SYNTHESES OF DIFFERENTIAL GAMES AND PSEUDO-RICCATI EQUATIONS

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For differential games of fixed duration of linear dynamical systems with nonquadratic payoff functionals, it is proved that the value and the optimal strategies as saddle point exist whenever the associated pseudo-Riccati equation has a regular solution $P(t,x)$. Then the closed-loop optimal strategies are given by $u(t) = -R^{-1}B^*P(t,x(t))$, $v(t) = -S^{-1}C^*P(t,x(t))$. For differential game problems of Mayer type, the existence of a regular solution to the pseudo-Riccati equation is proved under certain assumptions and a constructive expression of that solution can be found by solving an algebraic equation with time parameter.

1. Introduction

The theory of differential games has been developed for several decades. The early results of differential games of a fixed duration can be found in [2, 3, 5], and the references therein. For linear-quadratic differential and integral games of distributed systems, the closed-loop syntheses have been established in various ways and cases in [6, 8, 10], and most generally in terms of causal synthesis [12, 14].

In another relevant arena, the synthesis results for nonquadratic optimal control problems of linear dynamical systems have been obtained in [11, 13], and some of the references therein. The key issue is how to find and implement nonlinear closed-loop optimal controls with nonquadratic criteria, which have been solved with the aid of a quasi-Riccati equation.

In this paper, we investigate nonquadratic differential games of a finite-dimensional linear system, with a remark that the generalization of the obtained results to infinite-dimensional distributed systems has no essential difficulty. Here the primary objective is to explore whether the linear-nonquadratic

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differential game problem has a value and whether a saddle point of optimal strategies exists and can be found in terms of an explicit state feedback.

Since the players' sets of choices are not compact for such a differential game of fixed duration and (unlike the quadratic optimal control problems) its payoff functional has no convexity or concavity in general, the existences of a value, a saddle point, and most importantly a feedback implementation of optimal strategies in a constructive manner for this type of games are still open issues. We will tackle these issues with a new idea of pseudo-Riccati equation.

Let $T > 0$ be finite and fixed. Consider a linear system of differential equations:

$$\frac{dx}{dt} = Ax + Bu(t) + Cv(t), \quad x(0) = x_0, \tag{1.1}$$

where the state function $x(t)$ and initial data x_0 take values in $\mathbb{R}^n$, $u(t)$ as the control of the player (I) takes value in $\mathbb{R}^m$ and governed by a strategy (which is denoted simply by u), and $v(t)$ as the control of the player (II) takes value in $\mathbb{R}^k$ and governed by a strategy (which is denoted simply by v). The inner products in $\mathbb{R}^n$, $\mathbb{R}^m$, and $\mathbb{R}^k$ will be denoted by $\langle \cdot, \cdot \rangle$, which will be clear in the context. Define function spaces $\mathbf{X} = L^2(0,T;\mathbb{R}^n)$, $\mathbf{X}_c = C([0,T];\mathbb{R}^n)$, $\mathbf{U} = L^2(0,T;\mathbb{R}^m)$, and $\mathbf{V} = L^2(0,T;\mathbb{R}^k)$. Assume that A, B, and C are, respectively, $n \times n$, $n \times m$, and $n \times k$ constant matrices. Any pair of strategies $\{u, v\} \in \mathbf{U} \times \mathbf{V}$ is called admissible strategies.

Set a nonquadratic payoff functional

$$\begin{aligned} J(x_0, u, v) = M(x(T)) + \int_0^T \Big[Q(x(T)) + \frac{1}{2} \langle Ru(t), u(t) \rangle \\ + \frac{1}{2} \langle Sv(t), v(t) \rangle \Big] \, dt, \end{aligned} \tag{1.2}$$

where M and Q are functions in $C^2(\mathbb{R}^n)$, R is an $m \times m$ positive definite matrix, and S is a $k \times k$ negative definite matrix. The game problem is to find a pair of optimal strategies $\{\hat{u}, \hat{v}\} \in \mathbf{U} \times \mathbf{V}$ in the following sense of saddle point:

$$J(x_0, \hat{u}, v) \le J(x_0, \hat{u}, \hat{v}) \le J(x_0, u, \hat{v}), \tag{1.3}$$

for any admissible strategies $u \in \mathbf{U}$ and $v \in \mathbf{V}$. If

$$\sup_v \inf_u J(x_0, u, v) = \inf_u \sup_v J(x_0, u, v), \tag{1.4}$$

then the number given by (1.4) is denoted by $J^*(x_0)$ and called the value of this game. It is seen that whenever a pair of optimal strategies exists, the game has a value and indeed $J^*(x_0) = J(x_0, \hat{u}, \hat{v})$.

We denote by $L(E_1, E_2)$ the space of bounded linear operators from Banach space E_1 to Banach space E_2 with the operator norm. If $E_1 = E_2$, then this operator space is denoted by $L(E_1)$. Any matrix with superscript $*$ means its transposed matrix and any bounded linear operator with superscript $*$ means its adjoint operator. Here, we mention the following relation between a Fréchet differentiable mapping f defined in a convex, open set of a Banach space and its Fréchet derivative Df,

$$f(x+h) - f(x) = \int_0^1 Df(x+sh)h\,ds. \tag{1.5}$$

All the concepts and results in nonlinear analysis used in this paper, such as gradient operator and proper mapping, can be found in [1, 9].

2. Pseudo-Riccati equations

To study the solvability of the nonquadratic differential game problem described by (1.1), (1.2), and (1.3), we consider the following pseudo-Riccati equation associated with the game problem:

$$\begin{aligned} &P_t(t,x) + P_x(t,x)Ax + A^*P(t,x) \\ &\quad - P_x(t,x)\left(BR^{-1}B^* + CS^{-1}C^*\right)P(t,x) + Q'(x) \\ &\qquad = 0, \quad \text{for } (t,x) \in [0,T] \times \mathbb{R}^n, \end{aligned} \tag{2.1}$$

with the terminal condition

$$P(T,x) = M'(x), \quad \text{for } x \in \mathbb{R}^n. \tag{2.2}$$

The unknown of the pseudo-Riccati equation is a nonlinear mapping $P(t,x)$: $[0,T] \times \mathbb{R}^n \to \mathbb{R}^n$. We use P_t and P_x to denote the partial derivatives of P with respect to t and x, respectively. This pseudo-Riccati equation (2.1) with determining condition (2.2) will be denoted by (PRE).

Definition 2.1. A mapping $P(t,x) : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ is called a regular solution of the (PRE) if P satisfies the following conditions:

(i) $P(t,x)$ is continuous in (t,x) and continuously differentiable, respectively, in t and in x, and P satisfies (2.1) and condition (2.2);
(ii) for $0 \le t \le T$, $P(t,\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is a gradient operator;
(iii) the initial value problem

$$\frac{dx}{dt} = Ax - \left(BR^{-1}B^* + CS^{-1}C^*\right)P(t,x), \quad x(0) = x_0, \tag{2.3}$$

has a unique global solution $x \in \mathbf{X}_c$ for any given $x_0 \in \mathbb{R}^n$.

