/2 + 1 \geq 2$. Then (2.10) implies that g_j is a multiple of g_2 for every $j \geq 2$. In particular, if $g_2 = 0$ then g (and hence f) is a linear function.

If $g_2 \neq 0$ we may assume that $g_2 = 1$. In this case (2.10) implies $g_j = (-1)^j$ for every $j \geq 2$. This follows from (2.9) and the identity

$$\sum_{s=1}^{j} \binom{s}{j-s} 2^{2s-j} (-1)^{s-j} = j+1, \quad j \geq 2, \tag{2.11}$$

which is the special case $\alpha = 2$, $\beta = -1$ of the formula

$$A_j(\alpha,\beta) := \sum_{s=0}^{j} \binom{s}{j-s} \alpha^{2s-j} \beta^{j-s} = \sum_{k=0}^{j} \lambda_1^k \lambda_2^{j-k}, \quad j \geq 0, \tag{2.12}$$

where $\alpha, \beta \in \mathbb{C}$ and $\lambda_{1,2}$ are the zeros of $\lambda^2 - \alpha\lambda - \beta$. Hence, it is enough to verify (2.12). For $j = 0$ and $j = 1$, (2.12) is obvious. Next, for $j \geq 1$

$$\begin{aligned}
A_{j+1}(\alpha,\beta) &= \sum_{s=1}^{j+1} \binom{s}{j+1-s} \alpha^{2s-j-1} \beta^{j+1-s} \\
&= \alpha \sum_{s=1}^{j+1} \binom{s-1}{j-(s-1)} \alpha^{2(s-1)-j} \beta^{j-(s-1)} \\
&\quad + \beta \sum_{s=1}^{j} \binom{s-1}{j-1-(s-1)} \alpha^{2(s-1)-(j-1)} \beta^{(j-1)-(s-1)} \\
&= \alpha A_j(\alpha,\beta) + \beta A_{j-1}(\alpha,\beta) \qquad (2.13)\\
&= \alpha \sum_{k=0}^{j} \lambda_1^k \lambda_2^{j-k} + \beta \sum_{k=0}^{j-1} \lambda_1^k \lambda_2^{j-1-k} \\
&= \sum_{k=0}^{j} \left(\lambda_1^{k+1}\lambda_2^{j-k} + \lambda_1^k \lambda_2^{j+1-k}\right) - \sum_{k=0}^{j-1} \lambda_1^{k+1} \lambda_2^{j-k} \\
&= \sum_{k=0}^{j+1} \lambda_1^k \lambda_2^{j+1-k},
\end{aligned}$$

where we used the identities

$$\binom{s}{j+1-s} = \binom{s-1}{j-s+1} + \binom{s-1}{j-s}, \quad s = 1, \ldots, j, \tag{2.14}$$

$\lambda_1 + \lambda_2 = \alpha$, $\lambda_1\lambda_2 = -\beta$, and induction with respect to j.

Since $g_j = (-1)^j$, $j \geq 2$, it follows from (2.7) that $g(t) = g_2/(1+t) + At + B$ for $t \in (-r, r)$. We have proved that for every $x_0 \in D$ there exists $r(x_0) > 0$ such that $f(x) = c/x + ax + b$, $|x - x_0| < r(x_0)$. We have to show that the coefficients a, b, and c are independent of $x_0 \in D$. Let I_1 and I_2 be two open subinterval of D such that

$$f(x) = \frac{c_\nu}{x} + a_\nu x + b_\nu, \quad x \in I_\nu, \ \nu = 1, 2. \tag{2.15}$$

From (2.5) we get

$$f[x,y] = \frac{\left(c_2 + a_2 y^2 + b_2 y\right)x - \left(c_1 + a_1 x^2 + b_1 x\right)y}{(y-x)xy} = G(xy), \quad x \in I_1, \ y \in I_2. \tag{2.16}$$

Let $C = x_0 y_0$ for some $x_0 \in I_1$ and $y_0 \in I_2$, and let $\gamma_C = (I_1 \times I_2) \cap \{(x,y) : xy = C\}$. Then

$$f[x,y] = \frac{\left(c_2 + a_2 y^2 + b_2 y\right)C - \left(c_1 y^2 + a_1 C^2 + b_1 C y\right)}{\left(y^2 - C\right)C} = G(C), \quad (x,y) \in \gamma_C. \tag{2.17}$$

Since γ_C is a continuous curve, (2.17) implies that $(b_2 - b_1)C = 0$, $a_2 C - c_1 = G(C)C$, and $(c_2 - a_1 C)C = -G(C)C^2$. Using that $C \neq 0$ we get $b_1 = b_2$ and $a_1 C - c_2 = G(C)C = a_2 C - c_1$, hence, $(a_2 - a_1)C = c_1 - c_2$. Unless $a_1 = a_2$ we can choose $C \neq (c_1 - c_2)/(a_2 - a_1)$ and that choice would give us a contradiction. Thus $a_1 = a_2$ and then $c_1 = c_2$.

