MAXIMUM PRINCIPLES FOR A FAMILY OF NONLOCAL BOUNDARY VALUE PROBLEMS

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We study a family of three-point nonlocal boundary value problems (BVPs) for an nth-order linear forward difference equation. In particular, we obtain a maximum principle and determine sign properties of a corresponding Green function. Of interest, we show that the methods used for two-point disconjugacy or right-disfocality results apply to this family of three-point BVPs.

1. Introduction

The disconjugacy theory for forward difference equations was developed by Hartman [15] in a landmark paper which has generated so much activity in the study of difference equations. Sturm theory for a second-order finite difference equation goes back to Fort [12], which also serves as an excellent reference for the calculus of finite differences. Hartman considers the nth-order linear finite difference equation

$$Pu(m) = \sum_{j=0}^{n} \alpha_j(m)u(m+j) = 0, \tag{1.1}$$

$\alpha_n\alpha_0 \neq 0$, $m \in I = \{a, a+1, a+2, \ldots\}$. To illustrate the analogy of (1.1) to an nth-order ordinary differential equation, introduce the finite difference operator Δ by

$$\Delta u(m) = u(m+1) - u(m), \qquad \Delta^0 u(m) \equiv u(m),$$
$$\Delta^{i+1} u(m) = \Delta(\Delta^i u)(m), \quad i \geq 1. \tag{1.2}$$

Clearly, P can be algebraically expressed as an nth-order finite difference operator.

Let m_1, b denote two positive integers such that $n - 2 \leq m_1 < b$. In this paper, we assume that $a = 0$ for simplicity, and we consider a family of three-point boundary conditions of the form

$$u(0) = 0, \ldots, u(n-2) = 0, \qquad u(m_1) = u(b). \tag{1.3}$$

Clearly, the boundary conditions (1.3) are equivalent to the boundary conditions

$$\Delta^i u(0) = 0, \quad i = 0, \dots, n-2, \qquad u(m_1) = u(b). \tag{1.4}$$

There is a current flurry to study nonlocal boundary conditions of the type described by (1.3). In certain sectors of the literature, such boundary conditions are referred to as multipoint boundary conditions. Study was initiated by Il'in and Moiseev [16, 17]. These initial works were motivated by earlier work on nonlocal linear elliptic boundary value problems (BVPs) (see, e.g., [3, 4]). Gupta and coauthors have worked extensively on such problems; see, for example, [13, 14]. Lomtatidze [18] has produced early significant work. We point out that Bobisud [5] has recently developed a nontrivial application of such problems to heat transfer. For the rest of the paper, we will use the term nonlocal boundary conditions, initiated by Il'in and Moiseev [16, 17].

We motivate this paper by first considering the equation

$$Pu(m) = \Delta^n u(m) = 0, \quad m = 0, \dots, b-n. \tag{1.5}$$

In this preliminary discussion, we employ the natural family of polynomials, $m^{(k)} = m(m-1)\cdots(m-k+1)$ so that $\Delta m^{(k)} = km^{(k-1)}$.

A Green function, $G(m_1, m, s)$ for the BVP (1.5), (1.3) exists for $(m_1, m, s) \in \{n-2, \dots, b-1\} \times \{0, \dots, b\} \times \{0, \dots, b-n\}$. It can be constructed directly and has the form

$$G(m_1; m, s) = \begin{cases} \dfrac{a(m_1; s) m^{(n-1)}}{(n-1)!}, & 0 \le m \le s \le b-n, \\[2mm] \dfrac{a(m_1; s) m^{(n-1)} + (m - (s+1))^{(n-1)}}{(n-1)!}, & 0 \le s+1 \le m \le b, \end{cases} \tag{1.6}$$

where

$$a(m_1; s) = -\frac{(b - (s+1))^{(n-1)}}{b^{(n-1)} - m_1^{(n-1)}}, \quad m_1 \le s,$$

$$a(m_1; s) = -\frac{(b - (s+1))^{(n-1)} - (m_1 - (s+1))^{(n-1)}}{b^{(n-1)} - m_1^{(n-1)}}, \quad s+1 \le m_1. \tag{1.7}$$

