ON A CLASS OF FOURTH-ORDER NONLINEAR DIFFERENCE EQUATIONS

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We consider a class of fourth-order nonlinear difference equations. The classification of nonoscillatory solutions is given. Next, we divide the set of solutions of these equations into two types: F_+- and F_--solutions. Relations between these types of solutions and their nonoscillatory behavior are obtained. Necessary and sufficient conditions are obtained for the difference equation to admit the existence of nonoscillatory solutions with special asymptotic properties.

1. Introduction

Consider the difference equation

$$\Delta(a_n\Delta(b_n\Delta(c_n\Delta y_n))) + f(n, y_n) = 0, \quad n \in \mathbb{N}, \tag{1.1}$$

where $\mathbb{N} = \{0, 1, 2, \dots\}$, Δ is the forward difference operator defined by $\Delta y_n = y_{n+1} - y_n$, and (a_n), (b_n), and (c_n) are sequences of positive real numbers. Function $f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$. By a solution of (1.1) we mean a sequence (y_n) which satisfies (1.1) for n sufficiently large. We consider only such solutions which are nontrivial for all large n. A solution of (1.1) is called nonoscillatory if it is eventually positive or eventually negative. Otherwise it is called oscillatory.

In the last few years there has been an increasing interest in the study of oscillatory and asymptotic behavior of solutions of difference equations. Compared to second-order difference equations, the study of higher-order equations, and in particular fourth-order equations (see, e.g., [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]), has received considerably less attention. An important special case of fourth-order difference equations is the discrete version of the Schrödinger equation.

The purpose of this paper is to establish some necessary and sufficient conditions for the existence of solutions of (1.1) with special asymptotic properties.

Throughout the rest of our investigations, one or several of the following assumptions will be imposed:

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(H1) $\sum_{i=1}^{\infty}(1/a_i) = \sum_{i=1}^{\infty}(1/b_i) = \sum_{i=1}^{\infty}(1/c_i) = \infty$;
(H2) $yf(n,y) > 0$ for all $y \neq 0$ and $n \in \mathbb{N}$;
(H3) the function $f(n,y)$ is continuous on $\mathbb{R}$ for each fixed $n \in \mathbb{N}$.

In [14] we can find the following existence theorem (some modification of Schauder's theorem) which will be used in this paper.

Lemma 1.1. *Suppose Ω is a Banach space and K is a closed, bounded, and convex subset. Suppose T is a continuous mapping such that $T(K)$ is contained in K, and suppose that $T(K)$ is uniformly Cauchy. Then T has a fixed point in K.*

2. Main results: existence of nonoscillatory solutions

In this section, we obtain necessary and sufficient conditions for the existence of nonoscillatory solutions of (1.1) with certain asymptotic properties. We start with the following Lemma.

Lemma 2.1. *Assume that (H1) and (H2) hold. Let (y_n) be an eventually positive solution of (1.1). Then exactly one of the following statements holds for all sufficiently large n:*
(i) $y_n > 0$, $\Delta y_n > 0$, $\Delta(c_n\Delta y_n) > 0$, *and* $\Delta(b_n\Delta(c_n\Delta y_n)) > 0$;
(ii) $y_n > 0$, $\Delta y_n > 0$, $\Delta(c_n\Delta y_n) < 0$, *and* $\Delta(b_n\Delta(c_n\Delta y_n)) > 0$.

Proof. Let (y_n) be an eventually positive solution of (1.1). Then, by assumption (H2), $(\Delta(a_n\Delta(b_n\Delta(c_n\Delta y_n))))$ is eventually negative. Therefore, it is easy to see that the sequences $(a_n\Delta(b_n\Delta(c_n\Delta y_n)))$, $(b_n\Delta(c_n\Delta y_n))$, and $(c_n\Delta y_n)$ are all monotone and of one sign, say for $n \geq n_1$.

Suppose that $a_{n_2}\Delta(b_{n_2}\Delta(c_{n_2}\Delta y_{n_2})) = -c_1 < 0$ for some $n_2 \geq n_1$. Hence,

$$a_n\Delta(b_n\Delta(c_n\Delta y_n)) \leq -c_1 \quad \text{for } n \geq n_2, \tag{2.1}$$

then

$$\Delta(b_n\Delta(c_n\Delta y_n)) \leq -\frac{c_1}{a_n}. \tag{2.2}$$

Summing both sides of the last inequality from n_2 to $n-1$, we have

$$b_n\Delta(c_n\Delta y_n) - b_{n_2}\Delta(c_{n_2}\Delta y_{n_2}) \leq -\sum_{i=n_2}^{n-1}\frac{c_1}{a_i}. \tag{2.3}$$

Then $b_n\Delta(c_n\Delta y_n) \leq -\sum_{i=n_2}^{n-1}(c_1/a_i)$, which tends to $-\infty$ as $n \to \infty$.

