# ON THE LOCATION OF THE PEAKS OF LEAST-ENERGY SOLUTIONS TO SEMILINEAR DIRICHLET PROBLEMS WITH CRITICAL GROWTH

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We study the location of the peaks of solution for the critical growth problem $-\varepsilon^2\Delta u + u = f(u) + u^{2^*-1}$, $u > 0$ in $\Omega$, $u = 0$ on $\partial\Omega$, where $\Omega$ is a bounded domain; $2^* = 2N/(N-2)$, $N \geq 3$, is the critical Sobolev exponent and $f$ has a behavior like $u^p$, $1 < p < 2^* - 1$.

## 1. Introduction

In this paper, we will study the location of the peaks of *least-energy* solution for the problem

$$\begin{aligned} -\varepsilon^2\Delta u + u &= f(u) + u^{2^*-1} \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N$, $\varepsilon > 0$, and $f$ is a function satisfying some subcritical conditions. Here $2^* = 2N/(N-2)$, $N \geq 3$, is the critical Sobolev exponent.

By *least-energy* solution for problem (1.1) we mean a critical point at the *Mountain-Pass* level of the associated *energy* functional

$$J_\varepsilon(u) = \frac{1}{2}\int_\Omega \left(\varepsilon^2|\nabla u|^2 + u^2\right) dz - \int_\Omega \left[F(u) + \frac{1}{2^*}\left(u^+\right)^{2^*}\right] dz, \tag{1.2}$$

(where $u^+ = \max\{u, 0\}$), defined on the Hilbert space $H_o^1(\Omega)$ endowed with the norm

$$\|u\|_\varepsilon^2 = \int_\Omega \left(\varepsilon^2|\nabla u|^2 + u^2\right) dz. \tag{1.3}$$


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The Mountain-Pass level of $J_\varepsilon$ is defined by

$$c_\varepsilon = \inf_{g\in\Gamma}\max_{0\le t\le 1} J_\varepsilon(g(t)), \tag{1.4}$$

where $\Gamma$ is the set of all continuous paths joining the origin and a fixed nonzero element $e$ in $H_o^1(\Omega)$, such that $e \neq 0$ and $J_\varepsilon(e) \le 0$. Under suitable hypothesis (e.g., $(f_1)$, $(f_4)$, $(f_5)$ below), it is not hard to check that $c_\varepsilon > 0$ does not depend on the element $0 \neq v \in H_o^1(\Omega)$ and $u$ is a *least-energy* solution if and only if $J_\varepsilon(u) = c$ and $J_\varepsilon'(u) = 0$, and $J_\varepsilon(u) \le J_\varepsilon(v)$ for all $v \neq 0$ such that $J_\varepsilon'(v) = 0$.

The existence of least-energy solution of problem (1.1) was given in Brézis and Nirenberg in [3, Theorem 2.1] (see Lemma 2.4 in this paper).

In this paper, we will study some properties of the least-energy solution $u_\varepsilon$ of problem (1.1) when $\varepsilon$ is small. In order to describe these properties, we introduce the hypotheses on the function $f$.

Suppose that $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a $C^{1,\alpha}$ function such that

$(f_1)$ $f(0) = f'(0) = 0$;

$(f_2)$ there is $q_1 \in (1, (N+2)/(N-2))$ such that

$$\lim_{s\to\infty} \frac{f(s)}{s^{q_1}} = 0; \tag{1.5}$$

$(f_3)$ there are $q_2 \in (1, (N+2)/(N-2))$ and $\lambda > 0$ such that

$$f(s) \ge \lambda s^{q_2}, \quad \forall s > 0 \tag{1.6}$$

(when $N = 3$, we need $q_2 > 2$, otherwise we require a sufficiently large $\lambda$);

$(f_4)$ if $F(s) = \int_o^s f(t)\,dt$, for some $\theta \in (2, q_1 + 1)$ we have

$$0 < \theta F(s) \le f(s)s, \quad \forall s > 0; \tag{1.7}$$

$(f_5)$ the function $f(s)/s$ is increasing for $s > 0$.

Since our interest is on positive solutions we define $f(s) = 0$, in $s \le 0$.

Now we will state our main result.

Theorem 1.1. *Suppose that $\Omega$ is a bounded domain in $\mathbb{R}^N$; $f$ satisfies* $(f_1)$, $(f_2)$, $(f_3)$, $(f_4)$, $(f_5)$; *and let $u_\varepsilon$ be the least-energy solution of (1.1). Then, there is a $\varepsilon_o > 0$ such that*

(i) *$u_\varepsilon$ attains only one local maximum at some $z_\varepsilon \in \Omega$ (hence global maximum), for all $\varepsilon \in (0, \varepsilon_o]$;*

(ii) *$u_\varepsilon$ converges uniformly to zero over compact subsets of $\Omega \setminus \{z_\varepsilon\}$ as $\varepsilon \to 0$;*

(iii) $\operatorname{dist}(z_\varepsilon, \partial\Omega) \to \max_{z\in\Omega} \operatorname{dist}(z, \partial\Omega)$ .

This statement is analogous to the one given by Ni and Wei in [8], in the subcritical case

$$-\varepsilon^2\Delta u + u = h(u), \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega, \tag{1.8}$$

where $h$ satisfies the following hypothesis:

(i) $(f_1)$, $(f_2)$, $(f_4)$, and $(f_5)$ hold;

(ii) the global problem

$$-\Delta u + u = h(u), \quad \text{in } \mathbb{R}^N \tag{1.9}$$

has a unique positive solution in $H^1(\mathbb{R}^N)$;

(iii) this solution is *nondegenerate* in the sense that

$$-\Delta v + v = h'(u)v, \quad \text{in } \mathbb{R}^N \tag{1.10}$$

has no nontrivial spherically symmetric solution in $L^2(\mathbb{R}^N)$.