Associated with the BVP (1.5), (1.3) are two extreme cases. At $m_1 = n - 2$, we have the boundary conditions

$$u(0) = 0, \dots, u(n-2) = 0, \quad u(n-2) = u(b), \tag{1.8}$$

which are equivalent to the two-point conjugate conditions [15]

$$u(0) = 0, \dots, u(n-2) = 0, \qquad u(b) = 0. \tag{1.9}$$

At $m_1 = b - 1$, we have the boundary conditions

$$u(0) = 0, \dots, u(n-2) = 0, \qquad u(b-1) = u(b), \tag{1.10}$$

which are equivalent to the two-point "in between conditions" [9]

$$u(0) = 0,\ldots,u(n-2) = 0, \qquad \Delta u(b-1) = 0. \tag{1.11}$$

The following inequalities have been previously obtained [10, 15]:

$$0 > G(n-2;m,s) > G(b-1;m,s), \tag{1.12}$$

$(m,s) \in \{n-1,\ldots,b\} \times \{0,\ldots,b-n\}$.
   The following theorem is obtained directly from the representation (1.6) of $G(m_1; m,s)$.

THEOREM 1.1. $G(m_1;m,s)$ *is decreasing as a function of* m_1; *that is,*

$$0 \geq G(m_1;m,s) > G(m_1+1;m,s), \tag{1.13}$$

$(m_1,m,s) \in \{n-2,\ldots,b-2\} \times \{n-1,\ldots,b\} \times \{0,\ldots,b-n\}$. *The first inequality is strict, except in the conjugate case,* $m_1 = n-2$, *at* $m = b$.

   The purpose of this paper is to obtain Theorem 1.1 for a more general finite difference equation, $Pu(m) = 0$. Note that even for the specific BVP (1.5), (1.3), the calculations to show that G is decreasing in m_1 are tedious. The method exhibited in the next section allows one to bypass the tedious calculations. We will need to assume a condition that implies disconjugacy. We will then argue that similar results are obtained if the nonlocal boundary condition has the form

$$\Delta^j u(m_1) = \Delta^j u(b-j), \quad j \in \{0,\ldots,n-1\}. \tag{1.14}$$

The similar results will be valid if we assume that $Pu(m) = 0$ is right-disfocal [2].

2. A general disconjugate equation

Hartman [15] defined the disconjugacy of (1.1) on $I = \{0,\ldots,b\}$. First recall the definition of a generalized zero [15]. $m = 0$ is a generalized zero of u if $u(0) = 0$. $m > 0$ is a generalized zero of u if $u(m) = 0$, or there exists an integer $k \geq 1$ such that $m - k \geq 0$, $u(m-k+1) = \cdots = u(m-1) = 0$, and $(-1)^k u(m-k)u(m) > 0$. Then (1.1) is disconjugate on I if u is a solution of (1.1) on I and that u has at least n generalized zeros on I implies that $u \equiv 0$ on I. A condition related to and stronger than disconjugacy is that of right-disfocality [1, 8]; (1.1) is right-disfocal on I if u is a solution of (1.1) on I and that $\Delta^j u$ has a generalized zero at s_j, $0 \leq s_0 \leq s_1 \leq \cdots \leq s_{n-1} \leq b-n+1$, implies that $u \equiv 0$ on I. For this particular paper, a concept of right $(n-1;j)$ disfocality would be appropriate; (1.1) is right $(n-1;j)$ disfocal on I if u is a solution of (1.1) on I and that u has at least $n-1$ generalized zeros at $s_0,\ldots,s_{n-2}$, $\Delta^j u$ has a generalized zero at s_{n-1}, $\max\{s_0,\ldots,s_{n-2}\} \leq s_{n-1} \leq b-j$, imply that $u \equiv 0$ on I.
   Hartman [15] showed the equivalence of disconjugacy and a Frobenius factorization in the discrete case; in particular, $Pu = 0$ is disconjugate on $\{0,\ldots,b\}$ if and only if there