Suppose P is a regular solution of the (PRE). According to the definition of gradient operators (cf. [1]), for any $t \in [0,T]$ there exist anti-derivatives $\Phi(t,x)$ of $P(t,x)$, which are nonlinear functionals $\Phi(t,x) : [0,T]\times\mathbb{R}^n \to \mathbb{R}$ such that

$$\Phi_x(t,x) = P(t,x), \quad (t,x) \in [0,T]\times\mathbb{R}^n. \tag{2.4}$$

Since anti-derivatives may be different only up to a constant, we can set the following condition to fix the constant:

$$\Phi(t,0) \equiv M(0), \quad 0 \le t \le T. \tag{2.5}$$

Lemma 2.2. *Let $x(\cdot)$ be any state trajectory corresponding to an initial state x_0 and any admissible strategies $\{u,v\}$. If $P(t,x)$ is a regular solution of the (PRE) given by (2.1) and (2.2), and $\Phi(t,x)$ is the anti-derivative of $P(t,x)$ with (2.5) being satisfied, then $\Phi(\cdot,x(\cdot))$ is an absolutely continuous function on $[0,T]$, which is denoted by $\Phi(\cdot,x(\cdot)) \in AC([0,T];\mathbb{R})$.*

Proof. From the expression of any state trajectory,

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}\big[Bu(s)+Cv(s)\big]\,ds, \quad t\in[0,T], \tag{2.6}$$

it is seen that $x \in AC([0,T];\mathbb{R})\cap C_{\text{lip}}([0,T];\mathbb{R})$. From (1.5), (2.4), and (2.5) it follows that

$$\Phi\big(t,x(t)\big) = \int_0^1 \big\langle P\big(t,sx(t)\big),x(t)\big\rangle\,ds + M(0), \quad t\in[0,T]. \tag{2.7}$$

Let Ω be a closed and convex set defined by

$$\Omega = \text{Cl. conv.}\big\{sx(t) \mid 0\le s\le 1,\ t\in[0,T]\big\}, \tag{2.8}$$

where x is a trajectory as above. According to Definition 2.1, $P(t,x)$, $P_t(t,x)$, and $P_x(t,x)$ are all uniformly bounded in their norms over the convex, compact set $[0,T]\times\Omega$. By the mean value theorem, it follows that $P(t,x)$ satisfies the uniform Lipschitz condition with respect to $(t,x)\in[0,T]\times\Omega$.

These facts imply that $\Phi(\cdot,x(\cdot)) \in AC([0,T];\mathbb{R})$ by the following straightforward estimation based on (2.7):

$$\begin{aligned}
&\big|\Phi\big(t,x(t)\big)-\Phi\big(\tau,x(\tau)\big)\big| \\
&\quad= \left|\int_0^1 \big\langle P\big(t,sx(t)\big),x(t)\big\rangle\,ds - \int_0^1 \big\langle P\big(\tau,sx(\tau)\big),x(\tau)\big\rangle\,ds\right| \\
&\quad\le \left|\int_0^1 \big\langle P_t\big(\xi,sx(t)\big)(t-\tau)+P_x(\tau,s\eta)\big(x(t)-x(\tau)\big),x(t)\big\rangle\,ds\right| \\
&\qquad+ \left|\int_0^1 \big\langle P\big(\tau,sx(\tau)\big),x(t)-x(\tau)\big\rangle\,ds\right| \\
&\quad\le K\big(|t-\tau|+\big|x(t)-x(\tau)\big|\big),
\end{aligned} \tag{2.9}$$

for any t, $\tau \in [0,T]$, where ξ is between t and τ, η is between $x(t)$ and $x(\tau)$, and K is a constant only depending on $\{x_0, u, v, T\}$. Since we have shown earlier that $x \in C_{\text{lip}}([0,T];\mathbb{R})$, this implies that $\Phi(\cdot, x(\cdot)) \in AC([0,T];\mathbb{R})$. The proof is completed. □

Now we prove a key lemma which addresses the connection of the (PRE) and the concerned differential game problem.

Lemma 2.3. *Under the same assumptions as in Lemma 2.2, it holds that*

$$\begin{aligned}\frac{d}{dt}\Phi\big(t,x(t)\big) &= Q(0) - Q\big(x(t)\big) + \big\langle Bu(t) + Cv(t), P\big(t,x(t)\big)\big\rangle \\ &\quad + \int_0^1 \big\langle P_x\big(t,sx(t)\big)\big(BR^{-1}B^* + CS^{-1}C^*\big)P\big(t,sx(t)\big), x(t)\big\rangle\,ds,\end{aligned} \tag{2.10}$$

for almost every $t \in [0,T]$.

Proof. As a consequence of Lemma 2.2, $\Phi(t,x(t))$ is a.e. differentiable with respect to t in $[0,T]$. On the other hand, from the proof of Lemma 2.2 it is seen that the integrand function $\langle P(t,sx(t)), x(t)\rangle$ in (2.7) is uniformly Lipschitz continuous with respect to t and a Lipschitzian constant can be made independent of the integral variable $s \in [0,1]$.

According to the differentiation theorem for Lebesgue integrals with parameters, we can differentiate two sides of (2.7) to obtain

$$\begin{aligned}\frac{d}{dt}\Phi\big(t,x(t)\big) &= \int_0^1 \frac{d}{dt}\big\langle P\big(t,sx(t)\big), x(t)\big\rangle\,ds \\ &= \int_0^1 \Big[\big\langle P_t\big(t,sx(t)\big), x(t)\big\rangle \\ &\qquad + \Big\langle P_x\big(t,sx(t)\big)\frac{dx}{dt}, sx(t)\Big\rangle + \Big\langle P\big(t,sx(t)\big), \frac{dx}{dt}\Big\rangle\Big]\,ds \\ &= \int_0^1 \big[\big\langle P_t\big(t,sx(t)\big), x(t)\big\rangle + \big\langle P_x\big(t,sx(t)\big)Asx(t), x(t)\big\rangle \\ &\qquad + \big\langle A^*P\big(t,sx(t)\big), x(t)\big\rangle\big]\,ds \\ &\quad + \int_0^1 \big\langle P_x\big(t,sx(t)\big)\big(Bu(t) + Cv(t)\big), x(t)\big\rangle s\,ds \\ &\quad + \Big\langle \int_0^1 P\big(t,sx(t)\big)\,ds, Bu(t) + Cv(t)\Big\rangle \\ &= \int_0^1 \big\langle P_x\big(t,sx(t)\big)\big(BR^{-1}B^* + CS^{-1}C^*\big)P\big(t,sx(t)\big), x(t)\big\rangle\,ds \\ &\quad - \int_0^1 \big\langle Q'\big(sx(t)\big), x(t)\big\rangle\,ds\end{aligned}$$

$$
\begin{aligned}
&+\left\langle Bu(t)+Cv(t), \int_0^1 \left[P_x(t,sx(t))sx(t)+P(t,sx(t))\right] ds\right\rangle \\
&= \int_0^1 \langle P_x(t,sx(t))(BR^{-1}B^*+CS^{-1}C^*)P(t,sx(t)),x(t)\rangle\, ds \\
&\quad - Q(x(t))+Q(0)+\Big\langle Bu(t)+Cv(t), \\
&\qquad\qquad \int_0^1 \left[P_x(t,sx(t))sx(t)+P(t,sx(t))\right] ds\Big\rangle,
\end{aligned}
\tag{2.11}
$$

where in the penultimate equality we used the pseudo-Riccati equation (2.1) and the fact that $P_x(t,x)$ is a selfadjoint operator (symmetric matrix), $P_x(t,x) = P_x(t,x)^*$, because $P(t,x)$ is a gradient operator with respect to x (cf. [1, Theorem 2.5.2]).

Then using the integration by parts to treat the term at the end of (2.11), we have

$$
\begin{aligned}
&\left\langle Bu(t)+Cv(t), \int_0^1 P_x(t,sx(t))sx(t)\, ds\right\rangle \\
&\quad = \left\langle Bu(t)+Cv(t), \left(\int_0^s P_x(t,\sigma x(t))x(t)\, d\sigma\cdot s\right)\Big|_{s=0}^{s=1}\right\rangle \\
&\qquad - \left\langle Bu(t)+Cv(t), \int_0^1\int_0^s P_x(t,\sigma x(t))x(t)\, d\sigma\, ds\right\rangle \\
&\quad = \left\langle Bu(t)+Cv(t), \int_0^1 P_x(t,sx(t))x(t)\, ds\right\rangle \\
&\qquad - \left\langle Bu(t)+Cv(t), \int_0^1\int_0^s P_x(t,\sigma x(t))x(t)\, d\sigma\, ds\right\rangle
\end{aligned}
\tag{2.12}
$$

and by (1.5) the inner integral in the last term of (2.12) can be rewritten as follows:

$$
\begin{aligned}
\int_0^s P_x(t,\sigma x(t))x(t)\, d\sigma &= \int_0^1 P_x(t,\eta sx(t))sx(t)\, d\eta \quad (\text{let } \sigma = \eta s) \\
&= P(t,sx(t)) - P(t,0).
\end{aligned}
\tag{2.13}
$$