In [8], Ni and Wei also have described the asymptotic profile (in $\varepsilon$) of $u_\varepsilon$, giving a detailed description for $\varepsilon$ small. Here in the critical case, the solutions have the same profile.

In this work we will show that a ground state solution of the critical problem (1.1) is also solution of a subcritical problem (1.8) by showing that for small $\varepsilon$ we have a uniform bound for the $L^\infty$ norm of $u_\varepsilon$.

The difficulty here lies in finding an upper bound for $\|u_\varepsilon\|_{L^\infty(\Omega)}$ by obtaining a bound for $u_\varepsilon$ in $L^p(\Omega)$ norm, for all $p \geq 2$. In the subcritical case this boundedness is obtained since the family $u_\varepsilon$ is bounded in $H^1(\Omega)$ but this argument does not work in the critical case. Here, we obtain an $L^\infty$-bound for $u_\varepsilon$ through the estimate below, which is based on Moser's iteration technique (see [11]) and is essentially due to Brézis and Kato [2].

Proposition 1.2. *Let $\Lambda$ be an open subset and $q \in L^{N/2}(\Lambda)$. Suppose that $g : \Lambda \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function satisfying*

$$|g(x,s)| \leq (q(x) + C_g)|s|, \quad \forall s \in \mathbb{R},\ x \in \Lambda \text{ and for some } C_g > 0. \tag{1.11}$$

*Then, if $v \in H_o^1(\Lambda)$ is such that*

$$-\Delta v = g(x,v), \quad \text{in } \Lambda \tag{1.12}$$

*we have $v \in L^p(\Lambda)$ for all $2 \leq p < \infty$. Moreover, there is a positive constant $C_p = C(p, C_g, q)$ such that*

$$\|v\|_{L^{2^*(p+1)}(\Lambda)} \leq C_p \|v\|_{L^{2(p+1)}(\Lambda)}. \tag{1.13}$$

*Remark 1.3.* The dependence on $q$ of $C_p$ can be given uniformly on a family of functions $\{q_\varepsilon\}_{\varepsilon>0}$ such that $q_\varepsilon$ converges in $L^{N/2}$ (see the appendix).

We have organized this paper as follows: the next section contains the proof of Theorem 1.1. This proof consists in a series of lemmas which show the $L^\infty$-bound for $u_\varepsilon$, where these functions are solutions of a class of subcritical problems (1.8). The third section is an appendix proving Proposition 1.2, for the sake of completeness.

## 2. Proof of Theorem 1.1

Before proving Theorem 1.1, let us fix some notation and preliminaries.

*Remark 2.1.* Throughout this section, we use the equivalent characterization of $c_\varepsilon$, which is more adequate to our purposes, given by

$$c_\varepsilon = \inf_{v \in H_o^1(\Omega)\setminus\{0\}} \max_{t\geq 0} J_\varepsilon(tv). \tag{2.1}$$

(see Willem [13, Theorem 4.2]).

We denote by $J : H^1(\mathbb{R}^N) \to \mathbb{R}$ the functional given by

$$J(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} \left[F(u) + \frac{1}{2^*}(u_+)^{2^*}\right] dx, \tag{2.2}$$

where

$$\|u\|^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)\, dx, \tag{2.3}$$

associated with the problem

$$-\Delta u + u = f(u) + |u|^{2^*-2}u, \quad \text{in } \mathbb{R}^N. \tag{2.4}$$

It is known that under assumptions $(\mathrm{f}_1)$, $(\mathrm{f}_2)$, $(\mathrm{f}_3)$, $(\mathrm{f}_4)$, $(\mathrm{f}_5)$, and (2.4) possesses a *ground state* solution $\omega$ in the level

$$c = J(\omega) = \inf_{v \in H^1(\mathbb{R}^N)\setminus\{0\}} \max_{t\geq 0} J(tv), \tag{2.5}$$

(see [1]).

*Remark 2.2.* It is easy to check that for each nonzero $v$ in $H^1(\mathbb{R}^N)$, there is a unique $t_o = t(v)$ such that

$$J(t_o v) = \max_{t\geq 0} J(tv). \tag{2.6}$$

Indeed, since

$$J(tv) = \frac{t^2}{2}\|v\|^2 - \int_{\mathbb{R}^N} \left[F(tv) - \frac{t^{2^*}}{2^*}(v^+)^{2^*}\right] dx, \quad \text{for } t \geq 0, \tag{2.7}$$

the maximum point $t_o$ of $J(tv)$ is given by

$$\|v\|^2 = \int_{\mathbb{R}^N} \left[ t_o^{-1} v f(t_o v) + t_o^{2^*-2} (v^+)^{2^*} \right] dx. \tag{2.8}$$

We assume, without loss of generality that $0 \in \Omega$. Set $\Omega_\varepsilon = \{x \in \mathbb{R}^N;\ \varepsilon x \in \Omega\}$.