exist positive functions v_i defined on $\{0,\ldots,b-i+1\}$ such that

$$Pu(m) = \left(\frac{1}{v_{n+1}}\right)\Delta\left(\left(\frac{1}{v_n}\right)\Delta\left(\cdots\Delta\left(\left(\frac{1}{v_2}\right)\Delta\left(\frac{u}{v_1}\right)\right)\cdots\right)\right)(m), \tag{2.1}$$

$m \in \{0,\ldots,b-n\}$. Define quasidifferences

$$P_0u(m) = \left(\frac{u}{v_1}\right)(m),$$

$$P_ju(m) = \left(\frac{1}{v_{j+1}}\right)\Delta\left(\left(\frac{1}{v_j}\right)\Delta\left(\cdots\Delta\left(\left(\frac{1}{v_2}\right)\Delta\left(\frac{u}{v_1}\right)\right)\cdots\right)\right)(m) \tag{2.2}$$

$$= \left(\frac{1}{v_{j+1}}\right)\Delta(P_{j-1}u)(m),$$

$m \in \{0,\ldots,b-j\}$, $j = 0,\ldots,n$. We will now consider a family of nonlocal boundary conditions of the form

$$P_ju(0) = 0, \quad j = 0,\ldots,n-2, \qquad P_0u(m_1) = P_0u(b). \tag{2.3}$$

We will remind the reader of a version of Rolle's theorem.

LEMMA 2.1. *Let u be a sequence of reals defined on a set of integers. If P_ju has generalized zeros at μ_1 and μ_2, where $\mu_1 < \mu_2$, then $P_{j+1}u$ has a generalized zero in $\{\mu_1,\ldots,\mu_2-1\}$.*

Proof. Hartman [15] proved that $v_{j+2}P_{j+1}u$ has a generalized zero in the set $\{\mu_1,\ldots,\mu_2-1\}$. The lemma follows since v_{j+2} is positive. $\square$

THEOREM 2.2. *Assume that $Pu = 0$ is right $(n-1;1)$ disfocal on $\{0,\ldots,b\}$. Then there exists a uniquely determined Green function $G(m_1;m,s)$ for the BVPs (1.1), (2.3).*

Proof. Let v denote the solution of the initial value problem (IVP) (1.1), satisfying initial conditions

$$P_jv(0) = 0, \quad j = 0,\ldots,n-2, \qquad P_{n-1}v(0) = 1. \tag{2.4}$$

Let $\chi(m,s)$ denote the Cauchy function for (1.1); that is, χ, as a function of m, is the solution of the IVP (1.1), with the initial conditions

$$\chi(s+1+j,s) = 0, \quad j = 0,\ldots,n-2, \qquad \chi(s+1+n-1,s) = 1. \tag{2.5}$$

Set

$$G(m_1;m,s) = \begin{cases} a(m_1;s)v(m), & 0 \le m \le s \le b-n, \\ a(m_1;s)v(m) + \chi(m,s), & 0 \le s+1 \le m \le b. \end{cases} \tag{2.6}$$

Force G to satisfy the nonlocal condition $P_0(m_1)u(m_1) = P_0(b)u(b)$; in particular, solve algebraically for $a(m_1;s)$ to obtain

$$
\begin{aligned}
a(m_1;s) &= \frac{-P_0\chi(b,s)}{P_0v(b) - P_0v(m_1)}, \quad m_1 \le s, \\
a(m_1;s) &= \frac{P_0\chi(m_1,s) - P_0\chi(b,s)}{P_0v(b) - P_0v(m_1)}, \quad s+1 \le m_1.
\end{aligned}
\tag{2.7}
$$