Substituting (2.12) with (2.13) into (2.11), we obtain

$$
\begin{aligned}
\frac{d}{dt}\Phi(t,x(t)) &= Q(0)-Q(x(t)) \\
&\quad + \int_0^1 \langle P_x(t,sx(t))(BR^{-1}B^*+CS^{-1}C^*)P(t,sx(t)),x(t)\rangle\, ds
\end{aligned}
$$

$$
\begin{aligned}
&+\left\langle Bu(t)+Cv(t),\int_0^1 P_x\big(t,sx(t)\big)x(t)\,ds\right\rangle\\
&-\left\langle Bu(t)+Cv(t),\int_0^1 \big[P\big(t,sx(t)\big)-P(t,0)\big]\,ds\right\rangle\\
&+\left\langle Bu(t)+Cv(t),\int_0^1 P\big(t,sx(t)\big)\,ds\right\rangle\\
={}&Q(0)-Q\big(x(t)\big)\\
&+\int_0^1 \langle P_x\big(t,sx(t)\big)\big(BR^{-1}B^*+CS^{-1}C^*\big)P\big(t,sx(t)\big),x(t)\rangle\,ds\\
&+\langle Bu(t)+Cv(t),P\big(t,x(t)\big)-P(t,0)\rangle\\
&-\left\langle Bu(t)+Cv(t),\int_0^1 \big[P\big(t,sx(t)\big)-P(t,0)\big]\,ds\right\rangle\\
&+\left\langle Bu(t)+Cv(t),\int_0^1 P\big(t,sx(t)\big)\,ds\right\rangle\\
={}&Q(0)-Q\big(x(t)\big)\\
&+\int_0^1 \langle P_x\big(t,sx(t)\big)\big(BR^{-1}B^*+CS^{-1}C^*\big)P\big(t,sx(t)\big),x(t)\rangle\,ds\\
&+\langle Bu(t)+Cv(t),P\big(t,x(t)\big)\rangle,\quad \text{for a.e. } t\in[0,T]. \qquad (2.14)
\end{aligned}
$$

Therefore, (2.10) is satisfied for a.e. $t \in [0,T]$. □

3. Closed-loop optimal strategies

Under the assumption that there is a regular solution of the pseudo-Riccati equation (2.1), (2.2), we can show the existence, uniqueness, and closed-loop expressions of a pair of optimal strategies as well as the existence of the value of this differential game. It is one of the main results of this work.

Theorem 3.1. *Assume that there exists a regular solution $P(t,x)$ of the (PRE). Then, for any given $x_0 \in \mathbb{R}^n$, the differential game described by (1.1), (1.2), and (1.3) has a value and a unique pair of optimal strategies in the saddle-point sense. Moreover, the optimal strategies are given by the following closed-loop expressions,*

$$
\hat{u}(t) = -R^{-1}B^*P\big(t,x(t)\big),\quad \hat{v}(t) = -S^{-1}C^*P\big(t,x(t)\big),\quad t\in[0,T], \qquad (3.1)
$$

where x stands for the corresponding state trajectory of (1.1).

Proof. Let $\Phi(t,x)$ be the anti-derivative of $P(t,x)$ such that (2.5) is satisfied. For any given x_0 and any admissible strategies $\{u,v\}$, from Lemma 2.3 we have

$$\begin{aligned}
&\frac{d}{dt}\Phi\big(t,x(t)\big)+Q\big(x(t)\big)+\frac{1}{2}\langle Ru(t),u(t)\rangle+\frac{1}{2}\langle Sv(t),v(t)\rangle\\
&\qquad=Q(0)+\frac{1}{2}\langle Ru(t),u(t)\rangle+\frac{1}{2}\langle Sv(t),v(t)\rangle+\big\langle Bu(t)+Cv(t),P\big(t,x(t)\big)\big\rangle\\
&\qquad\quad+\int_0^1\langle P_x\big(t,sx(t)\big)\big(BR^{-1}B^*+CS^{-1}C^*\big)P\big(t,sx(t)\big),x(t)\rangle\,ds.
\end{aligned}\tag{3.2}$$

Let $\beta(t,x)$ be the function defined by

$$\beta(t,x)=\frac{1}{2}\big[\langle R^{-1}B^*P(t,x),B^*P(t,x)\rangle+\langle S^{-1}C^*P(t,x),C^*P(t,x)\rangle\big].\tag{3.3}$$

Then we have

$$\frac{\partial\beta}{\partial x}(t,x)=P_x(t,x)\big(BR^{-1}B^*+CS^{-1}C^*\big)P(t,x).\tag{3.4}$$

From (3.2) and (3.4), we can get

$$\begin{aligned}
&\frac{d}{dt}\Phi\big(t,x(t)\big)+Q\big(x(t)\big)+\frac{1}{2}\langle Ru(t),u(t)\rangle+\frac{1}{2}\langle Sv(t),v(t)\rangle\\
&\qquad=Q(0)+\frac{1}{2}\langle Ru(t),u(t)\rangle+\frac{1}{2}\langle Sv(t),v(t)\rangle+\big\langle Bu(t)+Cv(t),P\big(t,x(t)\big)\big\rangle\\
&\qquad\quad+\int_0^1\left\langle\frac{\partial\beta}{\partial x}\big(t,sx(t)\big),x(t)\right\rangle ds\\
&\qquad=Q(0)+\frac{1}{2}\langle Ru(t),u(t)\rangle\\
&\qquad\quad+\frac{1}{2}\langle Sv(t),v(t)\rangle+\big\langle Bu(t)+Cv(t),P\big(t,x(t)\big)\big\rangle+\beta\big(t,x(t)\big)-\beta(t,0)\\
&\qquad=Q(0)-\beta(t,0)+\frac{1}{2}\big\langle R\big[u(t)+R^{-1}B^*P\big(t,x(t)\big)\big],\\
&\qquad\qquad\qquad u(t)+R^{-1}B^*P\big(t,x(t)\big)\big\rangle\\
&\qquad\quad+\frac{1}{2}\big\langle S\big[v(t)+S^{-1}C^*P\big(t,x(t)\big)\big],v(t)+S^{-1}C^*P\big(t,x(t)\big)\big\rangle.
\end{aligned}\tag{3.5}$$

Now integrating the expressions at the two ends of equality (3.5) in t over $[0,T]$, since $\Phi(\cdot,x(\cdot))$ is an absolutely continuous function, we end up with

$$\begin{aligned}
&\Phi\big(T,x(t)\big)-\Phi\big(0,x_0\big)+\int_0^T\Big[Q\big(x(t)\big)+\frac{1}{2}\langle Ru(t),u(t)\rangle+\frac{1}{2}\langle Sv(t),v(t)\rangle\Big]dt\\
&\quad=Q(0)T-\int_0^T\beta(t,0)\,dt\\
&\qquad+\frac{1}{2}\int_0^T\big\langle R\big[u(t)+R^{-1}B^*P\big(t,x(t)\big)\big],u(t)+R^{-1}B^*P\big(t,x(t)\big)\big\rangle\,dt\\
&\qquad+\frac{1}{2}\int_0^T\big\langle S\big[v(t)+S^{-1}C^*P\big(t,x(t)\big)\big],v(t)+S^{-1}C^*P\big(t,x(t)\big)\big\rangle\,dt.
\end{aligned}\tag{3.6}$$

Note that (2.2) and (2.5) imply $\Phi(T,x)-\Phi(T,0)=M(x)-M(0)$ and

$$\Phi(T,x)=M(x),\quad \forall x\in\mathbb{R}^n. \tag{3.7}$$

Then, with (3.7) substituted, (3.6) can be written as

$$\begin{aligned} J(x_0,u,v) &= M(x(T))+\int_0^T\left[Q(x(t))+\frac{1}{2}\langle Ru(t),u(t)\rangle+\frac{1}{2}\langle Sv(t),v(t)\rangle\right]dt \\ &= W(x_0,T)+\frac{1}{2}\int_0^T\langle R[u(t)+R^{-1}B^*P(t,x(t))], \\ &\qquad\qquad u(t)+R^{-1}B^*P(t,x(t))\rangle\,dt \\ &\quad+\frac{1}{2}\int_0^T\langle S[v(t)+S^{-1}C^*P(t,x(t))],v(t)+S^{-1}C^*P(t,x(t))\rangle\,dt, \end{aligned} \tag{3.8}$$

where

$$W(x_0,T)=\Phi(0,x_0)+Q(0)T-\int_0^T\beta(t,0)\,dt. \tag{3.9}$$

Note that (3.8) holds for any admissible strategies $\{u,v\}$.

