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Research Article

Robust Impulsive Synchronization of Discrete Dynamical Networks

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We aim to study robust impulsive synchronization problem for uncertain discrete dynamical networks. For the discrete dynamical networks with unknown but bounded network coupling, we will design some robust impulsive controllers which ensure that the state of a discrete dynamical network asymptotically synchronize with an arbitrarily assigned state of an isolate node of the network. Three representative examples are also worked through to illustrate our results.

1. Introduction

Since the 1990s, synchronization of chaotic systems has been a current and active research area. Numerous methods have been developed for chaos synchronization (see, e.g., [1–9]). More recently, synchronization of dynamical networks has been reported in the literature (see, e.g., [10–14]). The dynamical networks consist of coupled nodes, which are usually chaotic systems. It has been noticed that when synchronization is applied to the dynamical networks, the network coupling may cause the failure of a synchronization scheme. The network coupling functions may be unknown a priori and may be in form of linear or nonlinear functions. In order to deal with this problem, the robust synchronization for uncertain dynamical networks has become an important research topic. Although robust adaptive synchronization scheme can be used to synchronize nodes of the uncertain dynamical networks where the network coupling is an unknown but bounded nonlinear function (see, e.g., [14]), yet the controller for adaptive synchronization is usually complex. It has been proved in the study of chaotic synchronization that impulsive synchronization approach is effective and robust in synchronization of chaotic systems (see, e.g., [7, 8]), and has a relatively simple structure. Moreover, since the controller of impulsive synchronization is discontinuous, impulsive synchronization can be useful for digital secure communization systems [9]. But up to present, to the best knowledge of the authors, there are not any results about impulsive synchronization of discrete dynamical network.

In this paper, we aim to study the robust impulsive synchronization problem for an uncertain discrete dynamical network. By utilizing the ideas developed in [15, 16] for impulsive systems [15–23], we will derive several criteria under which robust impulsive synchronization is achieved for an uncertain discrete dynamical network, with the network coupling functions being unknown but bounded. It will be shown that impulsive synchronization approach of a dynamical network has the same good properties as those in impulsive synchronization of chaotic systems. Moreover, the impulsive controller is also easy to design.

The main contribution of this paper is a proposed new control approach, that is, impulsive control, for discrete dynamical network or general discrete system $x(k+1) = f(k, x(k)) + g(k, x(k))$. For the classical feedback control, $u(k)$ is in form of $u(k) = K(k, x(k))$. In this classical control scheme, the control signal is input into the system at all the time $k \in \mathbb{N}$. However, for some practical systems, it is not necessary and in some case is also impossible to input control signal into the system at all the time. In this paper, the classical control $u(k) = K(k, x(k))$ is replaced by the proposed impulsive control $u(k) = \sum_{m=1}^{\infty} \delta(k - t_m) I_m(k, x(k))$. Thus, the control signal is put into the discrete system just at the impulsive instances $\{t_k, k \in \mathbb{N}\}$, not at all the time series $\{k, k \in \mathbb{N}\}$, where function $\delta(t)$ satisfies

$$\delta(t) = \begin{cases} 1, & t = 0, \\ 0, & t \neq 0. \end{cases} \tag{1.1}$$

This kind of control scheme will be useful in control theory and applications. For example, it can be used for control and synthesis of the sampled-data control system, and so forth.

The organization of this paper is as follows. In Section 2, we introduce the concept of uniformly positive definite matrix function and some other notations. The robust impulsive synchronization scheme is also formulated for a dynamical network in Section 2. In Section 3, robust impulsive synchronization criteria are established. These criteria can be easily used for the design of a robust feedback controller. For illustration, some representative examples are given in Section 4. Section 5 concludes the paper.

2. Problem formulation

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space. Let $\mathbb{R}_+ = [0, +\infty)$, $\mathbb{N} = \{1, 2, \ldots\}$, and let $\|\cdot\|$ stand for the Euclidean norm in $\mathbb{R}^n$.