Then there exists $c_2 > 0$ and $n_3 \geq n_2$ such that

$$b_n\Delta(c_n\Delta y_n) \leq -c_2, \quad \text{for } n \geq n_3. \tag{2.4}$$

So,

$$\Delta(c_n\Delta y_n) \leq -\frac{c_2}{b_n}. \tag{2.5}$$

Summing both sides of the last inequality form n_3 to $n-1$, we obtain

$$c_n\Delta y_n - c_{n_3}\Delta y_{n_3} \leq -\sum_{i=n_3}^{n-1}\frac{c_2}{b_i}, \tag{2.6}$$

which tends to $-\infty$, as $n \to \infty$.

Then there exists $c_3 > 0$ and $n_4 \geq n_3$ such that $(c_n\Delta y_n) \leq -c_3$, for $n \geq n_4$. Hence, $\Delta y_n \leq -c_3/c_n$. A final summation yields $y_n - y_{n_3} \leq -\sum_{i=n_4}^{n-1}(1/c_i) \to -\infty$, which implies $\lim_{n\to\infty} y_n = -\infty$. This contradiction implies $a_n\Delta(b_n\Delta(c_n\Delta y_n)) > 0$ eventually.

Next, assume that there exists $n_5 \in \mathbb{N}$ such that $b_n\Delta(c_n\Delta y_n) < 0$, for $n \geq n_5$, then $(c_n\Delta y_n)$ must be eventually positive for otherwise we are again led to conclude that $c_n\Delta y_n - c_{n_3}\Delta y_{n_3} \leq -\sum_{i=n_3}^{n-1}(c_2/b_i)$, $\lim_{n\to\infty} y_n = -\infty$. Thus, case (ii) is verified.

Next, suppose that $b_n\Delta(c_n\Delta y_n) > 0$ for all $n \geq n_1$. Then $b_n\Delta(c_n\Delta y_n) > b_{n_1}\Delta(c_{n_1}\Delta y_{n_1}) = c_4 > 0$.

Divide the above inequality by b_n and sum from n_1 to $n-1$ to get

$$c_n\Delta y_n - c_{n_1}\Delta y_{n_1} > c_4\sum_{i=n_1}^{n-1}\frac{1}{b_i} \longrightarrow \infty, \tag{2.7}$$

as $n \to \infty$. Hence, (Δy_n) is eventually positive. □

Now we introduce an operator which divides the set of solutions of a special case of (1.1) into two disjoint subsets. We will prove that, for nonoscillatory solution, the first of them equals type (ii) solution and the second equals type (i) solution. We assume that $c_n = a_{n+1}$. Hence (1.1) takes the form

$$\Delta(a_n\Delta(b_n\Delta(a_{n+1}\Delta y_n))) = -f(n, y_n). \tag{2.8}$$

We introduce an operator as follows:

$$F_n = x_{n-1}(a_n\Delta(b_n\Delta(a_{n+1}\Delta x_n))) - (a_n\Delta x_{n-1})(b_n\Delta(a_{n+1}\Delta x_n)). \tag{2.9}$$

Hence

$$\Delta F_n = x_n\Delta(a_n\Delta(b_n\Delta a_{n+1}\Delta x_n)) - b_{n+1}\Delta(a_n\Delta x_{n-1})\Delta(a_{n+2}\Delta x_{n+1}). \tag{2.10}$$

It is clear, by (H2), that the operator F_n is nonincreasing for every nonoscillatory solution (y_n) of (2.8).

If $F_n \geq 0$ for all $n \in N$, then a solution (y_n) of (2.8) is called an F_+-solution. If $F_n < 0$ for some n, then (y_n) is called an F_--solution.

The operator F divides the set of solutions of (2.8) into two disjoint subsets: F_+- and F_--solutions.

Theorem 2.2. *Assume that (b_n) is a bounded sequence. Let y be an F_+-solution of (2.8), then*

$$\sum_{n=1}^{\infty} b_{n+1}\Delta(a_n\Delta x_{n-1})\Delta(a_{n+2}\Delta x_{n+1}) < \infty, \tag{2.11}$$

$$\lim_{n\to\infty} b_n\Delta(a_{n+1}\Delta y_n) = 0. \tag{2.12}$$

Proof. Let (y_n) be an F_+-solution of (2.8). Then, from (2.8), we obtain

$$\Delta F_k = -y_k f(k, y_k) - b_{k+1}\Delta(a_k\Delta y_{k-1})\Delta(a_{k+2}\Delta y_{k+1}). \tag{2.13}$$

By summation, we obtain

$$F_n = F_1 - \sum_{k=1}^{n-1} y_k f(k, y_k) - \sum_{k=1}^{n-1} b_{k+1}\Delta(a_k\Delta y_{k-1})\Delta(a_{k+2}\Delta y_{k+1}). \tag{2.14}$$

Since $F_n \geq 0$, we have

$$\sum_{k=1}^{n-1} b_{k+1}\Delta(a_k\Delta y_{k-1})\Delta(a_{k+2}\Delta y_{k+1}) \leq F_1. \tag{2.15}$$

Therefore $\sum_{k=1}^{n-1} b_{k+1}\Delta(a_k\Delta y_{k-1})\Delta(a_{k+2}\Delta y_{k+1}) < \infty$.