According to Definition 2.1, the initial value problem (2.3) has a global solution $x(\cdot)\in\mathbf{X}_c$ over $[0,T]$. Hence, the strategies given by the state feedback expressions in (3.1) are admissible strategies. And (3.8) shows that

$$J(x_0,\hat{u},\hat{v})=W(x_0,T), \tag{3.10}$$

which depends on x_0 and T only. For any other admissible strategies $\{u,v\}$, (3.8) implies

$$\begin{aligned} J(x_0,\hat{u},v) &= W(x_0,T)+\frac{1}{2}\int_0^T\langle S[v(t)+S^{-1}C^*P(t,x(t))], \\ &\qquad\qquad v(t)+S^{-1}C^*P(t,x(t))\rangle\,dt \\ &\le W(x_0,T)=J(x_0,\hat{u},\hat{v}) \\ &\le W(x_0,T)+\frac{1}{2}\int_0^T\langle R[u(t)+R^{-1}B^*P(t,x(t))],u(t) \\ &\qquad\qquad +R^{-1}B^*P(t,x(t))\rangle\,dt \\ &= J(x_0,u,\hat{v}), \end{aligned} \tag{3.11}$$

since R is positive definite and S is negative definite. This proves that there exists a unique pair of optimal strategies $\{\hat{u},\hat{v}\}$, given by (3.1), and that the value of this game exists. In fact, the value is $J^*(x_0)=J(x_0,\hat{u},\hat{v})=W(x_0,T)$. □

Remark 3.2. In the above argument which goes from (3.8) to (3.11), it is important to clearly distinguish the following two concepts: one is the strategy u and

v used by each player, and the other is the control function $u(t)$ and $v(t)$ of the time variable $t \in [0, T]$. The strategy is a pattern like the feedback shown in (3.1) or any other admissible feedback. When a strategy is implemented, then u and v become concrete functions of time variable, which are usually called control functions for the players.

When a pair of strategies $\{u, v\}$ is different from the pair $\{\hat{u}, \hat{v}\}$, certainly the state trajectories $x(t, x_0, \hat{u}, v)$, $x(t, x_0, \hat{u}, \hat{v})$, and $x(t, x_0, u, \hat{v})$ are different functions in general, but as long as the optimal strategy patterns are shown by (3.1), then we have

$$\begin{aligned} &\int_0^T \langle R[\hat{u}(t) + R^{-1}B^*P(t, x(t))], \hat{u}(t) + R^{-1}B^*P(t, x(t))\rangle\, dt = 0, \\ &\int_0^T \langle S[\hat{v}(t) + S^{-1}C^*P(t, x(t))], \hat{v}(t) + S^{-1}C^*P(t, x(t))\rangle\, dt = 0 \end{aligned} \tag{3.12}$$

in the derivation of (3.11).

4. Mayer problem: solution to the pseudo-Riccati equation

In this section, we assume that $Q(x) \equiv 0$. Then the payoff functional reduces to

$$J(x_0, u, v) = M(x(T)) + \frac{1}{2}\int_0^T [\langle Ru(t), u(t)\rangle + \langle Sv(t), v(t)\rangle]\, dt. \tag{4.1}$$

This type of differential games described by (1.1), (4.1), and (1.3) can be referred to as the Mayer problem, according to its counterpart in optimal control theory and in calculus of variations. Since a general problem (1.1), (1.2) can be reduced to a Mayer problem by augmenting the state variable with additional one dimension, it is without loss of generality to consider Mayer problems only.

Associated with this Mayer problem, we will consider a nonlinear algebraic equation with one parameter τ, $0 \le \tau \le T$, as follows:

$$y + G(T-\tau)M'(y) = e^{A(T-\tau)}x, \quad \forall x \in \mathbb{R}^n, \tag{4.2}$$

where

$$G(t) = \int_0^T e^{As}(BR^{-1}B^* + CS^{-1}C^*)e^{A^*s}\, ds, \quad t \in [0, T]. \tag{4.3}$$

Here, (4.2) has an unknown $y \in \mathbb{R}^n$ and a parameter $\tau \in [0, T]$. Equation (4.2) can also be written as

$$y + G(t)M'(y) = e^{At}x, \quad \text{for any } x \in \mathbb{R}^n, \tag{4.4}$$

with $t = T - \tau$, $0 \le t \le T$. Note that $G(t)$ is a symmetric matrix for each $t \in [0, T]$. However, unlike the optimal control problems, here $G(t)$ is in general neither nonnegative, nor nonpositive due to the assumptions on R and S.

First consider a family of differential games defined over a time interval $[\tau, T]$, where $0 \le \tau \le T$ is arbitrarily fixed. We use $(\mathrm{DGP})_\tau$ to denote the differential game problem for the linear system

$$\frac{dx}{dt} = Ax + Bu(t) + Cv(t), \qquad x(\tau) = x_0, \tag{4.5}$$

with respect to the payoff functional

$$J_\tau(x_0, u, v) = M\big(x(T)\big) + \frac{1}{2}\int_\tau^T \big[\langle Ru(t), u(t)\rangle + \langle Sv(t), v(t)\rangle\big]\, dt \tag{4.6}$$

in the sense of saddle point, that is,

$$J_\tau(x_0, \hat{u}, v) \le J_\tau(x_0, \hat{u}, \hat{v}) \le J_\tau(x_0, u, \hat{v}), \tag{4.7}$$

where A, B, C, M, R, and S satisfy the same assumptions made in Section 1.

We first investigate the solution of (4.2) and then find out its connection to a regular solution of the pseudo-Riccati equation (PRE). The entire process will go through several lemmas as follows.

Assumption 4.1. *Assume that for every $\tau \in [0, T]$, there exists a pair of saddle-point strategies to $(\mathrm{DGP})_\tau$ defined by (4.5), (4.6), and (4.7).*

Lemma 4.2. *Under Assumption 4.1, there exists a solution of (4.2) for any given $x \in \mathbb{R}^n$ and for every $\tau \in [0, T]$.*

Proof. Suppose that $\{\hat{u}, \hat{v}\}$ is a pair of saddle-point strategies with respect to $(\mathrm{DGP})_\tau$. Then one has

$$J_\tau(x_0, \hat{u}, \hat{v}) = \min\big\{J_\tau(x_0, u, \hat{v}) \mid \text{admissible } u\big\}, \tag{4.8}$$

$$J_\tau(x_0, \hat{u}, \hat{v}) = \max\big\{J_\tau(x_0, \hat{u}, v) \mid \text{admissible } v\big\}. \tag{4.9}$$

In other words, $\hat{u}$ is the minimizer of $J_\tau(x_0, u, \hat{v})$ subject to constraint (4.5) with $v = \hat{v}$, and $\hat{v}$ is the maximizer of $J_\tau(x_0, \hat{u}, v)$ subject to constraint (4.5) with $u = \hat{u}$. Thus one can apply the Pontryagin maximum principle (cf. [7]). Since the Hamiltonians in these two cases are, respectively,

$$\begin{aligned} H^1(x, \varphi, u) &= \langle Ax + Bu + C\hat{v}, \varphi\rangle + \frac{1}{2}\big[\langle Ru, u\rangle + \langle S\hat{v}, \hat{v}\rangle\big], \\ H^2(x, \psi, u) &= \langle Ax + B\hat{u} + Cv, \psi\rangle + \frac{1}{2}\big[\langle R\hat{u}, \hat{u}\rangle + \langle Sv, v\rangle\big], \end{aligned} \tag{4.10}$$

the co-state function φ associated with the optimal control $\hat{u}$ in (4.8) satisfies the following terminal value problem:

$$\begin{aligned} \frac{d\varphi}{dt} &= -\frac{\partial H^1}{\partial x}(x, \varphi, u) = -A^*\varphi, \qquad \tau \le t \le T, \\ \varphi(T) &= M'\big(x(T)\big), \end{aligned} \tag{4.11}$$

and the co-state function ψ associated with the optimal control $\hat{v}$ in (4.9) satisfies the *same* terminal value problem (4.11), with the same value $x(T)$ that corresponds to the control functions $\{\hat{u}, \hat{v}\}$. Therefore, one has

$$\varphi(t) = \psi(t) = e^{A^*(T-t)} M'\big(x(T)\big), \quad \tau \le t \le T. \tag{4.12}$$