The restriction of $J$ to $H_o^1(\Omega_\varepsilon)$ is the energy functional,

$$J(u) = \frac{1}{2} \int_{\Omega_\varepsilon} (|\nabla u|^2 + u^2)\, dx - \int_{\Omega_\varepsilon} \left[ F(u_+) + \frac{1}{2^*} u_+^{2^*} \right] dx, \quad u \in H_o^1(\Omega_\varepsilon), \tag{2.9}$$

associated with the problem

$$\begin{gathered} -\Delta u + u = f(u) + u^{2^*-1} \quad \text{in } \Omega_\varepsilon, \\ u = 0 \quad \text{on } \partial\Omega_\varepsilon. \end{gathered} \tag{2.10}$$

If $u_\varepsilon$ is a critical point of $J_\varepsilon$, the family

$$v_\varepsilon(x) = u_\varepsilon(z) = u_\varepsilon(\varepsilon x), \quad z = \varepsilon x \tag{2.11}$$

is such that each $v_\varepsilon$ is a critical point of functional $J$ restricted to $H_o^1(\Omega_\varepsilon)$ at the level

$$b_\varepsilon = J(v_\varepsilon) = \inf_{v \in H_o^1(\Omega_\varepsilon)\setminus\{0\}} \max_{t \geq 0} J(tv). \tag{2.12}$$

It is easy to check that $b_\varepsilon = \varepsilon^{-N} c_\varepsilon$ and from the definition of $c$ it follows that $b_\varepsilon \geq c$ for all $\varepsilon > 0$.

We will start with the following property of $\{b_\varepsilon\}_{\varepsilon>0}$.

Lemma 2.3. *For* $\{b_\varepsilon\}_{\varepsilon>0}$, $\lim_{\varepsilon\to 0} b_\varepsilon = c$.

*Proof.* Fix $\omega$ a ground state solution of problem (2.4) and let $\psi_\varepsilon(x) = \varphi(\varepsilon x)\omega(x)$, where $\varphi$ is a $C^1$-function such that

$$\varphi(x) = \begin{cases} 1 & \text{if } x \in B_1, \\ 0 & \text{if } x \notin B_2, \end{cases} \tag{2.13}$$

$B_1 = B_\rho(0)$, $B_2 = B_{2\rho}(0) \subset \Omega$. Observe that $\psi_\varepsilon \to \omega$ in $H^1(\mathbb{R}^N)$ and the support of $\psi_\varepsilon$ is in $\Omega_\varepsilon$. By definition of $b_\varepsilon$, we have $t_\varepsilon > 0$ such that

$$b_\varepsilon \leq \max_{t>0} J(t\psi_\varepsilon) = J(t_\varepsilon \psi_\varepsilon). \tag{2.14}$$

From (2.8) and condition ($f_3$) it follows that

$$\begin{aligned} \|\psi_\varepsilon\|^2 &= \int_{\mathbb{R}^N}\left[t_\varepsilon^{-1}\psi_\varepsilon f(t_\varepsilon\psi_\varepsilon)+t_\varepsilon^{2^*-2}\psi_\varepsilon^{2^*}\right]dx \\ &\geq \int_{\mathbb{R}^N}\left[\lambda t_\varepsilon^{q_2-1}\psi_\varepsilon^{q_2+1}+t_\varepsilon^{2^*-2}\psi_\varepsilon^{2^*}\right]dx, \end{aligned} \tag{2.15}$$

so that, $t_\varepsilon$ is bounded. Equality (2.15) and Remark 2.2 show that $t_\varepsilon \to t(\omega)=1$, as $\varepsilon \to 0$. Then we have $t_\varepsilon\psi_\varepsilon \to \omega$ in $H^1(\mathbb{R}^N)$ and

$$\lim_{\varepsilon\to 0} J(t_\varepsilon\psi_\varepsilon) = J(\omega) = c. \tag{2.16}$$

Combining (2.14), (2.16), and the inequality $b_\varepsilon \geq c$, for all $\varepsilon > 0$, we have proved this lemma. □

Lemma 2.4. *The inequality $c < (1/N)S^{N/2}$ holds, where $S$ is the best Sobolev constant for the embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$.*

*Proof.* For each $h > 0$, consider the function

$$\phi_h(x) = \frac{[N(N-2)h]^{(N-2)/4}}{(h+|x|^2)^{(N-2)/2}}. \tag{2.17}$$

We recall that $\phi_h$ satisfies the problem

$$\begin{gathered} -\Delta u = u^{2^*-1} \quad \text{in } \mathbb{R}^N, \\ u(x) > 0, \qquad \int_{\mathbb{R}^N}|\nabla u|^2 dx < \infty, \\ \int_{\mathbb{R}^N}|\nabla\phi_h|^2 dx = \int_{\mathbb{R}^N}\phi_h^{2^*}dx = S^{N/2} \quad \text{(see Talenti [12])}. \end{gathered} \tag{2.18}$$

Now, consider $\psi_h(x) = \varphi\phi_h(x)/\|\varphi\phi_h\|_{L^{2^*}(\mathbb{R}^N)}$, where $\varphi$ is the function defined in the proof of Lemma 2.3. From condition ($f_3$) we have

$$J(t\psi_h) \leq \frac{t^2}{2}\int_{B_2}\left(|\nabla\psi_h|^2+\psi_h^2\right)dx - \frac{\lambda t^{q_2+1}}{q_2+1}\int_{B_2}\psi_h^{q_2+1}dx - \frac{t^{2^*}}{2^*}. \tag{2.19}$$

Using arguments as in [7], there exists $h > 0$ such that

$$\max_{t\geq 0}\left\{\frac{t^2}{2}\int_{B_2}\left(|\nabla\psi_h|^2+\psi_h^2\right)dx - \frac{\lambda t^{q_2+1}}{q_2+1}\int_{B_2}\psi_h^{q_2+1}dx - \frac{t^{2^*}}{2^*}\right\} < \frac{1}{N}S^{N/2}. \tag{2.20}$$

Therefore, from (2.19) and (2.20) we have that

$$\max_{t\geq 0} J(t\psi_h) < \frac{1}{N} S^{N/2}, \tag{2.21}$$

and the proof of the lemma is completed. □

Notice that the same proof of Lemma 2.4 can be used to show that $b_\varepsilon < (1/N)S^{N/2}$, for all $\varepsilon > 0$. Using [3, Theorem 2.1], this inequality implies the existence of $v_\varepsilon$ and then the existence of $u_\varepsilon$.