Because (b_n) is a bounded sequence, then $(1/b_n)$ is bounded away from zero. Hence, from (2.11) and the equality

$$b_{n+1}\Delta(a_n\Delta y_{n-1})\Delta(a_{n+2}\Delta y_{n+1}) = \frac{1}{b_{n-1}}\big(b_{n-1}\Delta(a_n\Delta y_{n-1})\big)\big(b_{n+1}\Delta(a_{n+2}\Delta y_{n+1})\big), \tag{2.16}$$

we obtain

$$\lim_{n\to\infty} b_{n-1}\Delta(a_n\Delta y_{n-1}) = 0, \tag{2.17}$$

then

$$\lim_{n\to\infty} b_n\Delta(a_{n+1}\Delta y_n) = 0. \tag{2.18}$$

□

THEOREM 2.3. *Assume that (b_n) is a bounded sequence. Every nonoscillatory solution (y_n) of (2.8) is an F_+-solution if and only if (y_n) is type (ii) solution.*

Proof. We prove this theorem for an eventually positive solution.

Let (y_n) be an eventually positive F_+-solution. Suppose for the sake of contradiction that it is type (i) solution.

Then from $\Delta(b_n\Delta(a_{n+1}\Delta y_n)) > 0$, we get $b_n\Delta(a_{n+1}\Delta y_n) > b_M\Delta(a_{M+1}\Delta y_M) > 0$ for sufficiently large M and $n > M$.

This inequality contradicts condition (2.12) of Theorem 2.2. So, (y_n) is type (ii) solution.

Let (y_n) be type (ii) solution. We will show the positivity of the operator F on the whole sequence. Choose m sufficiently large. Then, from the definition of type (ii) solution, we have $F_n > 0$ for $n \geq m$. Because the operator F is nonincreasing, hence $F_j \geq F_m > 0$ for all $j < m$. Since m was taken arbitrary, then $F_n > 0$ for all $n \in \mathbb{N}$. So, (y_n) is an F_+-solution. □

Remark 2.4. Assume that (b_n) is a bounded sequence. Then every nonoscillatory solution of (2.8) is an F_--solution if and only if (y_n) is type (i) solution.

Now we turn our attention to (1.1). We introduce the notation

$$P_{n,N} = \sum_{k=N+2}^{n} \frac{1}{a_k} \sum_{j=N+2}^{k} \frac{1}{b_j} \sum_{i=N+2}^{j} \frac{1}{c_i}, \qquad Q_{n,N} = \sum_{k=N}^{n-1} \frac{1}{c_k} \sum_{j=N}^{k-1} \frac{1}{b_j} \sum_{i=N}^{j-1} \frac{1}{a_i}. \tag{2.19}$$

Note that $Q_{n,N}$ can be written in the form

$$Q_{n,N} = \sum_{i=N}^{n-1} \frac{1}{a_i} \sum_{j=i+1}^{n-1} \frac{1}{b_j} \sum_{k=j+1}^{n-1} \frac{1}{c_k}. \tag{2.20}$$

Lemma 2.5. *Assume conditions (H1) and (H2) hold. If (y_n) is an eventually positive solution of (1.1), then there exist positive constants C_1 and C_2 and integer N such that*

$$C_1 \le y_n \le C_2 Q_{n,N}, \tag{2.21}$$

for $n \ge N+3$.

Proof. Let (y_n) be an eventually positive solution of (1.1). Then $y_n > 0$ for large n. From Lemma 1.1, $\Delta y_n > 0$ eventually, and so $y_n \ge C_1 > 0$.

Now we prove the right-hand side of (2.21). From (1.1) and (H2), there exists N such that

$$\Delta\left(a_n \Delta\left(b_n \Delta\left(c_n \Delta y_n\right)\right)\right) < 0, \quad \text{for } n \ge N. \tag{2.22}$$

Summing the above inequality form N to $n-1$, we get

$$\Delta\left(b_n \Delta\left(c_n \Delta y_n\right)\right) < \frac{A_0}{a_n}, \quad \text{for } n \ge N+1, \tag{2.23}$$

where A_0 is a constant.

Summing again, we have

$$b_n \Delta\left(c_n \Delta y\right) < A_0 \sum_{i=N}^{n-1} \frac{1}{a_i} + b_N \Delta\left(c_N \Delta y_N\right), \tag{2.24}$$

and therefore,

$$\Delta\left(c_n \Delta y_n\right) < \frac{A_0}{b_n} \sum_{i=N}^{n-1} \frac{1}{a_i} + \frac{A_1}{b_n}, \quad \text{for } n \ge N+1. \tag{2.25}$$

Summing the last inequality, we obtain

$$c_n \Delta y_n < A_0 \sum_{j=N}^{n-1} \frac{1}{b_j} \sum_{l=N}^{j-1} \frac{1}{a_i} + A_1 \sum_{j=1}^{n-1} \frac{1}{b_j} + A_2, \quad n \ge N+2, \tag{2.26}$$

where A_1 and A_2 are constants.