By the maximum principle, the saddle-point strategies can be expressed as the following functions of the time variable t:

$$\begin{aligned} \hat{u}(t) &= -R^{-1}B^*\varphi(t) = -R^{-1}B^* e^{A^*(T-t)} M'\big(x(T)\big), \\ \hat{v}(t) &= -S^{-1}C^*\psi(t) = -S^{-1}C^* e^{A^*(T-t)} M'\big(x(T)\big). \end{aligned} \quad t \in [\tau, T], \tag{4.13}$$

Hence the state trajectory x corresponding to the saddle-point strategies $\{\hat{u}, \hat{v}\}$ satisfies the following equation, for $t \in [\tau, T]$,

$$\begin{aligned} x(t) &= e^{A(t-\tau)}x_0 - \int_\tau^t e^{A(t-s)}\big(BR^{-1}B^* + CS^{-1}C^*\big) e^{A^*(T-s)} M'\big(x(T)\big)\, ds \\ &= e^{A(t-\tau)}x_0 - \int_0^{t-\tau} e^{A(t-\tau-s)}\big(BR^{-1}B^* + CS^{-1}C^*\big) \\ &\qquad\qquad \times e^{A^*(T-\tau-s)}\, ds M'\big(x(T)\big). \end{aligned} \tag{4.14}$$

Let $t = T$ in (4.14) and change variable in the integral by renaming $T - \tau - s$ as s. Then we obtain

$$x(T) + G(T-\tau)M'\big(x(T)\big) = e^{A(T-\tau)}x_0. \tag{4.15}$$

Equation (4.15) shows that, since $x_0 \in \mathbb{R}^n$ is arbitrary, for any given $x = x_0 \in \mathbb{R}^n$ on the right-hand side of (4.2), there exists a solution y to (4.2), which is given by

$$y = x\big(T; x_0, \tau\big) = \text{(simply denoted by) } x(T), \tag{4.16}$$

where $x(T; x_0, \tau)$ represents the terminal value of the saddle-point state trajectory with the initial status $x(\tau) = x_0$. □

It is, however, quite difficult to address the issue of the uniqueness of solutions to (4.2). Now we will exploit a homotopy-type result in nonlinear analysis for this purpose, based on a reasonable assumption below. For each $\tau \in [0, T]$, define a mapping $K_\tau : \mathbb{R}^n \to \mathbb{R}^n$ by

$$K_\tau(y) = y + G(T-\tau)M'(y), \tag{4.17}$$

where $G(\cdot)$ is given by (4.3). Actually, $K_\tau(y)$ is the left-hand side of (4.2). Also let $K(y, \tau) = K_\tau(y)$. We make another assumption here.

Assumption 4.3. *Assume that M is an analytic function on $\mathbb{R}^n$, and K is uniformly coercive in the sense that $\|K(y,\tau)\| \to \infty$ uniformly in τ, whenever $\|y\| \to \infty$.*

An instrumental homotopy result for parametrized nonlinear operators is described in the following lemma, which was first established by R. Caccioppoli in 1932 (cf. [4, Theorem 6.3, page 41] and [9, Theorem 16.3, page 176]). The proof of Lemma 4.4 is shown in [4] and here it is omitted.

Lemma 4.4 (Caccioppoli). *Let X and Y be Banach spaces and let Z be a connected, compact metric space. Suppose that $f : X \times Z \to Y$ is a mapping and we write $f(x, \lambda) = f_\lambda(x)$ where $(x, \lambda) \in X \times Z$. Assume that the following conditions are satisfied:*

(a) *for every $x \in X$, there is an open neighborhood $O(x)$ in X such that $f_\lambda(O(x))$ is open in Y and $f_\lambda : O(x) \to f_\lambda(O(X))$ is isomorphic;*
(b) *the mapping f is proper;*
(c) *for some $\lambda = \lambda_0 \in Z$, the mapping f_{λ_0} is a homeomorphism.*

Then f_λ is a homeomorphism for every $\lambda \in Z$.

Note that a continuous mapping is called *proper* if the inverse image of any compact set in the range space is compact in the domain space. The following lemma is a corollary of [1, Theorem 2.7.1].

Lemma 4.5. *Let f be a continuous mapping from X to Y, where X and Y are both finite dimensional. Then the following statements are equivalent:*

(i) *f is coercive in the sense that $\|f(x)\| \to \infty$ whenever $\|x\| \to \infty$;*
(ii) *f is a closed mapping, and the inverse image set $f^{-1}(p)$ is compact for any fixed $p \in Y$;*
(iii) *f is proper.*

Using the above two lemmas, we can study the uniqueness of solutions of (4.2) and the properties of the solution mapping based on the aforementioned assumptions.

Lemma 4.6. *Under Assumptions 4.1 and 4.3, for every $\tau \in [0, T]$, the mapping K_τ is a C^1 diffeomorphism on $\mathbb{R}^n$.*

Proof. We will check all the conditions in Lemma 4.4 and then apply that lemma to this case by setting $X = Y = \mathbb{R}^n$, $Z = [0, T]$, $f = K$, $f_\lambda = K_\tau$, and $(x, \lambda) = (y, \tau)$.

First, it is easy to see that K is a continuous mapping. By Lemma 4.5, Assumption 4.3 implies directly that $K : X \times [0, T] \to Y$ is coercive and proper. Hence, condition (b) of Lemma 4.4 is verified.

Second, Lemma 4.2 together with the linear homeomorphism $e^{A(T-\tau)}$ shows that the range of K is the entire space $\mathbb{R}^n$. For any given $y_0 \in \mathbb{R}^n$, let $p = K_\tau(y_0)$ be its image. By the continuity of the mapping K_τ, for any open neighborhood $N(p)$ of p, the preimage $K_\tau^{-1}(N(p))$ is an open set. Also note that by Lemma 4.5, $K_\tau^{-1}(p)$ is a compact set for any fixed p. Since the function $M(\cdot)$ is analytic, $K_\tau(\cdot)$ is an analytic function, so the compactness implies that the preimage set $K_\tau^{-1}(p)$ of this p must have no accumulation points, because otherwise K_τ would be a constant-valued function which contradicts Lemma 4.2. As a consequence, each point in $K_\tau^{-1}(p)$ must be isolated.

Therefore, there exists a sufficiently small open neighborhood $N_0(p)$ of p such that the component of $K_\tau^{-1}(N_0(p))$ containing y_0 is an open neighborhood $O(y_0)$ that has no intersection with any other components containing any other preimages (if any) of p. Moreover, as a consequence of this and the continuity, $K_\tau : O(y_0) \to K_\tau(O(y_0))$ is isomorphic. So condition (a) is satisfied.

Third, for $T \in [0,T]$, we have $K_T = I$, the identity mapping on $\mathbb{R}^n$, which is certainly a homeomorphism. Thus, condition (c) of Lemma 4.4 is satisfied.

Therefore, we apply Lemma 4.4 to conclude that for every $\tau \in [0,T]$, the mapping K_τ is a homeomorphism on $\mathbb{R}^n$. Finally, since M is analytic, it is clear that the mapping K_τ is a C^1 mapping. It remains to show that K_τ^{-1} is also a C^1 mapping. Indeed, due to (4.16) and the uniqueness of the solution to (4.2) just shown by the homeomorphism, we can assert that

$$K_\tau^{-1}(p) = x\big(T; e^{-A(T-\tau)}p, \tau\big), \tag{4.18}$$

where $x(t)$, $\tau \le t \le T$, satisfies (4.14) or equivalently $\{x, \varphi\}$ satisfies the following differential equations and the initial-terminal conditions:

$$\begin{gathered} \frac{dx}{dt} = Ax - \big(BR^{-1}B^* + CS^{-1}C^*\big)\varphi, \qquad \frac{d\varphi}{dt} = -A^*\varphi, \\ x(\tau) = e^{-A(T-\tau)}p, \qquad \varphi(T) = M'\big(x(T)\big). \end{gathered} \tag{4.19}$$

By the differentiability of a solution of ODEs with respect to the initial data, or directly by the successive approximation approach, we can show that for any $\tau \le t \le T$,

$$\frac{\partial x\big(t; e^{-A(T-\tau)}p, \tau\big)}{\partial p} \text{ exists and is continuous in } p. \tag{4.20}$$

Let $t = T$, it shows that K_τ^{-1} is a C^{-1} mapping. Thus, we have proved K_τ is a C^1 diffeomorphism on $\mathbb{R}^n$. □

Corollary 4.7. *Under Assumptions 4.1 and 4.3, for every $\tau \in [0,T]$ and every $y \in \mathbb{R}^n$, the derivative*

$$DK_\tau(y) = I + G(T-\tau)M''(y) \tag{4.21}$$

is a nonsingular matrix.