Consider a discrete dynamical network consisting of N identical nodes (n-dimensional discrete systems) with uncertain network coupling:

$$x_i(k+1) = A(k)x_i(k) + \varphi\big(k, x_i(k)\big) + g_i\big(x_1(k), x_2(k), \ldots, x_N(k)\big), \quad n \in \mathbb{N}, \ i = 1, 2, \ldots, N, \tag{2.1}$$

where $A(k) \in \mathbb{R}^{n\times n}$, $k \in \mathbb{N}$, $\varphi : \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}^n$ is smooth nonlinear vector-valued function, and $g_i : \mathbb{R}^m \to \mathbb{R}^n$ are smooth but unknown network coupling functions, where $m = nN$.

Clearly, the isolated node of the network is in form of

$$y(k+1) = A(k)y(k) + \varphi\big(k, y(k)\big), \quad k \in \mathbb{N}. \tag{2.2}$$

It is assumed that the solution of (2.2) exists and is unique under any given initial condition $y(0) = y_0$.

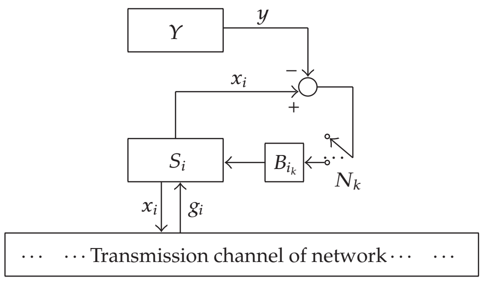

Figure 1: The impulsive synchronization control for the ith node S_i.

Remark 2.1. When the network achieves synchronization, namely, the state $x_1(k) = x_2(k) = \cdots = x_N(k) = y(k)$, as $k \to \infty$, the coupling terms should vanish: $g_i(y, y, \ldots, y) = 0$.

The robust impulsive synchronization scheme for the discrete network (2.1) is to design impulsive controllers $\{N_k, B_{i_k}\}$ such that the state of the following system (2.3) synchronizes with the state of (2.2):

$$\begin{aligned} x_i(n+1) &= A(n)x_i(n) + \varphi(n, x_i(n)) + g_i(x_1(n), x_2(n), \ldots, x_N(n)), \quad n \neq N_k, \\ \triangle x_i(n+1) &= B_{i_k}(x_i(n) - y(n)), \quad n = N_k,\ k \in \mathbb{N},\ i = 1, 2, \ldots, N, \end{aligned} \tag{2.3}$$

where $\Delta x_i(N_k + 1) = x_i(N_k + 1) - x_i(N_k)$. The sequence $\{N_k\}$ satisfies

(i) $0 = N_0 < N_1 < N_2 < \cdots$, with $\lim_{k\to\infty} N_k = \infty$;

(ii) for all $k \in \mathbb{N}$, $N_{k+1} - N_k \geq 2$.

Figure 1 depicts the entire impulsive synchronization scheme subject to network coupling, where S_i stands for ith node, Y is the isolated node (2.2), and g_i is the uncertain network coupling of ith node, $i = 1, 2, \ldots, N$.

Remark 2.2. It should be noticed that the mathematical modeling of this paper is basically the discrete impulsive systems, in which the impulses occur in a discrete system at some instances. But they are different from the discrete systems with inputs $u(n)$, in which the input signals $u(n)$ are input into system at every instance $n = 1, 2, \ldots$. In this impulsive control discrete system (2.3), the input signals are input into system only at some instances N_k, $k = 1, 2, \ldots$.

Remark 2.3. The synchronization scheme given by (2.1)–(2.3) is some similar to the one used in [16, 24] for impulsive synchronization of continuous dynamical networks, but it is different from that in [16, 24] and it is more significant than that in [16, 24] because of the following reasons.

(i) In the practical networks, the signals, which are used to transmit, receive, and sample, are often in form of discrete signals, not continuous forms. Hence, it is more practically significant to study the synchronization problem of discrete networks than that for continuous networks.

(ii) The mathematical modeling is also different from that in [16, 24]. Here, we use the impulsive difference equation (discrete impulsive system) to depict the impulsive synchronization scheme, while in [16, 24], the impulsive differential equation is used. Although significant progress has been made in the stability theory of impulsive differential equations, the corresponding theory for discrete impulsive systems has not been fully developed; see [25]. It is a new research topic. Hence, the work in this paper is not a trivial extension of the previous work in [16, 24].