Hence,

$$\Delta y_n < \frac{A_0}{c_n}\sum_{j=N}^{n-1}\frac{1}{b_j}\sum_{i=N}^{j-1}\frac{1}{a_i} + \frac{A_1}{c_n}\sum_{j=1}^{n-1}\frac{1}{b_j} + \frac{A_2}{c_n}. \tag{2.27}$$

A final summation yields

$$y_n < A_0\sum_{k=N}^{n-1}\frac{1}{c_k}\sum_{j=N}^{k}\frac{1}{b_j}\sum_{i=N}^{j-1}\frac{1}{a_i} + A_1\sum_{k=N}^{n-1}\frac{1}{c_k}\sum_{j=N}^{k-1}\frac{1}{b_k} + A_2\sum_{k=N}^{n-1}\frac{1}{c_k} + A_3, \quad n \geq N+3. \tag{2.28}$$

It is easy to see that every term on the right-hand side of the last inequality is less than $Q_{n,N}$. Therefore, we obtain $y_n \leq C_2 Q_{n,N}$ for $n \geq N+3$, where C_2 is a positive constant. □

We say that a nonoscillatory solution (y_n) of (1.1) is asymptotically constant if there exist some positive constant α such that $y_n \to \alpha$ and asymptotically $Q_{n,N}$ if there is some positive constant β such that $y_n/Q_{n,N} \to \beta$.

According to Lemma 2.5, we may regard an asymptotically constant solution as a "minimal" solution, and an asymptotically $Q_{n,N}$ solution as a "maximal" solution.

Now, we present a necessary and sufficient condition for the existence of an asymptotically $Q_{n,N}$ solution.

Theorem 2.6. *Assume that (H1), (H2), and (H3) hold and f is a nondecreasing function in another argument, that is, "$f(n,t1) \leq f(n,t2)$ for $t1 < t2$ and each fixed n." Then a necessary and sufficient condition for (1.1) to have a solution (y_n) satisfying*

$$\lim_{n\to\infty}\frac{y_n}{Q_{n,N}} = \beta \neq 0 \tag{2.29}$$

is that

$$\sum_{n=1}^{\infty}\left|f(n, CQ_{n,N})\right| < \infty, \tag{2.30}$$

for some integer $N \geq 1$ and some nonzero constant C.

Proof

Necessity. Let (y_n) be a nonoscillatory solution of (1.1) which satisfies (2.29). Without loss of generality, we may assume that $\beta > 0$. Then there exist positive numbers d_1 and d_2 such that

$$d_1 Q_{n,N} \leq y_n \leq d_2 Q_{n,N}, \quad n \geq N+3, \tag{2.31}$$

where N is a sufficiently large integer. Then

$$f(n, y_n) \geq f(n, d_1 Q_n). \tag{2.32}$$

On the other hand, summing (1.1) from N to $n-1$, and from Lemma 1.1, we get

$$0 < a_n\Delta(b_n\Delta(c_n\Delta y_n)) = a_N\Delta(b_N\Delta(c_N\Delta y_N)) - \sum_{i=N}^{n-1} f(i, y_i), \tag{2.33}$$

which implies that

$$\sum_{i=N}^{\infty} f(i, y_i) \le a_N\Delta(b_N\Delta(c_N\Delta y_N)) < \infty. \tag{2.34}$$

So, by (2.32), we have

$$\sum_{i=N}^{\infty} f(i, d_1 Q_{i,N}) < \infty. \tag{2.35}$$

Sufficiency. Assume that (2.30) holds with $C > 0$ since a similar argument holds if $C < 0$. Let N be large enough that

$$\sum_{i=N-3}^{\infty} f(i, CQ_{n,N}) < \frac{1}{8}C. \tag{2.36}$$

Consider the Banach space B_N of all real sequences $y = (y_n)$ defined for $n \ge N+3$ such that

$$\|y\| = \sup_{n\ge N+3} \frac{|y_n|}{Q_{n,N}^2} < \infty. \tag{2.37}$$

Let S be the subset of B_N defined by

$$S = \left\{(y_n) \in B_N : \frac{C}{2}Q_{n,N} \le y_n \le CQ_{n,N},\ n \ge N+3\right\}. \tag{2.38}$$

It is not difficult to see that S is a bounded, convex, and closed subset of B_N.

We define a mapping $T : S \to B_N$ as follows:

$$\begin{aligned}(Ty)_n = {} & \frac{C}{2}Q_{n,N} + Q_{n,N}\sum_{i=n-1}^{\infty} F(i) + \sum_{j=N}^{n-1} F(j-1)Q_{j,N} \\ & + \sum_{i=N}^{n-1}\frac{1}{c_i}\sum_{j=N}^{i-1}F(j)\sum_{k=N}^{j-1}\frac{1}{b_k}\sum_{s=N}^{k-1}\frac{1}{a_s} \\ & + \sum_{i=N}^{n-1}\frac{1}{c_i}\sum_{j=N}^{i-1}\frac{1}{b_j}\sum_{k=N}^{j-1}F(k+1)\sum_{s=N}^{k-1}\frac{1}{a_s}, \quad \text{for } n \ge N+3,\end{aligned} \tag{2.39}$$

where we have used the notation $F(k)$ for denoting $f(k-2, y_{(k-2)})$.