The inverse matrix of (4.21) will be denoted by $[I + G(T-\tau)M''(y)]^{-1}$. Corollary 4.7 is a direct consequence of Lemma 4.6 and the chain rule (cf. [1] or [9]). Thanks to Lemma 4.6 and the linear homeomorphism $e^{-A(T-\tau)}$, there exists a unique solution y of (4.2) for any given $\tau \in [0,T]$ and any given $x \in \mathbb{R}^n$.

This solution y can be written as a mapping $H(T-\cdot,\cdot):[0,T]\times\mathbb{R}^n\to\mathbb{R}^n$, namely,

$$y=H(T-\tau,x),\quad \tau\in[0,T],\ x\in\mathbb{R}^n. \tag{4.22}$$

This mapping H will be referred to as the *solution mapping* of (4.2).

We are going to show the properties of the nonlinear mapping H, which will be used later in proving the main theorem of this section.

Lemma 4.8. *Under Assumptions 4.1 and 4.3, the solution mapping $y=H(T-t,x)$ is continuously differentiable with respect to $(t,x)\in[0,T]\times\mathbb{R}^n$. Moreover, we have*

$$\begin{aligned} H_t(T-t,x)=\big[I+G(T-\tau)M''\big(H(T-t,x)\big)\big]^{-1}&e^{A(T-t)}\\ \times\big[-Ax+\big(BR^{-1}B^*+CS^{-1}C^*\big)&e^{A^*(T-t)}M'\big(H(T-t,x)\big)\big], \end{aligned} \tag{4.23}$$

$$H_x(T-t,x)=\big[I+G(T-\tau)M''\big(H(T-t,x)\big)\big]^{-1}e^{A(T-t)}, \tag{4.24}$$

where H_t and H_x stand for the partial derivatives of H with respect to t and x, respectively.

Proof. Define a mapping

$$E(t,y,x)=y+G(T-\tau)M'(y)-e^{A(T-\tau)}x=K_\tau(y)-e^{A(T-\tau)}x, \tag{4.25}$$

where $(t,y,x)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^n$. Obviously, E is C^1 mapping and $E_y=DK_\tau(y)$ is invertibly due to Corollary 4.7, for any τ and y. Note that (4.2) is exactly $E(t,y,x)=0$. Then by the implicit function theorem and its corollary (cf. [1]), the solution mapping $H(T-t,x)$ of (4.2) (renaming $\tau=t$) is a C^1 mapping with respect to (t,x). Its partial derivatives are given by

$$H_t(T-t,x)=-\big[E_y\big(t,x,H(T-t,x)\big)\big]^{-1}E_t\big(t,x,H(T-t,x)\big), \tag{4.26}$$

$$H_x(T-t,x)=-\big[E_y\big(t,x,H(T-t,x)\big)\big]^{-1}E_x\big(t,x,H(T-t,x)\big). \tag{4.27}$$

Directly calculating E_t and E_x from (4.25) and (4.3) and then substituting them into (4.26) and (4.27), we obtain (4.23) and (4.24). □

Before presenting the main result of this section, we need a lemma which provides some properties of the inverses of some specific types of operators. These properties will be used to prove the self-adjointness of concerned operators in the main result.

Lemma 4.9. *Let X and Y be Banach spaces and $A_0\in L(X)$, $B_0\in L(Y,X)$, $C_0\in L(X,Y)$, and $D_0\in L(Y)$. Then the following statements hold:*

(a) *if A_0, D_0, and $D_0-C_0A_0^{-1}B_0$ are boundedly invertible, then $A_0-B_0D_0^{-1}C_0$ is boundedly invertible and its inverse operator is given by*

$$\big(A_0-B_0D_0^{-1}C_0\big)^{-1}=A_0^{-1}+A_0^{-1}B_0\big(D_0-C_0A_0^{-1}B_0\big)^{-1}C_0A_0^{-1}; \tag{4.28}$$

(b) *suppose* $P_0 \in L(Y,X)$ *and* $Q_0 \in L(X,Y)$. *If* $I_X + P_0Q_0$ *is boundedly invertible, then* $I_Y + Q_0P_0$ *is also boundedly invertible and the following equality holds:*

$$\left(I_X + P_0Q_0\right)^{-1}P_0 = P_0\left(I_Y + Q_0P_0\right)^{-1}. \tag{4.29}$$

Proof. The proof is similar to the matrix case, so it is omitted. □

Now we can present and prove the main result of this section.

Theorem 4.10. *Under Assumptions 4.1 and 4.3, there exists a regular solution* $P(t,x)$ *of the pseudo-Riccati equation (2.1) with the terminal condition (2.2). This regular solution is given by*

$$P(t,x) = e^{A^*(T-t)}M'\big(H(T-t,x)\big), \tag{4.30}$$

where $(t,x) \in [0,T] \times \mathbb{R}^n$, *and* $H(T-t,x)$ *is the solution mapping of (4.2) defined in (4.22).*

Proof. It is easy to verify that the terminal condition (2.2) is satisfied by this $P(t,x)$ because

$$P(T,x) = M'\big(H(0,x)\big) = M'(x). \tag{4.31}$$

Step 1. It is clear that $P(t,x)$ defined by (4.30) is continuous in (t,x) and, due to Lemma 4.8, $P(t,x)$ is continuously differentiable in t and in x, respectively. Now we show that this P satisfies the pseudo-Riccati equation (2.1). In fact, by (4.23) and (4.24) and using the chain rule, we can get

$$P_t(t,x) = -A^*P(t,x) + e^{A^*(T-t)}M''\big(H(T-t,x)\big)H_t(T-t,x), \tag{4.32}$$

$$P_x(t,x) = e^{A^*(T-t)}M''\big(H(T-t,x)\big)H_x(T-t,x). \tag{4.33}$$

From (4.23), (4.24), (4.32), and (4.33), it follows that

$$\begin{aligned}
&P_t(t,x) + P_x(t,x)Ax + A^*P(t,x) - P_x(t,x)\left(BR^{-1}B^* + CS^{-1}C^*\right)P(t,x)\\
&= e^{A^*(T-t)}M''\big(H(T-t,x)\big)H_t(T-t,x)\\
&\quad + e^{A^*(T-t)}M''\big(H(T-t,x)\big)H_x(T-t,x)Ax\\
&\quad - e^{A^*(T-t)}M''\big(H(T-t,x)\big)H_x(T-t,x)\\
&\quad \cdot \left(BR^{-1}B^* + CS^{-1}C^*\right)e^{A^*(T-t)}M'\big(H(T-t,x)\big)\\
&= e^{A^*(T-t)}M''\big(H(T-t,x)\big)\left[I + G(T-\tau)M''\big(H(T-t,x)\big)\right]^{-1}\\
&\quad \cdot \Big\{e^{A(T-t)}\left[-Ax + \left(BR^{-1}B^* + CS^{-1}C^*\right)e^{A^*(T-t)}M'\big(H(T-t,x)\big)\right]\\
&\qquad + e^{A(T-t)}Ax - e^{A(T-t)}\left(BR^{-1}B^* + CS^{-1}C^*\right)e^{A^*(T-t)}M'\big(H(T-t,x)\big)\Big\}\\
&= 0, \quad \text{for } (t,x) \in [0,T] \times \mathbb{R}^n.
\end{aligned} \tag{4.34}$$

Equation (4.34) shows that the nonlinear matrix function $P(t,x)$ given by (4.30) satisfies pseudo-Riccati equation (2.1), with $Q(x)=0$ for the Mayer problem.