Defining the synchronization error as $e_i(n) = x_i(n) - y(n)$, then one has an error dynamical system of the form

$$\begin{aligned} e_i(n+1) &= A(n)e_i(n) + \widetilde{\varphi}(n, x_i(n), y(n)) + \widetilde{g}_i(x(n), y(n)), \quad n \neq N_k, \\ \triangle e_i(n+1) &= B_{i_k} e_i(n), \quad n = N_k, \ k \in \mathbb{N}, \ i = 1, 2, \ldots, N, \end{aligned} \tag{2.4}$$

where $\widetilde{\varphi}(t, x_i, y) = \varphi(t, x_i) - \varphi(t, y)$, $\widetilde{g}_i(x, y) = g_i(x_1, x_2, \ldots, x_N) - g_i(y, y, \ldots, y)$, and $B_{i_k} \in \mathbb{R}^{n\times n}$.

Clearly, the network (2.1) synchronizes robustly with system (2.2) by impulsive controllers $\{N_k, B_{i_k}\}$ if and only if the error system (2.4) is robustly asymptotically stable.

Assumption 2.4. There exist positive constants $r_{ij} > 0$, $i, j = 1, 2, \ldots, N$, such that

$$\|g_i(x_1, x_2, \ldots, x_N)\| \leq \sum_{j=1}^{N} r_{ij} \|e_j\|, \quad i = 1, 2, \ldots, N. \tag{2.5}$$

Assumption 2.5. Assume that there exists an attractive domain $\mathbb{U} \subseteq \mathbb{R}^n$ for the isolated node (2.2) and for any $x_i, y \in \mathbb{U}$, there exist positive constants $L_{ik} > 0$ such that for $n \in (N_k, N_{k+1}]$,

$$\|\varphi(n, x_i) - \varphi(n, y)\| \leq L_{ik} \|x_i - y\|, \quad i = 1, 2, \ldots, N, \ k \in \mathbb{N}. \tag{2.6}$$

Remark 2.6. (i) Assumption 2.4 is based on $g_i(y, y, \ldots, y) = 0$, for $i = 1, 2, \ldots, N$, and any $y \in \mathbb{R}^n$. Also, Assumption 2.5 is based on the fact that the chaotic system is ultimate bounded.

(ii) In recent published paper [25], by using interval matrix decomposition method and comparing method (for detail, see [25]), the robust stability is investigated for interval linear discrete impulsive systems and a class of affine discrete impulsive systems. In this paper, by employing Lyapunov function approach, we focus on the stability of error system (2.4), which is a large-scale discrete impulsive system. Based on the stability results of (2.4), the impulsive synchronization can be achieved on the isolated node's attraction domain. Hence, the stability issue studied in this paper is different from that in [25].

Definition 2.7. Let $X : \mathbb{N} \to \mathbb{R}^{n\times n}$ be an $n \times n$ matrix function. Then, $X(k)$ is said to be

(i) a positive definite matrix function if for any $k \in \mathbb{N}$, $X(k)$ is a positive definite matrix;

(ii) a positive definite matrix function bounded from above if it is a positive definite matrix function and there exists a positive real number $M > 0$ such that

$$\lambda_{\max}(X(k)) \leq M, \quad k \in \mathbb{N}, \tag{2.7}$$

where $\lambda_{\max}(\cdot)$ is the maximum eigenvalue;

(iii) a uniformly positive definite matrix function if it is a positive definite matrix function and there exists a positive real number $m > 0$ such that

$$\lambda_{\min}(X(k)) \geq m, \quad k \in \mathbb{N}, \tag{2.8}$$

where $\lambda_{\min}(\cdot)$ is the minimum eigenvalue of matrix $(\cdot)$.