We first show that $T(S) \subset S$. Indeed, if $y \in S$, it is clear from (2.39) that $(Ty)_n \geq (C/2)Q_{n,N}$ for $n \geq N+3$. Furthermore, for $n \geq N+3$, we have

$$\begin{aligned}
(Ty)_n &\leq \frac{C}{2}Q_{n,N} + Q_{n,N}\sum_{i=n-1}^{\infty} F(i) + Q_{n,N}\sum_{j=N}^{n-1} F(j-1) \\
&\quad + \sum_{i=N}^{n-1}\frac{1}{c_i}\sum_{k=N}^{i-1}\frac{1}{b_k}\sum_{s=N}^{k-1}\frac{1}{a_s}\sum_{j=k+1}^{i-1} F(j) \\
&\quad + \sum_{i=N}^{n-1}\frac{1}{c_i}\sum_{j=N}^{i-1}\frac{1}{b_j}\sum_{s=N}^{j-1}\frac{1}{a_s}\sum_{k=s+1}^{j-1} F(k+1) \\
&\leq \frac{C}{2}Q_{n,N} + Q_{n,N}\sum_{i=N-3}^{\infty} F(i+2) + Q_{n,N}\sum_{j=N-3}^{\infty} F(j+2) \\
&\quad + Q_{n,N}\sum_{j=N-1}^{\infty} F(j+2) + Q_{n,N}\sum_{k=N}^{\infty} F(k+2) \\
&\leq \frac{C}{2}Q_{n,N} + 4Q_{n,N}\sum_{i=N-3}^{\infty} F(i+2).
\end{aligned} \tag{2.40}$$

So, we have

$$(Ty)_n \leq \frac{C}{2}Q_{n,N} + 4Q_{n,N}\sum_{i=N-3}^{\infty} f(i, y_i). \tag{2.41}$$

Therefore, by (2.36), we get

$$(Ty)_n \leq \frac{C}{2}Q_{n,N} + 4Q_{n,N}\sum_{i=N-3}^{\infty} f(i, CQ_{i,N}) \leq CQ_{n,N}. \tag{2.42}$$

Thus T maps S into itself.

Next we prove that T is continuous. Let $(y^{(m)})$ be a sequence in S such that $y^{(m)} \to y$ as $m \to \infty$. Because S is closed, $y \in S$. Now, by (2.41), we get

$$\left|(Ty^{(m)})_n - (Ty)_n\right| \leq 4Q_{n,N}\sum_{i=N-3}^{\infty}\left|f(i, y_i^{(m)}) - f(i, y_i)\right|, \quad n \geq N+3, \tag{2.43}$$

and therefore,

$$\left\|(Ty^{(m)})_n - (Ty)_n\right\| \leq \frac{4}{Q_{n,N}}\sum_{i=N-3}^{\infty}\left|f(i, y_i^{(m)}) - f(i, y_i)\right|. \tag{2.44}$$

Since

$$\lim_{m\to\infty} \left| f(i, y_i^{(m)}) - f(i, y_i) \right| = 0,$$
$$\left| f(i, y_i^{(m)}) - f(i, y_i) \right| \leq 2f(i, CQ_{i,N}), \quad \text{for } i \geq N+3, \tag{2.45}$$

we see from Lebesgue's dominated convergence theorem that

$$\lim_{m\to\infty} \| Ty^{(m)} - Ty \| = 0. \tag{2.46}$$

This means that T is continuous.

Finally, we need to show that $T(S)$ is uniformly Cauchy. To see this, we have to show that, given any $\epsilon > 0$, there exists an integer N_1 such that, for $m > n > N_1$,

$$\left| \frac{(Ty)_m}{Q_{m,N}^2} - \frac{(Ty)_n}{Q_{n,N}^2} \right| < \epsilon, \tag{2.47}$$

for any $y \in S$. Indeed, by (2.41) and (2.36), we have

$$\left| \frac{(Ty)_m}{Q_{m,N}^2} - \frac{(Ty)_n}{Q_{n,N}^2} \right| \leq \frac{C}{Q_{n,N}} + \frac{8}{Q_{n,N}} \sum_{i=N-3}^{\infty} f(i, y_i) \leq \frac{2C}{Q_{n,N}} \longrightarrow 0. \tag{2.48}$$

Therefore, by Lemma 1.1, there exists $y \in S$ such that $y_n = (Ty)_n$, for $n \geq N+3$. It is easy to see that (y_n) is a solution of (1.1). Furthermore, by Stolz's theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{y_n}{Q_{n,N}} &= \lim_{n\to\infty} \frac{\Delta y_n}{\Delta Q_{n,N}} = \lim_{n\to\infty} \frac{c_n \Delta y_n}{c_n \Delta Q_{n,N}} = \lim_{n\to\infty} \frac{\Delta(c_n \Delta y_n)}{\Delta(c_n \Delta Q_{n,N})} \\
&= \lim_{n\to\infty} \frac{b_n \Delta(c_n \Delta y_n)}{b_n \Delta(c_n \Delta Q_{n,N})} = \lim_{n\to\infty} \frac{\Delta(b_n \Delta(c_n \Delta y_n))}{\Delta\left(\sum_{i=1}^{n-1} (1/a_i)\right)} \\
&= \lim_{n\to\infty} a_n \Delta(b_n \Delta(c_n \Delta y_n)),
\end{aligned} \tag{2.49}$$

so,

$$\lim_{n\to\infty} \frac{y_n}{Q_{n,N}} = \lim_{n\to\infty} \left(C + \sum_{s=n+2}^{\infty} G(s) \right) = \lim_{n\to\infty} \left(C + \sum_{s=n+2}^{\infty} f(i, y_i) \right) = C. \tag{2.50}$$

This completes the proof. □

Theorem 2.6 extends [13, Theorem 1] and [14, Theorem 3].