Now assume that Theorem 1.1 is true for some $n \geq 2$ and consider the case $n+1$. Let $x_1 \in D$ be fixed. We define $\tilde{f}(x) := f[x_1, x]$. Let $\{x_j\}_{j=2}^{n+1} \subset D \setminus \{x_1\}$ be distinct numbers. From (1.2) and (1.3) with $\tilde{\omega}_n(x) := \prod_{k=2}^{n+1}(x - x_k)$, we obtain

$$\begin{aligned} \tilde{f}\left[x_2, \ldots, x_{n+1}\right] &= \sum_{j=2}^{n+1} \frac{f\left(x_j\right) - f\left(x_1\right)}{\left(x_j - x_1\right)\tilde{\omega}_n'\left(x_j\right)} = \sum_{j=1}^{n+1} \frac{f\left(x_j\right) - f\left(x_1\right)}{\omega_{n+1}'\left(x_j\right)} \\ &= f\left[x_1, \ldots, x_{n+1}\right] = G\left(x_1 \prod_{j=2}^{n+1} x_j\right), \end{aligned} \tag{2.18}$$

where we also used that $f(x_1)[x_1, \ldots, x_{n+1}] = 0$. Hence, $\tilde{f}$ satisfies the conditions of Theorem 1.1 for n points. By the induction assumption

$$f\left[x_1, x\right] = \tilde{f}(x) = \sum_{j=-1}^{n-1} \tilde{a}_j\left(x_1\right)x^j, \quad x \in D \setminus \left\{x_1\right\}, \tag{2.19}$$

or, equivalently,

$$f(x) = f(x_1) + (x - x_1) \sum_{j=-1}^{n-1} \tilde{a}_j(x_1) x^j =: \sum_{j=-1}^{n} a_j(x_1) x^j, \quad x \in D \setminus \{x_1\}. \tag{2.20}$$

Equation (2.20) is true for $x = x_1$ as well. Furthermore, (2.20) is unique in the sense that the coefficients $a_j(x_1)$ are independent of $x_1 \in D$. Indeed,

$$x f(x) = \sum_{j=0}^{n+1} a_{j-1}(x_1) x^j = \sum_{j=0}^{n+1} a_{j-1}(x_2) x^j, \quad x \in D \tag{2.21}$$

implies $a_j(x_1) = a_j(x_2)$, $j = -1, \dots, n$, since a nonzero polynomial has finitely many zeros.

Now assume that $0 \in D$. Then $D_1 := D \setminus \{0\}$ is an open set. For every $n-1$ distinct numbers $\{x_j\}_{j=2}^{n} \subset D_1$, from (2.18) with $\tilde{f}(x) := f[0, x]$ and (1.3) we obtain

$$\tilde{f}[x_2, \dots, x_n] = f[0, x_2, \dots, x_n] = G(0). \tag{2.22}$$

Then $\tilde{f}(x)$ satisfies the conditions of Theorem 1.1 on the set D_1 and $0 \notin D_1$. Therefore, $\tilde{f}(x) = (f(x) - f(0))/x = a_{-1}/x + p(x)$, $p(x) \in \mathcal{P}_{n-2}$. Since $G \in C(D)$, (1.3) implies $f \in C(D)$ and $a_{-1} = 0$. Hence $f(x) = f(0) + x p(x) \in \mathcal{P}_{n-1}$.

The formula for $G(t)$ follows from the identities $x^k[x_1, \dots, x_n] = \delta_{n-1,k}$, $k = 0, \dots, n-1$, and

$$\frac{1}{x}[x_1, \dots, x_n] = \frac{(-1)^{n+1}}{x_1 \cdots x_n}, \quad n \geq 1. \tag{2.23}$$

The proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2. The proof of Theorem 1.2 follows the same arguments as the proof of Theorem 1.1 except that after verifying $f \in C^1(D)$ as in the proof of Theorem 1.1, we automatically obtain that $f \in C^\infty(D)$ via Cauchy's integral formula. □

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