Note that the right $(n-1;1)$ disfocality implies that $P_0v(b) - P_0v(m_1)$ is nonzero; in particular, $a(m_1;s)$ is well defined. Straightforward calculations show that

$$
\sum_{s=0}^{b-n} G(m_1;m,s)f(s)
\tag{2.8}
$$

is the unique solution of a nonhomogeneous BVP of the form $Pu(m) = f(m)$, $m \in \{0,\dots,b-n\}$, satisfying the boundary conditions (2.3).   □

THEOREM 2.3. *Assume that $Pu = 0$ is right $(n-1;1)$ disfocal on $\{0,\dots,b\}$. Then*

$$
G(m_1;m,s) \le 0,
\tag{2.9}
$$

$(m_1,m,s) \in \{n-2,\dots,b-1\} \times \{n-1,\dots,b\} \times \{0,\dots,b-n\}$. *The inequality is strict, except in the conjugate case, $m_1 = n-2$, at $m = b$.*

Remark 2.4. We consider a specific set of nonlocal boundary conditions in this paper to illustrate that theory and methods from disconjugacy theory apply to families of nonlocal BVPs. Because of the specific nonlocal boundary conditions, it is the case that P_1u has a generalized zero in $\{m_1,\dots,b-1\}$. Hence, the argument we produce below is precisely the general argument for the conjugate boundary conditions given in [6, Section 8.8], after Rolle's theorem has been applied one time.

Proof. It is known that (2.9) is valid in the extreme cases, $m_1 = n-2$ [15] and $m_1 = b-1$ [10]. Let $m_1 \in \{n-1,\dots,b-2\}$ be fixed. We first show that G is of fixed sign for

$$
(m,s) \in \{n-1,\dots,b\} \times \{0,\dots,b-n\}.
\tag{2.10}
$$

Let $s \in \{0,\dots,b-n\}$ be fixed. By construction, G, as a function of m, satisfies the boundary conditions (2.3).

Assume for the sake of contradiction that G has an additional generalized zero at m_0 for some $m_0 \in \{n-1,\dots,b\}$. Then P_0G takes on an additional generalized zero at m_0 since v_1 is of strict sign. Perform a count on the number of generalized zeros of each P_jG. (Since m_1 and s are fixed, P_jG is a function of m. We suppress the argument for simplicity of notation.)

Note, from the boundary conditions, that $P_1 G$ has a generalized zero at $n - 3$. The first point of the argument is to argue that $P_1 G$ has two more generalized zeros in $\{n - 2, \ldots, b - 1\}$. First assume that $m_0 \leq m_1$. Apply Rolle's theorem, Lemma 2.1. Then $P_1 G$ has two more generalized zeros, $m_{11} < m_{12}$, where $m_{11} \in \{n - 2, \ldots, m_0 - 1\}$ and $m_{12} \in \{m_1, \ldots, b - 1\}$. Second, assume that $m_1 < m_0$. With this assumption, $P_0 G(m_1) \neq 0$. Apply Rolle's theorem to see that $P_1 G$ has a generalized zero at $m_{11} \in \{n - 2, \ldots, m_0 - 1\}$. If $m_{11} < m_1$, then Rolle's theorem can be applied to the nonlocal conditions to obtain a second generalized zero $m_{12} \in \{m_1, \ldots, b - 1\}$. So, we come to the last subcase, $m_1 \leq m_{11}$. Assume without loss of generality that m_0 is the smallest generalized zero of $P_0 G$ to the right of m_1. Then $P_0 G(m_1) P_0 G(m_0) \leq 0$. This implies that $P_0 G(m_0) P_1 G(m_0 - 1) \geq 0$. These two inequalities imply that $P_1 G$ has a generalized zero in $\{m_0, \ldots, b - 1\}$. If not, then $\Delta P_0 G$ has a fixed sign on $\{m_0, \ldots, b - 1\}$, which agrees with the sign of $P_1 G$ at $m_0 - 1$. Recall the identity