Example 2.7. Consider the difference equation

$$\Delta\left(\frac{1}{n} \Delta((n-1)\Delta((n-1)\Delta y_n)) \right) + \frac{1}{n^{5/3}(n+1)} y_n^{1/3} = 0, \quad \text{for } n \geq 2. \tag{2.51}$$

It is easy to calculate that $Q_{n,N} = (1/8)n(n+1)$, $n \geq 4$. Hence the above equation has a solution (y_n) such that $\lim_{n\to\infty}(y_n/Q_{n,N}) = C \neq 0$. In fact, $y_n = n^2$ is a solution of this equation with $\lim_{n\to\infty}(y_n/Q_{n,N}) = 8$.

Next we derive a necessary and sufficient condition for the existence of an asymptotically constant solution of (1.1).

Theorem 2.8. *Assume that (H1), (H2), and (H3) hold and the function f is a monotonic function in the second argument. Then a necessary and sufficient condition for (1.1) to have a solution (y_n) which satisfies*

$$\lim_{n\to\infty} y_n = \alpha \neq 0 \tag{2.52}$$

is that

$$\sum_{i=1}^{\infty} P_{i,N} \left| f(i,c) \right| < \infty, \tag{2.53}$$

for some integer $N \geq 1$ and some nonzero constant c.

Proof

Necessity. Without loss of generality, we assume that (y_n) is an eventually positive solution of (1.1) such that

$$\lim_{n\to\infty} y_n = \alpha > 0. \tag{2.54}$$

Then there exist positive constants d_3 and d_4 such that

$$d_3 \leq y_n \leq d_4, \quad \text{for large } n. \tag{2.55}$$

Let $z_n = b_n\Delta(c_n\Delta y_n)$. It is clear that if condition (H1) is satisfied, then solution (y_n) of (1.1) of type (i) tends to infinity. Since (y_n) satisfies condition (ii) of Lemma 2.1, hence $y_n > 0$, $z_n < 0$, $\Delta y_n > 0$, and $\Delta z_n > 0$ eventually. Let N be so large that (2.55) and (ii) hold for $n \geq N$. We will use (1.1) in the following form:

$$\Delta\left(a_{n-2}\Delta z_{n-2}\right) = -f\left(n-2, y_{(n-2)}\right). \tag{2.56}$$

Multiplying the above equation by $P_{i-2,N-2}$, and summing both sides of it from $i = N$ to $n-2$, we obtain

$$\sum_{i=N}^{n-2} P_{i-2,N-2} f\left(i-2, y_{(i-2)}\right) = -\sum_{i=N}^{n-2} P_{i-2,N-2}\Delta\left(a_{i-2}\Delta z_{i-2}\right). \tag{2.57}$$

Hence, by the formula $\sum_{i=K}^{n-2} y_i \Delta x_i = x_i y_i |_{i=K}^{n-1} - \sum_{i=K}^{n-2} x_{i+1} \Delta y_i$, we get

$$\begin{aligned}
&\sum_{i=N}^{n-2} P_{i-2,N-2} f\left(i-2, y_{(i-2)}\right) \\
&\quad = -P_{i-2,N-2} a_{i-2} \Delta z_{i-2} |_{i=N}^{n-1} + \sum_{i=N}^{n-2} \left(\Delta P_{i-2,N-2}\right) a_{i-1} \Delta z_{i-1} \\
&\quad = -P_{n-3,N-2} a_{n-3} \Delta z_{n-3} + \sum_{i=N}^{n-2} \frac{1}{a_{i-1}} \sum_{j=N}^{i-1} \frac{1}{b_j} \sum_{k=N}^{j} \frac{1}{c_k} a_{i-1} \Delta z_{i-1} \\
&\quad < \sum_{i=N}^{n-2} \left(\sum_{j=N}^{i-1} \frac{1}{b_j} \sum_{k=N}^{j} \frac{1}{c_k} \right) \Delta z_{i-1} \\
&\quad = \sum_{j=N}^{i-1} \frac{1}{b_j} \sum_{k=N}^{j} \frac{1}{c_k} z_{i-1} |_{i=N}^{n-1} - \sum_{i=N}^{n-2} \left(\frac{1}{b_i} \sum_{j=N}^{i} \frac{1}{c_j} \right) \Delta z_i \\
&\quad = \sum_{j=N}^{n-2} \frac{1}{b_j} \sum_{k=N}^{j} \frac{1}{c_k} z_{n-1} - \sum_{i=N}^{n-2} \frac{1}{b_i} \sum_{j=N}^{i} \frac{1}{c_j} b_i \Delta\left(c_i \Delta y_i\right) \\
&\quad < -\sum_{i=N}^{n-2} \left(\sum_{j=N}^{i} \frac{1}{c_j} \right) \Delta\left(c_i \Delta y_i\right) \\
&\quad = -\sum_{j=N}^{i} \frac{1}{c_j} \left(c_i \Delta y_i\right) |_{i=N}^{n-1} + \sum_{i=N}^{n-2} \Delta y_{i+1} \\
&\quad = -\sum_{j=N}^{n-1} \frac{1}{c_j} \left(c_{n-1} \Delta y_{n-1}\right) + \Delta y_N + \Delta y_{n-1} + \cdots + \Delta y_{N+1} \\
&\quad < \Delta y_N + y_n - y_{N+1} < \Delta y_N + y_n;
\end{aligned} \tag{2.58}$$