Step 2. We now prove that for every $t\in[0,T]$, $P(t,\cdot):\mathbb{R}^n\to\mathbb{R}^n$ is a gradient operator. By [1, Theorem 2.5.2] it suffices to show that for every fixed $t\in[0,T]$, $P_x(t,x)$ is selfadjoint. From (4.33) and (4.24), we find

$$P_x(t,x)=e^{A^*(T-t)}M''\big(H(T-t,x)\big)\cdot\big[I+G(T-\tau)M''\big(H(T-t,x)\big)\big]^{-1}e^{A(T-t)}. \tag{4.35}$$

Applying Lemma 4.9(b) to this case with $P_0=G(T-\tau)$ and $Q_0=M''(H(T-t,x))$, we know that $I+M''(H(T-t,x))G(T-\tau)$ is also boundedly invertible. In order to show that $P_x(t,x)$ in (4.35) is selfadjoint, it is enough to show that

$$M''\big(H(T-t,x)\big)\big[I+G(T-\tau)M''\big(H(T-t,x)\big)\big]^{-1} \tag{4.36}$$

is selfadjoint. Since $G(T-\tau)^*=G(T-\tau)$ and $M''(H(T-t,x))^*=M''(H(T-t,x))$, we have

$$\begin{aligned}
&\Big\{M''\big(H(T-t,x)\big)\big[I+G(T-\tau)M''\big(H(T-t,x)\big)\big]^{-1}\Big\}^*\\
&\quad=\Big\{\big[I+G(T-\tau)M''\big(H(T-t,x)\big)\big]^{-1}\Big\}^*M''\big(H(T-t,x)\big)^*\\
&\quad=\big[I+M''\big(H(T-t,x)\big)^*G(T-\tau)^*\big]^{-1}M''\big(H(T-t,x)\big)^*\\
&\quad=\big[I+M''\big(H(T-t,x)\big)G(T-\tau)\big]^{-1}M''\big(H(T-t,x)\big)\\
&\quad=M''\big(H(T-t,x)\big)\big[I+G(T-\tau)M''\big(H(T-t,x)\big)\big]^{-1},
\end{aligned} \tag{4.37}$$

where the last equality follows from Lemma 4.9(b) and (4.29). Hence, $P_x(t,x)$ is selfadjoint and, consequently, $P(t,\cdot)$ is a gradient operator for every $t\in[0,T]$.

Step 3. Finally we show the existence of a global solution $x(\cdot)\in\mathbf{X}_c$ to the initial value problem (2.3) over $[0,T]$, for any given $x_0\in\mathbb{R}^n$. Indeed, by Lemma 4.2, there exists a trajectory $x(\cdot)\in\mathbf{X}_c$ corresponding to a saddle-point pair of strategies of $(\mathrm{DGP})_{\tau=0}$ with any given initial state x_0. Then by (4.14), the terminal value of this trajectory satisfies

$$\begin{aligned}
x(T)&=e^{AT}x_0-\int_0^T e^{A(T-s)}\big(BR^{-1}B^*+CS^{-1}C^*\big)e^{A^*(T-s)}M'\big(x(T)\big)\,ds\\
&=e^{A(T-t)}x(t)-\int_t^T e^{A(T-s)}\big(BR^{-1}B^*+CS^{-1}C^*\big)e^{A^*(T-s)}M'\big(x(T)\big)\,ds,
\end{aligned} \tag{4.38}$$

which is equivalent to

$$x(T)+G(T-t)M'\big(x(T)\big)=e^{A(T-t)}x(t), \tag{4.39}$$

and that in turn implies $x(T)$ is a solution to (4.2) with the right-hand side being $e^{A(T-t)}x(t)$. By the uniqueness of (4.2) shown in Lemma 4.6, we have

$$x(T)=H\big(T-t,x(t)\big),\quad \text{for } t\in[0,T]. \tag{4.40}$$

Substituting (4.40) into the constant-of-variation formula and by (4.30), we get

$$\begin{aligned} x(t) &= e^{At}x_0 - \int_0^t e^{A(t-s)}\left(BR^{-1}B^* + CS^{-1}C^*\right)e^{A^*(T-s)} \\ &\qquad\qquad \times M'\left(H\left(T-s, x(s)\right)\right)ds \\ &= e^{At}x_0 - \int_0^t e^{A(t-s)}\left(BR^{-1}B^* + CS^{-1}C^*\right)P\left(s, x(s)\right)ds, \end{aligned} \tag{4.41}$$

for $t \in [0, T]$. Equation (4.41) shows that the initial value problem (2.3) with $P(t, x)$ given by (4.30) has a global solution in $\mathbf{X}_c$ for any given $x_0 \in \mathbb{R}^n$. Certainly, since the local Lipschitz condition is satisfied by the right-side function of (2.3) with this $P(t, x)$, this global solution is unique.

Therefore, we conclude that $P(t, x)$ given by (4.30) is a regular solution of the (PRE). The proof is completed. □

Corollary 4.11. *Assume that the following conditions are satisfied:*

(a) *for every $\tau \in [0, T]$ and every $y \in \mathbb{R}^n$, $I + G(T-\tau)M''(y)$ is a nonsingular (i.e., bounded invertible) matrix;*

(b) *the initial value problem*

$$\begin{gathered} \frac{dx}{dt} = Ax - \left(BR^{-1}B^* + CS^{-1}C^*\right)e^{A^*(T-t)}M'\left(H\left(T-t, x(t)\right)\right), \\ x(0) = x_0, \end{gathered} \tag{4.42}$$

has a unique global solution $x \in \mathbf{X}_c$ for any given $x_0 \in \mathbb{R}^n$, where $H(T-\tau, x)$ defined by (4.22) is the solution mapping of (4.2).

Then there exists a regular solution $P(t, x)$ of the pseudo-Riccati equation (2.1) with the terminal condition (2.2). This regular solution is given by

$$P(t, x) = e^{A^*(T-t)}M'\left(H(T-t, x)\right), \tag{4.43}$$

where $(t, x) \in [0, T] \times \mathbb{R}^n$.

Proof. By condition (a) and the implicit function theorem, (4.2) has a unique solution for every τ and x so that the solution mapping $H(T-\tau, x)$ in (4.22) is well defined. Note that Lemma 4.8 depends on the invertibility of $I + G(T-\tau)M''(y)$ only, and Lemma 4.9 is independent. Therefore, Steps 1 and 2 in the proof of Theorem 4.10 remain valid since they depend only on Lemmas 4.8 and 4.9. Since Step 3 is entirely covered by condition (b) in this corollary, we have the same conclusion as in Theorem 4.10. □

In an example of one-dimensional differential equation we will present in Section 5, the two conditions in Corollary 4.11 can be verified. Especially condition (b) can be shown by conducting a priori estimates. If one uses Theorem 4.10, then the essential thing is to verify Assumption 4.1, that may involve the existence theory of saddle points such as some generalized von Neumann theorems (cf. [15, Section 2.13]).

5. Summary and example

In this paper, we present a new method to study the synthesis issue of nonquadratic differential games of linear dynamical systems. The new idea is by the pseudo-Riccati equations approach. In this direction, we have proved two results.

The first result is Theorem 3.1. It states that under very general conditions, as long as there exists a regular solution of the (PRE), then the differential game has a unique pair of optimal strategies in the saddle-point sense and the optimal strategies can be expressed explicitly as real-time state feedbacks given in (3.1). Regarding this result, we have the following remarks.

(i) This result is a substantial generalization of the synthesis of nonquadratic optimal control via the quasi-Riccati equations (cf. [11, 13]).

(ii) This result can be potentially generalized to nonquadratic differential games of infinite-dimensional linear dynamical systems. It can also be generalized to the nonautonomous cases.

(iii) Since the Hamilton-Jacobi equations are related to the synthesis of optimal control problems and the Issac equations are related to the synthesis of differential game problems, the relationship between the pseudo-Riccati equations and these two equations is to be explored.