Lemma 2.5. *There are $\varepsilon_o > 0$; a family $\{y_\varepsilon\}_{\{0<\varepsilon\leq\varepsilon_o\}} \subset \mathbb{R}^N$, $y_\varepsilon \in \Omega_\varepsilon$; constants $R > 0$ and $\beta > 0$ such that*

$$\int_{B_R(y_\varepsilon)} v_\varepsilon^2\, dx \geq \beta > 0, \quad \forall 0 < \varepsilon \leq \varepsilon_o, \tag{2.22}$$

$$\lim_{\varepsilon\to o} d(y_\varepsilon, \partial\Omega_\varepsilon) = \infty. \tag{2.23}$$

*Proof.* Start by showing that there is a family satisfying inequality (2.22). Arguing to the contrary, there is $\varepsilon_n \searrow 0$ such that for all $R > 0$

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}^N} \int_{B_R(x)} v_{\varepsilon_n}^2\, dx = 0. \tag{2.24}$$

Using (Lions [6, Lemma I.1]) we have

$$\int_{\mathbb{R}^N} v_{\varepsilon_n}^q\, dx = o_n(1), \quad \text{as } n \longrightarrow \infty,\ \forall 2 < q < 2^*, \tag{2.25}$$

and, from $(f_1)$ and $(f_2)$,

$$\int_{\mathbb{R}^N} F(v_{\varepsilon_n})\, dx = \int_{\mathbb{R}^N} v_{\varepsilon_n} f(v_{\varepsilon_n})\, dx = o_n(1). \tag{2.26}$$

Since $J'(v_{\varepsilon_n}) \cdot v_{\varepsilon_n} = 0$, we conclude from (2.26) that

$$\|v_{\varepsilon_n}\|^2 = \int_{\mathbb{R}^N} v_{\varepsilon_n}^{2^*}\, dx + o_n(1). \tag{2.27}$$

Let $\ell \geq 0$ be such that $\|v_{\varepsilon_n}\|^2 \to \ell$. Passing to the limit in $J(v_{\varepsilon_n}) = b_{\varepsilon_n}$ and using (2.26) we have

$$\ell = Nc \tag{2.28}$$

and hence $\ell > 0$. Now, using the definition of the constant $S$, we have

$$\|v_{\varepsilon_n}\|^2 \geq S\left(\int_{\mathbb{R}^N} v_{\varepsilon_n}^{2^*}\, dx\right)^{2/2^*}. \tag{2.29}$$

Taking the limit in the above inequalities, as $n \to \infty$, we achieve that

$$\ell \geq S\ell^{2/2^*}, \tag{2.30}$$

and by (2.28), that

$$c \geq \frac{1}{N} S^{N/2} \tag{2.31}$$

which contradicts Lemma 2.4 and then (2.22) holds.

Finally, to establish (2.23), suppose the contrary. That is, there exist $\varepsilon_n \to 0$ and $R > 0$ such that $\operatorname{dist}(y_{\varepsilon_n}, \partial\Omega_{\varepsilon_n}) \leq R$, hence $\operatorname{dist}(\varepsilon_n y_{\varepsilon_n}, \partial\Omega) \leq \varepsilon_n R$. Without loss of generality, we have $\varepsilon_n y_{\varepsilon_n} \to y_o$ for some $y_o \in \partial\Omega$. The arguments that follow can be found in [8].

Let $\nu$ be the unit interior normal to $\partial\Omega$ at $y_o$, and $\delta > 0$ such that $B_\delta(y_o + \delta\nu) \subset \Omega$ and $B_\delta(y_o - \delta\nu) \cap \Omega = \varnothing$. Let $\Omega_n = \{x \in \mathbb{R}^N : y_o + \varepsilon_n x \in \Omega\}$ and $w_n(x) = u_{\varepsilon_n}(y_o + \varepsilon_n x)$. This sequence $w_n$ is bounded in $H^1(\mathbb{R}^N)$, $-\Delta w_n + w_n = f(w_n) + w_n^{2^*-1}$ in $\Omega_n$,

$$\int_{B_{2R}(0)} w_n^2 \, dx \geq \int_{B_R(y_{\varepsilon_n})} v_{\varepsilon_n}^2 \, dx \geq \beta > 0, \quad \forall n, \tag{2.32}$$

and we have that $w_n$ converges weakly to some $w$ in $H^1(\mathbb{R}^N)$.

Let $\mathbb{R}^N_{+,\nu}$ be the half space $\{x \in \mathbb{R}^N : x \cdot \nu > 0\}$. Notice that $B_{\varepsilon_n^{-1}\delta}(\varepsilon_n^{-1}\delta\nu) \subset \Omega_n$ and $B_{\varepsilon_n^{-1}\delta}(-\varepsilon_n^{-1}\delta\nu) \cap \Omega_n = \varnothing$ and then we can prove that for all compacts $K_+ \subset \mathbb{R}^N_{+,\nu}$ and $K_- \subset \mathbb{R}^N_{-,\nu} = \mathbb{R}^N \setminus \overline{\mathbb{R}^N_{+,\nu}}$, we have $K_+ \subset \Omega_n$ and $K_- \cap \Omega_n = \varnothing$, for $n$ large.