which tends to $\Delta y_N + \alpha$ as $n \to \infty$. Therefore,

$$\sum_{i=N}^{\infty} P_{i-2,N-2} f\left(i-2, y_{(i-2)}\right) < \infty. \tag{2.59}$$

From the monotonicity of the function f, we get

$$f\left(i-2, y_{(i-2)}\right) \geq f\left(i-2, d_3\right) \tag{2.60}$$

when f is nondecreasing and

$$f\left(i-2, y_{(i-2)}\right) \geq f\left(i-2, d_4\right) \tag{2.61}$$

when f is nonincreasing. This implies that

$$\sum_{i=1}^{\infty} P_{i-2,N-2} f\left(i-2, d_k\right) < \infty, \tag{2.62}$$

where $k=3$ for f nondecreasing and $k=4$ for f nonincreasing. Then the necessity of this theorem holds.

Sufficiency. Next, let $M \geq 1$ be a large integer such that, for some nonzero constant c, $\sum_{i=M}^{\infty} P_{i,M} f(i,c) < \alpha/4$. From here,

$$\sum_{i=M+2}^{\infty} \frac{1}{c_i} \sum_{j=i}^{\infty} \frac{1}{b_j} \sum_{k=j}^{\infty} \frac{1}{a_k} \sum_{s=k}^{\infty} f(s,c) < \frac{\alpha}{4}. \tag{2.63}$$

Here $\alpha = \beta/2$ if f is nondecreasing and $\alpha = \beta$ when f is nonincreasing. Let B be the linear space of a bounded real sequences $(y_n)_N^{\infty}$ with the usual operations and supremum norm. Let S be the subset

$$S = \{(y_n) \in B;\ \alpha \leq y_n \leq 2\alpha,\ n \geq M+3\}. \tag{2.64}$$

It is easy to see that S is a bounded, convex, and closed subset of B.

We define operator T in the following way:

$$\begin{aligned}(Ty)_n = \alpha &+ \left(\sum_{s=M+2}^{n-1} \frac{1}{c_s} \sum_{k=n}^{\infty} \frac{1}{b_k} \sum_{j=k}^{\infty} \frac{1}{a_j} \sum_{i=j}^{\infty} f(i,y_i) \right) \\ &+ \left(\sum_{k=M+2}^{n-1} \frac{1}{b_k} \sum_{s=M+2}^{k} \frac{1}{c_s} \right) \sum_{j=n}^{\infty} \frac{1}{a_j} \sum_{i=j}^{\infty} f(i,y_i) \\ &+ \left(\sum_{j=M+2}^{n-1} \frac{1}{a_j} \sum_{k=M+2}^{j} \frac{1}{b_k} \sum_{s=M+2}^{k} \frac{1}{c_s} \right) \sum_{i=n}^{\infty} f(i,y_i) \\ &+ \sum_{i=M+2}^{n-1} \left[\left(\sum_{j=M+2}^{i} \frac{1}{a_j} \sum_{k=M+2}^{j} \frac{1}{b_k} \sum_{s=M+2}^{k} \frac{1}{c_s} \right) f(i,y_i) \right],\end{aligned} \tag{2.65}$$

for $n \geq M+3$. It is easy to see that T maps S into itself. Indeed, if $y_n \in S$, then $(Ty)_n \geq \alpha$. Furthermore, for $n \geq M+3$, we have