$$P_0 G(b) = P_0 G(m_0) + \sum_{\mu = m_0}^{b-1} \Delta P_0 G(\mu). \tag{2.11}$$

In particular, $P_0 G(m_1)$ and $P_0 G(b)$ have opposite signs which contradicts the nonlocal boundary conditions. Thus, there exists $m \in \{m_0, \ldots, b - 1\}$ such that $P_1 G(m_0 - 1) P_1 G(m) \leq 0$. In particular, there exists $m_{12} \in \{m_0, \ldots, b - 1\}$ such that $P_1 G$ has a generalized zero at m_{12}.

To summarize the purpose of the preceding paragraphs, we have shown that $P_1 G$ has at least three generalized zeros on $\{n - 3, \ldots, b - 1\}$. It now easily follows by induction and repeated applications of Lemma 2.1 and the boundary conditions that for each $j = 0, \ldots, n - 2$ $P_j G$ has at least 3 generalized zeros, one at $n - (j + 2)$ and other two satisfying $n - (j + 1) \leq m_{j1} < m_{j2} \leq b - j$.

Since $P_{n-2} G$ has at least three generalized zeros, $P_{n-2} G$ has at least two generalized zeros counting multiplicities for $m \leq s$ or for $s + 1 \leq m$. Either case will provide a contradiction.

Assume that $P_{n-2} G$ has at least two generalized zeros counting multiplicities for $m \leq s$. Then $P_{n-1} G$ has at least one generalized zero for $m \leq s$. By construction, $P_n G \equiv 0$ for $t \leq s$; thus, $v_n P_{n-1} G$ is of constant sign and has a generalized zero; in particular, $P_{n-1} G \equiv 0$ for $m \leq s$. Continue inductively and argue that $P_j G \equiv 0$ for $m \leq s$. In particular, $G = v \equiv 0$ for $m \leq s$. This clearly contradicts the construction of v.

Assume that $P_{n-2} G$ has at least two generalized zeros counting multiplicities for $s + 1 \leq m$. Then a similar argument gives that $G = v + \chi \equiv 0$ for $s + 1 \leq m$. If $v = -\chi$, then the disconjugacy is violated.

Thus, G is of strict sign on $\{n - 2, \ldots, b - 1\} \times \{n - 1, \ldots, b\} \times \{0, \ldots, b - n\}$.

To determine the sign of G, evaluate the sign of

$$h(m) = \sum_{s=0}^{b-n} G(m, s), \tag{2.12}$$

which is the unique solution of the BVP

$$Pu = 1, \quad m \in \{0,\ldots,b-n\}, \tag{2.13}$$

with boundary conditions (2.3). $P_j h$ has a generalized zero at $n - (j+2)$ because of the boundary conditions. In addition, because of the nonlocal boundary conditions and repeated applications of Rolle's theorem, $P_j h$ has a generalized zero at $m_{j,1}$, where

$$n - (j+2) < m_{j,1} < m_{j-1,1} < b \tag{2.14}$$

for $j = 1,\ldots,n-2$. Moreover, due to Rolle's theorem, $P_{n-1}h$ has precisely one generalized zero since $P_n u \equiv 1$.

   $P_n u \equiv 1$ implies that $v_n P_{n-1} u$ is increasing. From the above construction, $v_n P_{n-1} u$ has precisely one generalized zero at $0 < m_{n-1,1}$. Hence, $v_n P_{n-1} u < 0$ on $\{0,\ldots,m_{n-1,1} - 1\}$. Continue inductively. Initially, $v_{n-1} P_{n-2} u$ is decreasing and $P_{n-2} u(0) = 0$; so $P_{n-2} u(1) < 0$. Inductively, it follows that $P_j u(n-1-j) < 0$, $j = 0,\ldots,n-2$. In particular, $u(n-1) < 0$; since G does not change sign, u does not change sign. Thus, u negative implies that (2.9) is valid. $\square$