Then for each $\phi \in C_o^\infty(\mathbb{R}^N_{+,\nu})$ such that $\operatorname{supp}\phi \subset \Omega_n$, we have

$$\int_{\mathbb{R}^N_{+,\nu}} (\nabla w_n \nabla \phi + w_n \phi) \, dx = \int_{\mathbb{R}^N_{+,\nu}} (f(w_n) + w_n^{2^*-1}) \phi \, dx. \tag{2.33}$$

From (2.33), usual arguments show that $w \in H^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N_+)$ and satisfies $-\Delta w + w = f(w) + w^{2^*-1}$, in $\mathbb{R}^N_{+,\nu}$, and $w \equiv 0$ in $\mathbb{R}^N_{-,\nu}$. Theorem I.1, due to Esteban and Lions in [4], shows that $w \equiv 0$ which contradicts

$$\int_{B_{2R}(0) \cap \mathbb{R}^N_{+,\nu}} w^2 \, dx \geq \beta > 0. \tag{2.34}$$

This completes the proof of the lemma. □

Now we will consider the translation of $v_\varepsilon$, defined by $\omega_\varepsilon(x) = v_\varepsilon(x + y_\varepsilon) = u_\varepsilon(\varepsilon y_\varepsilon + \varepsilon x)$ in $\widetilde{\Omega}_\varepsilon = \{x \in \mathbb{R}^N; \varepsilon y_\varepsilon + \varepsilon x \in \Omega\}$ and $\omega_\varepsilon = 0$ outside $\widetilde{\Omega}_\varepsilon$. From (2.23), any compact subset of $\mathbb{R}^N$ is contained in $\widetilde{\Omega}_\varepsilon$, for $\varepsilon$ sufficiently small.

From Lemma 2.5,

$$\int_{B_R(0)} \omega_\varepsilon^2 \, dx \geq \beta > 0, \quad \forall 0 < \varepsilon \leq \varepsilon_o. \tag{2.35}$$

Consider a sequence $\varepsilon_n \searrow 0$ and set $\widetilde{\Omega}_n = \widetilde{\Omega}_{\varepsilon_n}$, $\omega_n = \omega_{\varepsilon_n}$, $v_n = v_{\varepsilon_n}$, $y_\varepsilon = y_{\varepsilon_n}$.

We will prove that $\omega_n$ is bounded in the $L^\infty$ norm. In that case, $u_\varepsilon$ is also bounded in $L^\infty(\Omega)$ norm and the proof of Theorem 1.1 follows from the subcritical case, as Lemma 2.8 will show.

Since the sequence $\omega_n$ a translation of $v_n$, we have a uniform bound for $\|\omega_n\|$ and there is a $\omega_o \in H^1(\mathbb{R}^N)$ which is weak limit of $\omega_n$ in $H^1(\mathbb{R}^N)$. From (2.35) we have $\omega_o \neq 0$. We can write limit (2.23) in the following form

$$\lim_{n\to\infty} d(0, \partial\widetilde{\Omega}_n) = \infty. \tag{2.36}$$

Then for each $\phi \in C_o^\infty(\mathbb{R}^N)$ and large $n$ such that $\operatorname{supp}\phi \subset \widetilde{\Omega}_n$, we have

$$\int_{\mathbb{R}^N} (\nabla\omega_n\nabla\phi + \omega_n\phi)\,dx = \int_{\mathbb{R}^N} \left(f(\omega_n) + \omega_n^{2^*-1}\right)\phi\,dx, \quad \forall n. \tag{2.37}$$

From (2.37), usual arguments show that $\omega_o$ is a solution of problem (2.4), hence a critical point of $J$, and $J(\omega_o) \geq c$.

Lemma 2.6. *The sequence $\omega_n$ converges to $\omega_o$ in $H^1(\mathbb{R}^N)$ and $J(\omega_o) = c$.*

*Proof.* This fact comes from Lemma 2.5 and Fatou's lemma applied in the positive sequence $\omega_n f(\omega_n) - \theta F(\omega_n)$. Observe that

$$\begin{aligned} b_{\varepsilon_n} &= J(v_n) - \frac{1}{\theta}J'(v_n)v_n \\ &= \left(\frac{\theta-2}{2\theta}\right)\|v_n\|^2 + \frac{1}{\theta}\int_{\mathbb{R}^N}\left[v_n f(v_n) - \theta F(v_n)\right] + \left(\frac{2^*-\theta}{2^*\theta}\right)\int_{\mathbb{R}^N} v_n^{2^*} \\ &= \left(\frac{\theta-2}{2\theta}\right)\|\omega_n\|^2 + \frac{1}{\theta}\int_{\mathbb{R}^N}\left[\omega_n f(\omega_n) - \theta F(\omega_n)\right] + \left(\frac{2^*-\theta}{2^*\theta}\right)\int_{\mathbb{R}^N} \omega_n^{2^*}. \end{aligned} \tag{2.38}$$

From (2.38)

$$\begin{aligned} c \leq J(\omega_o) &= J(\omega_o) - \frac{1}{\theta}J'(\omega_o)\omega_o \\ &= \left(\frac{\theta-2}{2\theta}\right)\|\omega_o\|^2 + \frac{1}{\theta}\int_{\mathbb{R}^N}\left[\omega_o f(\omega_o) - \theta F(\omega_o)\right] + \left(\frac{2^*-\theta}{2^*\theta}\right)\int_{\mathbb{R}^N} \omega_o^{2^*} \\ &\leq \liminf\left(\frac{\theta-2}{2\theta}\right)\|\omega_n\|^2 + \frac{1}{\theta}\int_{\mathbb{R}^N}\left[\omega_n f(\omega_n) - \theta F(\omega_n)\right] + \left(\frac{2^*-\theta}{2^*\theta}\right)\int_{\mathbb{R}^N} \omega_n^{2^*} \\ &= \lim_{n\to\infty} b_{\varepsilon_n} = c. \end{aligned} \tag{2.39}$$

We have proved that $J(\omega_o) = c$ and then (2.39) becomes an equality.