The second result is Theorem 4.10 which is valid for Mayer problems. It states that under two assumptions, which are reasonable though, there exists a regular solution of the pseudo-Riccati equation and its solution can be explicitly expressed as (4.30) in terms of the solution mapping of a parametrized, nonlinear algebraic equation (4.2). Therefore, the syntheses of the Mayer problems have been solved. Certainly Corollary 4.11 provides a (short-cut) alternative result with two different assumptions. Here we have two more remarks.

(iv) The proof of Theorem 4.10 involves substantial difficulties in comparison with the counterpart result associated with nonquadratic *optimal control* problems (cf. [11, 13]). In the optimal control cases since M is assumed to be convex so that $M''(y)$ is always nonnegative and $CS^{-1}C^* = 0$ so that $G(t)$ in (4.3) is always nonnegative too, one can easily claim that $DK_\tau(y) = I + G(T-\tau)M''(y)$ is nonsingular and its inverse has an explicit expression:

$$\begin{aligned}&\left[I+G(T-\tau)M''(y)\right]^{-1}\\&\quad = I - G(T-\tau)\sqrt{M''(y)}\left[I+\sqrt{M''(y)}G(T-\tau)\sqrt{M''(y)}\right]^{-1}\sqrt{M''(y)}.\end{aligned} \tag{5.1}$$

However, the above equality does not hold in the case of differential games, because none of $M''(y)$ and $G(t)$ can be assumed nonnegative. This difficulty is overcome by using a homotopy lemma due to Caccioppoli, Lemma 4.4, and showing the mapping K_τ is C^1 diffeomorphism directly.

(v) One can investigate different solution concepts of the pseudo-Riccati equations, such as the existence and uniqueness of viscosity solutions. But we did not address this issue since we do not need it for the main results.

Finally, we provide an example to illustrate the synthesis of a Mayer problem of nonquadratic differential game by using the results of Theorem 3.1 and Corollary 4.11 in this paper.

Example 5.1. Consider a differential game problem,

$$\begin{gathered}\frac{dx}{dt} = ax(t) + bu(t) + cv(t), \quad x(0) = x_0 \in \mathbb{R}, \\ J(x_0, u, v) = m|x(T)|^4 + \frac{1}{2}\int_0^T \left[R|u(t)|^2 - S|v(t)|^2\right] dt,\end{gathered} \tag{5.2}$$

in the saddle-point sense. Here $T > 0$ is a fixed number. Suppose that a, b, c, and m are *nonzero* constants that R and S are *positive* constants. We are going to show that Corollary 4.11 can be applied to this problem, with some restrictions on these parameters in (5.2). Then we will show that the regular solution of the corresponding pseudo-Riccati equation and the closed-loop optimal strategies can be found constructively.

The only condition we impose on the parameters is

$$m\left(b^2R^{-1} - c^2S^{-1}\right) > 0. \tag{5.3}$$

In this case, (4.2) takes the form

$$y + G(T-\tau)M'(y) = y + g(\tau)y^3 = e^{a(T-\tau)}x, \quad (\tau, x) \in [0, T] \times \mathbb{R}, \tag{5.4}$$

where

$$\begin{aligned}g(\tau) &= 4m\int_0^{T-\tau} e^{2a\sigma}\left(b^2R^{-1} - c^2S^{-1}\right) d\sigma \\ &= \frac{2m}{a}\left(b^2R^{-1} - c^2S^{-1}\right)\left(e^{2a(T-\tau)} - 1\right).\end{aligned} \tag{5.5}$$

Note that under assumption (5.3), since $a \neq 0$, we have $g(t) > 0$ for all $t \in [0, T]$. This implies that $I + G(T-\tau)M''(y) = 1 + 3g(\tau)y^2$ is boundedly invertible, namely,

$$0 < \left[I + G(T-\tau)M''(y)\right]^{-1} = \frac{1}{1 + 3g(\tau)y^2} \leq 1, \tag{5.6}$$

for all $(\tau, y) \in [0, T] \times \mathbb{R}$. Therefore, condition (a) of Corollary 4.11 is satisfied.

We use the same notation $H(T-\tau, x)$ to denote the solution mapping of (5.4). Then we have to verify condition (b) of Corollary 4.11. Here (4.42) turns out to be

$$\frac{dx}{dt} = ax - (b^2R^{-1} - c^2S^{-1})e^{a(T-t)}(4mH(T-t,x)^3). \tag{5.7}$$

Since the right-side function $f(t,x)$ of (5.7) is C^1 mapping in t and in x, as shown by Lemma 4.8, the local Lipschitz condition is satisfied by this function f. Hence, for any initial status $x(0) = x_0 \in \mathbb{R}$, (5.7) has a unique local solution $x(t)$, $t \in [0, t_1]$, for some $0 < t_1 \leq T$. It suffices to prove that this solution will not blow up, so that $t_1 = T$.

Let the maximal interval of the existence of this solution be $[0, t_\omega)$. Multiplying both sides of (5.7) by $x(t)$, we can get

$$\frac{1}{2}\frac{d}{dt}|x(t)|^2 = a|x(t)|^2 - 4m(b^2R^{-1} - c^2S^{-1})e^{a(T-t)}H(T-t,x(t))^3x(t), \tag{5.8}$$

for $t \in [0, t_\omega)$. Since $y = H(T-t,x(t))$ satisfies (5.4) with $\tau = t$ and $x = x(t)$ on the right-hand side, we have

$$y + g(t)y^3 = e^{a(T-t)}x(t). \tag{5.9}$$

Since $g(t) > 0$ for all $t \in [0, T]$, it follows that

$$y = H(T-t,x(t)) = \frac{e^{a(T-t)}x(t)}{1+g(t)y^2}, \quad t \in [0, t_\omega). \tag{5.10}$$

Substituting (5.10) into (5.8), we obtain

$$\frac{d}{dt}|x(t)|^2 = 2a|x(t)|^2 - 8m(b^2R^{-1} - c^2S^{-1})\frac{e^{4a(T-t)}}{(1+g(t)y^2)^3}|x(t)|^4. \tag{5.11}$$

Due to (5.3) and $1 + g(t)y^2 > 0$ for all t, we see that the second term on the right-hand side of (5.11) is ≤ 0. Hence, we get

$$\frac{d}{dt}|x(t)|^2 \leq 2a|x(t)|^2, \quad \text{for } t \in [0, t_\omega). \tag{5.12}$$

Therefore, this solution $x(t)$ satisfies

$$|x(t)|^2 \leq e^{2at}|x_0|^2, \quad \text{for } t \in [0, t_\omega). \tag{5.13}$$

Equation (5.13) confirms that there is no blow-up and the solution x exists globally on $[0, T]$. Thus, condition (b) of Corollary 4.11 is satisfied.

Then we can apply Corollary 4.11 and Theorem 3.1 to this problem and reach the following conclusion. For $0 \leq t \leq T$, define Δ_t by

$$\Delta_t = \left[-\frac{e^{a(T-t)}x}{2g(t)}\right]^2 + \left[\frac{1}{3g(t)}\right]^3. \tag{5.14}$$

Proposition 5.2. *Under assumption (5.3), the nonquadratic differential game (5.2) has a unique pair of optimal strategies in the saddle-point sense. The closed-loop optimal strategies are given by*

$$\begin{aligned}\hat{u}(t) &= -4mbR^{-1}e^{a(T-t)}H\big(T-t,x(t)\big)^3,\\ \hat{v}(t) &= -4mcS^{-1}e^{a(T-t)}H\big(T-t,x(t)\big)^3,\end{aligned}\qquad t\in[0,T], \tag{5.15}$$

where $H(T-t,x)$ is given by

$$H(T-t,x)=\left(\frac{1}{2g(t)}e^{a(T-t)}x+\sqrt{\Delta_t}\right)^{1/3}-\left(-\frac{1}{2g(t)}e^{a(T-t)}x+\sqrt{\Delta_t}\right)^{1/3}. \tag{5.16}$$

Proof. By Cardan's formula, the unique real root of (5.4) with $\tau = t$ is provided by (5.16). Then Corollary 4.11 guarantees that the pseudo-Riccati equation (2.1) with (2.2) has a regular solution

$$P(t,x) = 4me^{a(T-t)}H\big(T-t,x(t)\big)^3, \quad t\in[0,T]. \tag{5.17}$$

Finally, Theorem 3.1 is applied here to reach the conclusion of this proposition. □

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