THEOREM 2.5. *Assume that $Pu = 0$ is right $(n-1;1)$ disfocal on $\{0,\ldots,b\}$. Then G, as a function of m_1, is decreasing; that is, if $n - 2 \leq m_1 < m_2 \leq b - 1$, then*

$$G(m_2;m,s) < G(m_1;m,s) \leq 0, \tag{2.15}$$

$(m,s) \in \{n-1,\ldots,b\} \times \{0,\ldots,b-n\}$. *The second inequality is strict, except in the conjugate case, $m_1 = n - 2$, at $m = b$.*

   To prove the above comparison theorem, we first obtain a useful lemma. Let G_2 denote the quasidifference of G with respect to m; that is, let

$$G_2(m_1;m,s) = P_0 G(m_1;m+1,s) - P_0 G(m_1;m,s) = v_2 P_1 G. \tag{2.16}$$

LEMMA 2.6. *Let $m_1 \in \{n-2,\ldots,b-2\}$. Then*

$$G_2(m_1+1;m_1,s) < 0, \quad s \in \{0,\ldots,b-n\}. \tag{2.17}$$

Proof. The proof requires only a simple extension from the proof of Theorem 2.3. As summarized in the fourth paragraph of the proof of Theorem 2.3, we know that $P_1 G$ has precisely one generalized zero m_{11} to the right of $n - 3$. We also know by Rolle's theorem that $m_1 \leq m_{11}$.

   $G(n-2) = 0$, $G(n-1) < 0$ imply that $P_1 G(n-2) < 0$ which in turn implies that $P_1 G(m_1 - 1) < 0$. $\square$

Proof of Theorem 2.5. Let G_1 denote the difference of G with respect to m_1; that is, let

$$G_1(m_1;m,s) = G(m_1+1;m,s) - G(m_1;m,s). \tag{2.18}$$

In particular, we assume that $m_1 < b - 1$.

Note that G_1 is the unique solution of the BVP

$$
\begin{aligned}
Pu &= 0, \quad m \in \{0,\dots,b-n\}, \\
P_j u(0) &= 0, \quad j = 0,\dots,n-2, \\
P_0 u(b) - P_0 u(m_1) &= G_2(m_1+1;m_1,s) < 0.
\end{aligned}
\tag{2.19}
$$

G_1 satisfies the difference equation and each of the initial conditions at 0; this is clear since each term of $G(m_1+1;m,s)$, $G(m_1;m,s)$ satisfies the difference equation and the initial conditions. Simply calculate the nonlocal boundary condition

$$
\begin{aligned}
P_0\big(G(m_1+1;b,s) &- G(m_1;b,s)\big) - P_0\big(G(m_1+1;m_1,s) - G(m_1;m_1,s)\big) \\
&= \big(P_0 G(m_1+1;b,s) - P_0 G(m_1+1;m_1+1,s)\big) \\
&\quad + \big(P_0 G(m_1+1;m_1+1,s) - P_0 G(m_1+1;m_1,s)\big) \\
&= G_2(m_1+1;m_1,s).
\end{aligned}
\tag{2.20}
$$

In particular,

$$
P_0 u(b) < P_0 u(m_1).
\tag{2.21}
$$

The boundary conditions at 0 and the right $(n-1;1)$ disfocality imply that $P_0 u$ is monotone for $m > n-2$. $P_0 u(b) < P_0 u(m_1)$ implies that $P_0 u$ is monotone-decreasing and (2.15) is proved. $\qquad\square$

We end the paper with a brief general observation. Let $l \in \{0,\dots,n-2\}$. Let $m_1 \in \{n-2,\dots,b-l-1\}$. Consider the BVP

$$
Pu(m) = 0, \quad m \in \{0,\dots,b-n\},
\tag{2.22}
$$

with boundary conditions

$$
P_j u(0) = 0, \quad j = 0,\dots,n-2, \qquad P_l u(m_1) = P_l u(b-l).
\tag{2.23}
$$

We state without proof theorems analogous to Theorems 2.3 and 2.5. The observation to make now is that the BVP at $l = 0$, $m_1 = b-1$ is equivalent to the BVP with $l = 1$, $m_1 = n-3$. One can now begin an inductive argument on l and repeat the arguments in the paper.