Combining (2.39) with the three following inequalities:

$$\begin{gathered} \|\omega_o\|^2 \le \liminf \|\omega_n\|^2, \\ \int_{\mathbb{R}^N} [\omega_o f(\omega_o) - \theta F(\omega_o)]\, dx \le \liminf \int_{\mathbb{R}^N} [\omega_n f(\omega_n) - \theta F(\omega_n)]\, dx, \\ \int_{\mathbb{R}^N} \omega_o^{2^*}\, dx \le \liminf \int_{\mathbb{R}^N} \omega_n^{2^*}\, dx, \end{gathered} \tag{2.40}$$

we conclude that $\|\omega_n\| \to \|\omega_o\|$ and then $\omega_n \to \omega_o$ in $H^1(\mathbb{R}^N)$. □

We are ready to conclude the proof of our main result. From Proposition 1.2 and Remark 1.3 with $q(x) = \omega_n^{2^*-2} \in L^{N/2}$; $g(x,s) = f(s) + s^{2^*} - s$, we have $\omega_n \in L^t$ for all $t \ge 2$ and

$$\|\omega_n\|_{L^t} \le C_t, \tag{2.41}$$

where $C_t$ does not depend on $n$.

Now we will make use of a very particular version of [5, Theorem 8.17], due to Trudinger.

Proposition 2.7. *Suppose that $t > N$, $g \in L^{t/2}(\Lambda)$, and $u \in H_o^1(\Lambda)$ satisfies (in the weak sense)*

$$-\Delta u + u \le \tilde{g}(x), \tag{2.42}$$

*where $\Lambda$ is an open subset of $\mathbb{R}^N$. Then for any* ball $B_{2R}(y) \subset \Lambda$,

$$\sup_{B_R(y)} u \le C\left(\|u^+\|_{L^2(B_{2R}(y))} + \|g\|_{L^{t/2}(B_{2R}(y))}\right), \tag{2.43}$$

*where $C$ depends on $N$, $t$, and $R$.*

We know that each $\omega_n$ satisfies

$$-\Delta\omega_n + \omega_n = \omega_n^{2^*-1} + f(\omega_n), \quad \text{in } \tilde{\Omega}_n \tag{2.44}$$

and this implies that

$$-\Delta\omega_n + \omega_n \le g_n(x) = \omega_n^{2^*-1} + f(\omega_n), \quad \text{in } \mathbb{R}^N \tag{2.45}$$

in the weak sense.

Since (2.41) holds, $\|g_n\|_{L^t}$ is bounded from above for some $t > N$. Using Proposition 2.7 in (2.45) we have

$$\sup_{B_1(y)} \omega_n \le C\left(\|\omega_n\|_{L^2(B_{2R}(y))} + \|g_n\|_{L^t(B_{2R}(y))}\right) \tag{2.46}$$

for all $y \in \mathbb{R}^N$, which implies that there is a constant $a > 0$, independent of $n$, such that

$$\omega_n(x) \le a, \quad \forall x \in \mathbb{R}^N. \tag{2.47}$$

It follows that there is a $\varepsilon_o > 0$ such that

$$u_\varepsilon(z) \le a, \quad \forall z \in \Omega, \ \forall \varepsilon < \varepsilon_o. \tag{2.48}$$

To conclude the proof observe that $u_\varepsilon$ becomes a solution of the subcritical case (1.8) with $h$ given by

$$h(s) = \begin{cases} f(s) + s^{2^*-1}, & \text{if } s \le a, \\ f(s) + \dfrac{(2^*-1)}{(\theta-1)} a^{2^*-\theta} s^{\theta-1} - \dfrac{(2^*-\theta)}{(\theta-1)} a^{2^*-1}, & \text{if } s > a, \end{cases} \tag{2.49}$$

where $\theta > 2$ is that one fixed in condition $(f_4)$. It is easy to check that $h$ is a $C^{1,\alpha}$ function, $h$ and $H(s) = \int_o^s h(\tau)\,d\tau$ satisfy $(\mathrm{f}_1)$, $(\mathrm{f}_2)$, $(\mathrm{f}_3)$, $(\mathrm{f}_4)$, and $(\mathrm{f}_5)$. Let $\tilde{J}_\varepsilon$ be the $C^1$-functional on $H_o^1(\Omega)$ given by

$$\tilde{J}_\varepsilon(u) = \frac{1}{2} \int_\Omega \left(\varepsilon^2 |\nabla u|^2 + u^2\right) dz - \int_\Omega H(u)\, dz. \tag{2.50}$$

Since $f(s) + s^{2^*-1} \ge h(s)$ for all $s > 0$, we have that

$$J_\varepsilon(u) \le \tilde{J}_\varepsilon(u), \quad \forall u \in H_o^1(\Omega), \tag{2.51}$$

$J_\varepsilon(u_\varepsilon) = \tilde{J}_\varepsilon(u_\varepsilon)$, $J'_\varepsilon(u_\varepsilon) = \tilde{J}'_\varepsilon(u_\varepsilon) = 0$. We conclude that $u_\varepsilon$ is a least-energy solution of the subcritical problem (1.8).