$$\begin{aligned}\left(\sum_{s=M+2}^{n-1} \frac{1}{c_s} \right) \sum_{k=n}^{\infty} \frac{1}{b_k} \sum_{j=k}^{\infty} \frac{1}{a_j} \sum_{i=j}^{\infty} f(i,y_i) &\leq \sum_{s=M+2}^{\infty} \frac{1}{c_s} \sum_{k=n}^{\infty} \frac{1}{b_k} \sum_{j=k}^{\infty} \frac{1}{a_j} \sum_{i=j}^{\infty} f(i,y_i) \\ &\leq \sum_{s=M+2}^{\infty} \frac{1}{c_s} \sum_{k=s}^{\infty} \frac{1}{b_k} \sum_{j=k}^{\infty} \frac{1}{a_j} \sum_{i=j}^{\infty} f(i,y_i) < \frac{\alpha}{4}, \\ \left(\sum_{k=M+2}^{n-1} \frac{1}{b_k} \sum_{s=M+2}^{k} \frac{1}{c_s} \right) \sum_{j=n}^{\infty} \frac{1}{a_j} \sum_{i=j}^{\infty} f(i,y_i) &\leq \sum_{s=M+2}^{n-1} \frac{1}{c_s} \sum_{k=s}^{n-1} \frac{1}{b_k} \sum_{j=n}^{\infty} \frac{1}{a_j} \sum_{i=j}^{\infty} f(i,y_i) \\ &\leq \sum_{s=M+2}^{n-1} \frac{1}{c_s} \sum_{k=s}^{n-1} \frac{1}{b_k} \sum_{j=k}^{\infty} \frac{1}{a_j} \sum_{i=j}^{\infty} f(i,y_i) < \frac{\alpha}{4}.\end{aligned} \tag{2.66}$$

It follows that $(Ty)_n < \alpha/4 + \alpha/4 + \alpha/4 + \alpha/4 + \alpha = 2\alpha$.

Next we assert that T is continuous. Let $y^{(m)} \to y$ as $m \to \infty$. We derive that

$$(Ty)_n < \alpha + 4 \sum_{s=M+2}^{\infty} \frac{1}{c_s} \sum_{k=s}^{\infty} \frac{1}{b_k} \sum_{j=k}^{\infty} \frac{1}{a_j} \sum_{i=j}^{\infty} f(i, y_i) \tag{2.67}$$

for $n \geq M+3$.

We obtain

$$\begin{aligned} \left| (Ty^{(m)})_n - (Ty)_n \right| &< 4 \left| \sum_{s=M+2}^{\infty} \frac{1}{c_s} \sum_{k=s}^{\infty} \frac{1}{b_k} \sum_{j=k}^{\infty} \frac{1}{a_j} \sum_{i=j}^{\infty} f(i, y_i^{(m)}) \right. \\ &\quad \left. - \sum_{s=M+2}^{\infty} \frac{1}{c_s} \sum_{k=s}^{\infty} \frac{1}{b_k} \sum_{j=k}^{\infty} \frac{1}{a_j} \sum_{i=j}^{\infty} f(i, y_n) \right| \\ &< 4 \sum_{s=M+2}^{\infty} \frac{1}{c_s} \sum_{k=s}^{\infty} \frac{1}{b_k} \sum_{j=k}^{\infty} \frac{1}{a_j} \sum_{i=j}^{\infty} \left| f(i, y_i^{(m)}) - f(i, y_i) \right|. \end{aligned} \tag{2.68}$$

Hence,

$$\|Ty^{(m)} - Ty\| \leq 4 \sum_{s=M+2}^{\infty} \frac{1}{c_s} \sum_{k=s}^{\infty} \frac{1}{b_k} \sum_{j=k}^{\infty} \frac{1}{a_j} \sum_{i=j}^{\infty} \left| f(i, y_i^{(m)}) - f(i, y_i) \right|. \tag{2.69}$$

Since

$$\begin{gathered} \lim_{m \to \infty} \left| f(i, y_i^{(m)}) - f(i, y_i) \right| = 0, \\ \left| f(i, y_i^{(m)}) - f(i, y_i) \right| \leq 2 f(i, C), \end{gathered} \tag{2.70}$$

then, from Lebesgue's dominated convergence theorem, we see that

$$\lim_{m \to \infty} \|Ty^{(m)} - Ty\| = 0. \tag{2.71}$$

This proves our assertion.

It is easy to see that $T(S)$ is uniformly Cauchy.

Thus, by Lemma 1.1, the operator T has a fixed point in S. One can see that (y_n) is a solution of (1.1) for all large n, then $\lim_{n \to \infty} y_n \in [\alpha, 2\alpha]$ as required. □

Theorem 2.8 extends [13, Theorem 2] and [14, Theorem 4].

Example 2.9. Consider the difference equation

$$\Delta\left(n\Delta\left((n-1)\Delta\left(\frac{1}{(n-1)}\Delta y_n\right)\right)\right) + \frac{27n^3}{(n+4)(n+3)(n+2)(n+1)(n-1)^3} y_n^{\,3} = 0, \tag{2.72}$$

for $n \geq 2$. All assumptions of Theorem 2.8 are satisfied. Therefore, the above equation has solution (y_n) such that $\lim_{n \to \infty} y_n = \alpha \neq 0$. In fact, $y_n = 1 - 1/n$ is such a solution.

In conclusion, note that Theorems 2.6 and 2.8 can be easily extended to equations of the form

$$\Delta(a_n\Delta(b_n\Delta(c_n\Delta y_n))) + f(n, y_{\delta_n}) = 0, \quad n \in \mathbb{N}, \tag{2.73}$$

where δ is an integer-valued function defined on $\mathbb{N}$ such that $\lim_{n\to\infty}\delta_n = \infty$.

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