A Green function $G(l,m_1;m,s)$ for the BVP (1.1), (2.23) is readily constructed as in the proof of Theorem 2.2. So, from the above observation, we claim that

$$
G(0,b-1;m,s) = G(1,n-3;m,s).
\tag{2.24}
$$

Define the jth difference of G with respect to m by $\Delta_j G$. The proof of Theorem 2.3 generalizes readily to show that

$$
\Delta_l G(l,m_1;m,s) \le 0,
\tag{2.25}
$$

$(m_1, m, s) \in \{n - l - 2, \ldots, b - l - 1\} \times \{n - l - 1, \ldots, b - l\} \times \{0, \ldots, b - n\}$. We do not present the proofs because the arguments, applied to $\Delta_l G(l, m_1; m, s)$, go through in complete analogy; the inequalities for the lower-order differences are then obtained through repeated definite summations from $m = 0$, which are valid because of the boundary conditions.

THEOREM 2.7. *Assume that* $Pu = 0$ *is right* $(n - 1; l)$ *disfocal on* $\{0, \ldots, b\}$. *Then*

$$\Delta_j G(l, m_1; m, s) \le 0, \tag{2.26}$$

$(m_1, m, s) \in \{n - 2, \ldots, b - l - 1\} \times \{n - j - 1, \ldots, b - j\} \times \{0, \ldots, b - n\}$, $j = 0, \ldots, l$. *The inequality is strict, except in the case* $j = l$, $m_1 = n - l - 2$, *at* $m = b - l$.

THEOREM 2.8. *Assume that* $Pu = 0$ *is right* $(n - 1; l)$ *disfocal on* $\{0, \ldots, b\}$. *Then* $\Delta_j G$, *as a function of* m_1, *is decreasing; that is, if* $n - 2 \le m_1 < m_2 \le b - l - 1$, *then*

$$\Delta_j G(l, m_2; m, s) < \Delta_j G(l, m_1; m, s) \le 0, \tag{2.27}$$

$(m, s) \in \{n - j - 1, \ldots, b - j\} \times \{0, \ldots, b - n\}$, $j = 0, \ldots, l$. *The second inequality is strict, except in the case* $j = l$, $m_1 = n - l - 2$, *at* $m = b - l$.

Finally, in the spirit of the interesting comparison theorems first introduced by Elias [7] (see also [19] or [11]) and later discretized [10], we close with the following comparison theorem.

THEOREM 2.9. *Assume that* $Pu = 0$ *is right-disfocal on* $\{0, \ldots, b\}$. *Let* $l_1 < l_2$. *Then*

$$\Delta_j G(l_2, m_{l_2}; m, s) < \Delta_j G(l_1, m_{l_1}; m, s) \le 0, \tag{2.28}$$

$(m, s) \in \{n - j - 1, \ldots, b - j\} \times \{0, \ldots, b - n\}$, $j = 0, \ldots, l_1$. *The inequality is strict, except in the case* $j = l_1$, $m_1 = n - l_1 - 2$, *at* $m = b - l_1$. *If* $l_1 = l_2$ *and* $m_1 < m_2$, *then* $\Delta_j G(l, m_2; m, s) < \Delta_j G(l, m_1; m, s) \le 0$, $(m, s) \in \{n - j - 1, \ldots, b - j\} \times \{0, \ldots, b - n\}$, $j = 0, \ldots, l_1 = l_2$. *The inequality is strict, except in the case,* $j = l_1$, $m_1 = n - l_1 - 2$, *at* $m = b - l_1$.

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