Lemma 2.8. (i) *If $\tilde{c}_\varepsilon$ is the minimax level of $\tilde{J}_\varepsilon$, then $\tilde{c}_\varepsilon = c_\varepsilon$;*
(ii) *each $u_\varepsilon$ is a critical point of $\tilde{J}_\varepsilon$ in the minimax level and satisfies (1.8).*

Since global problem (1.9) has a unique nondegenerate positive solution (cf. [9, 10]), Theorem 1.1 comes from [8, Theorem 2.2] applied to the functional $\tilde{J}_\varepsilon$, and the asymptotic profile comes from [8, Theorem 2.3].

## Appendix

Let $\Lambda$ be some general domain in $\mathbb{R}^N$ (bounded or unbounded). We will start with the following lemma due to Brézis and Kato [2].

Lemma A.1. *Let $q \in L^{N/2}(\Lambda)$ be a nonnegative function. Then, for every $\varepsilon > 0$, there is a constant $\sigma_\varepsilon = \sigma(\varepsilon, q) > 0$ such that*

$$\int_\Lambda q(x) u^2\, dx \le \varepsilon \int_\Lambda |\nabla u|^2\, dx + \sigma_\varepsilon \int_\Lambda u^2\, dx, \quad \forall u \in H_o^1(\Lambda). \tag{A.1}$$

*Remark A.2.* If $q_k \to q$ in $L^{N/2}(\Lambda)$, we can choose a constant $\sigma_\varepsilon$ independent of $k$. That is, $\sigma(\varepsilon, q_k) = \sigma_\varepsilon$ and

$$\int_\Lambda q_k(x)u^2\,dx \le \varepsilon \int_\Lambda |\nabla u|^2\,dx + \sigma_\varepsilon \int_\Lambda u^2\,dx, \quad \forall u \in H_o^1(\Lambda),\ k \in \mathbb{N}. \tag{A.2}$$

*Proof.* Let $\sigma_\varepsilon = \sigma(\varepsilon, q) > 0$ be such that

$$\|q\|_{L^{N/2}(\{q \ge \sigma_\varepsilon\})} \le \varepsilon S, \tag{A.3}$$

where $S$ is a best constant in the Sobolev immersion $H_o^1(\Lambda) \hookrightarrow L^{2^*}(\Lambda)$, where $2^* = 2N/(N-2)$. For all $u \in H_o^1(\Lambda)$, we have

$$\begin{aligned}
\int_\Lambda q(x)u^2\,dx &= \int_{\{q \ge \sigma_\varepsilon\}} q(x)u^2\,dx + \int_{\{q \le \sigma_\varepsilon\}} q(x)u^2\,dx \\
&\le \sigma_\varepsilon \int_{\{q \le \sigma_\varepsilon\}} u^2\,dx + \int_{\{q \ge \sigma_\varepsilon\}} q(x)u^2\,dx \\
&\le \sigma_\varepsilon \int_\Lambda u^2\,dx + \|q\|_{L^{N/2}(\{q \ge \sigma_\varepsilon\})} \|u\|^2_{L^{2^*}(\{q \ge \sigma_\varepsilon\})}.
\end{aligned} \tag{A.4}$$

Inequality (A.1) follows from Sobolev estimate and the choice of $\sigma_\varepsilon$. □

Remark 1.3 follows from the proof of Lemma A.1 and the inequality

$$\int_\Lambda q_k(x)u^2\,dx \le \int_\Lambda q(x)u^2\,dx + \|q_k - q\|_{L^{N/2}(\Lambda)} \|u\|^2_{L^{2^*}(\Lambda)}. \tag{A.5}$$

*Proof of Proposition 1.2.* For any $n \in \mathbb{N}$ and $p > 0$, consider $A_n = \{x \in \Lambda : |v|^p \le n\}$, $B_n = \Lambda \setminus A_n$, and define $v_n$ by

$$v_n = v|v|^{2p} \quad \text{in } A_n, \qquad v_n = n^2 v \quad \text{in } B_n. \tag{A.6}$$

Observe that $v_n \in H_o^1(\Lambda)$, $v_n \le |v|^{2p+1}$ and

$$\nabla v_n = (2p+1)|v|^{2p}\nabla v \quad \text{in } A_n, \qquad \nabla v_n = n^2 \nabla v \quad \text{in } B_n. \tag{A.7}$$

So, using $v_n$ as a test function

$$\int_\Lambda \nabla v \nabla v_n\,dx = \int_\Lambda g(x,v)v_n\,dx. \tag{A.8}$$

Using (A.7), we have

$$\begin{aligned}
(2p+1)\int_{A_n} |v|^{2p}|\nabla v|^2\,dx + n^2 \int_{B_n} |\nabla v|^2\,dx \\
\le \int_\Lambda |g(x,v)v_n|\,dx \le \int_\Lambda (q(x) + C_g)\,|vv_n|\,dx.
\end{aligned} \tag{A.9}$$

Now consider

$$\omega_n = v|v|^p \quad \text{in } A_n, \qquad \omega_n = nv \quad \text{in } B_n. \tag{A.10}$$

Notice that $\omega_n^2 = vv_n \le |v|^{2(p+1)}$ and

$$\nabla\omega_n = (p+1)|v|^p\nabla v \quad \text{in } A_n, \qquad \nabla v_n = n\nabla v \quad \text{in } B_n. \tag{A.11}$$

Therefore,

$$\int_\Lambda |\nabla\omega_n|^2\,dx = (p+1)^2\int_{A_n}|v|^{2p}|\nabla v|^2\,dx + n^2\int_{B_n}|\nabla v|^2\,dx. \tag{A.12}$$

Combining (A.9) and (A.12), we obtain

$$\frac{2p+1}{(p+1)^2}\int_\Lambda |\nabla\omega_n|^2\,dx \le \int_\Lambda \big(q(x)+C_g\big)\omega_n^2\,dx. \tag{A.13}$$

Let $\sigma_p$ be given by Lemma A.1 with $\varepsilon = (2p+1)/2(p+1)^2$. Then

$$\int_\Lambda |\nabla\omega_n|^2\,dx \le \tilde{C}_p\int_\Lambda \omega_n^2\,dx, \tag{A.14}$$

where $\tilde{C}_n = (2(p+1)^2/(2p+1))(C_g+\sigma_p)$. Suppose that $v\in L^{2(p+1)}(\Lambda)$ for some $p\ge 2$. Applying Sobolev immersion in inequality (A.14) we have

$$\left[\int_{A_n}\omega_n^{2^*}\,dx\right]^{2/2^*} \le \left[\int_\Lambda \omega_n^{2^*}\,dx\right]^{2/2^*} \le S\tilde{C}_p\int_\Lambda |v|^{2(p+1)}\,dx \tag{A.15}$$

that is,

$$\left[\int_{A_n}|v|^{2^*(p+1)}\,dx\right]^{2/2^*}dx \le C_p\int_\Lambda |v|^{2(p+1)}\,dx, \tag{A.16}$$

where

$$C_p = \frac{2(p+1)^2}{2p+1}S(C_g+\sigma_p). \tag{A.17}$$

Now, passing to the limit in (A.16) we have $v\in L^{2^*(p+1)}(\Lambda)$ and

$$\|v\|_{L^{2^*(p+1)}(\Lambda)} \le C_p\|v\|_{L^{2(p+1)}(\Lambda)}. \tag{A.18}$$

The proof follows from the following iteration argument: let $p_1$ a positive such that $2(p_1+1) = 2^*$. It is easy to see that $0 < p_1$ and $v\in L^{2(p_1+1)}(\Lambda)$. Using inequality (A.18) we have

$$v\in L^{2^*(p_1+1)}(\Lambda). \tag{A.19}$$

Now choose $p_2$ such that $2(p_2+1) = 2^*(p_1+1)$. It is easy to see that $0 < p_1 < p_2$ and $v \in L^{p_2+1}(\Lambda)$. Using inequality (A.18) we have

$$v \in L^{2^*(p_2+1)}(\Lambda). \tag{A.20}$$

Continuing with this iteration we obtain an increasing sequence $p_k$ given by $2(p_{k+1}+1) = 2^*(p_k+1)$ such that $v \in L^{2(p_{k+1}+1)}(\Lambda)$ for all $k \in \mathbb{N}$. From

$$p_{k+1}+1 = \frac{N}{N-2}(p_k+1), \tag{A.21}$$

it follows that

$$p_{k+1}+1 = \left[\frac{N}{N-2}\right]^k 2^*. \tag{A.22}$$

This shows that $p_k$ goes to $\infty$ and therefore,

$$v \in L^p(\Lambda), \quad \forall p \geq 2. \tag{A.23}$$

□

*Remark A.3.* Proposition 1.2 is valid for positive subsolutions of problem (1.12) as we can check in its proof.

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## References

[1] C. O. Alves, D. C. de Morais Filho, and M. A. S. Souto, *Radially symmetric solutions for a class of critical exponent elliptic problems in* $\mathbf{R}^N$, Electron. J. Differential Equations (1996), no. 07, 1–12.

[2] H. Brézis and T. Kato, *Remarks on the Schrödinger operator with singular complex potentials*, J. Math. Pures Appl. (9) **58** (1979), no. 2, 137–151.

[3] H. Brézis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math. **36** (1983), no. 4, 437–477.

[4] M. J. Esteban and P.-L. Lions, *Existence and nonexistence results for semilinear elliptic problems in unbounded domains*, Proc. Roy. Soc. Edinburgh Sect. A **93** (1982/83), no. 1-2, 1–14.

[5] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, New York, 1998.

[6] P.-L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case. II*, Ann. Inst. H. Poincaré Anal. Non Linéaire **1** (1984), no. 4, 223–283.

[7] O. H. Miyagaki, *On a class of semilinear elliptic problems in* $\mathbf{R}^N$ *with critical growth*, Nonlinear Anal. **29** (1997), no. 7, 773–781.

[8] W.-M. Ni and J. Wei, *On the location and profile of spike-layer solutions to singularly perturbed semilinear Dirichlet problems*, Comm. Pure Appl. Math. **48** (1995), no. 7, 731–768.

[9] M. A. S. Souto, *Uniqueness of positive radial solutions of semilinear elliptic equations*, preprint, 1999.

[10] ———, *Uniqueness of positive radial solutions of problem* $-\Delta u + f(u) = 0$, Anais do $49^o$ Seminário Brasileiro de Análise, Universidade Estadual de Campinas, Campinas, 1999, pp. 235–251.

[11] M. Struwe, *Variational Methods. Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*, Springer-Verlag, Berlin, 1990.

[12] G. Talenti, *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl. (4) **110** (1976), 353–372.

[13] M. Willem, *Minimax Theorems*, Progress in Nonlinear Differential Equations and their Applications, vol. 24, Birkhäuser Boston, Massachusetts, 1996.

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