Lemma 2.8 (see [15]). *Let $X(k) \in \mathbb{R}^{n\times n}$ be a positive definite matrix function and $Y(k) \in \mathbb{R}^{n\times n}$ a symmetric matrix. Then, for any $x \in \mathbb{R}^n, k \in \mathbb{N}$, the following inequality holds:*

$$x^T Y(k)x \le \lambda_{\max}\left(X(k)^{-1}Y(k)\right)\cdot x^T X(k)x. \tag{2.9}$$

Proof. It follows from the properties of positive definite matrix. □

3. Robustly impulsive synchronization

In this section, we will derive the asymptotical stability criteria for the error system (2.4) such that the state of the discrete dynamical network synchronizes with an arbitrarily assigned state of an isolated node of the network by the robust impulsive controllers.

Theorem 3.1. *Suppose that Assumptions 2.4 and 2.5 hold, and assume that there exist uniformly positive definite matrix functions which are bounded from above, $P_i(n)$, $i = 1, 2, \dots, N$, and constants $\epsilon > 0$, $\gamma_i \ge 0$, $\alpha_{ik}(n) \ge 0$, where $n \in (N_k, N_{k+1})$ and $\beta_{ik}(N_k) \ge 0$, $i, k \in \mathbb{N}$, such that*

(i) *for all $n \in (N_k, N_{k+1}]$, $k \in \mathbb{N}$, the following inequalities hold:*

$$\begin{aligned}
A^T(n)P_i(n+1)A(n)&+2L_{i_k}\sqrt{\lambda_{\max}\left(P_i^{-1}(n+1)A^T(n)P_i(n+1)A(n)\right)}\cdot\sqrt{\frac{\lambda_{\max}\left(P_i(n+1)\right)}{\lambda_{\min}\left(P_i(n+1)\right)}}P_i(n+1)\\
&+\lambda_{\max}\left(P_i(n+1)\right)\left[(1+\nu_i)L_{i_k}^2+\left(1+\nu_i^{-1}\right)\sum_{j=1}^{N}r_{ij}^2\right]I\\
&+\sqrt{\lambda_{\max}\left(P_i^{-1}(n+1)A^T(n)P_i(n+1)A(n)\right)}\sum_{j=1}^{N}\left(\epsilon r_{ij}+\epsilon^{-1}r_{ji}\right)I\\
&\le \alpha_{ik}(n)P_i(n);
\end{aligned} \tag{3.1}$$

(ii) *for all $n = N_k$, $k \in \mathbb{N}$,*

$$\lambda_{\max}\left[P_i^{-1}(n+1)\left(I+B_{i_k}\right)^T P_i(n+1)\left(I+B_{i_k}\right)\right]P_i(n+1) \le \beta_{ik}(n)P_i(n); \tag{3.2}$$

(iii)

$$\sum_{j=0}^{\infty}\ln\gamma_j = -\infty, \tag{3.3}$$

where

$$\gamma_j = \begin{cases}\sqrt{\overline{\alpha}_k(j)}, & \text{if } j \in (N_k, N_{k+1}),\\ \sqrt{\overline{\beta}_k(j)}, & \text{if } j = N_k,\ k \in \mathbb{N},\end{cases} \tag{3.4}$$

and $\gamma_0 = 1$, $\overline{\alpha}_k(n) = \max_{1\le i\le N}\{\alpha_{ik}(n)\}$, $\overline{\beta}_k(N_k) = \max_{1\le i\le N}\{\beta_{ik}(N_k)\}$.

Then, for any initial conditions $x_i(0) = x_{i0}$, $y(0) = y_0 \in \mathbb{U}$, the uncertain discrete dynamical network (2.1) *is robust impulsive synchronization with system* (2.2) *by the impulsive controllers $\{N_k, B_{i_k}\}$.*

Proof. Let $V(n) = V(n, e_1, e_2, \dots, e_N) = \sum_{i=1}^{N} e_i^T P_i(n) e_i$. Denote $V_i(n) = e_i^T P_i(n) e_i,\ i = 1, 2, \dots, N$.

Since $P_i(n)$, $i = 1, 2, \dots, N$, are all uniformly positive definite matrix functions and bounded from above, there exist positive constants $a > 0$, $b > 0$ such that the following inequality holds:

$$a\sum_{i=1}^{N} e_i^T e_i \le N \min_{1\le i\le N}\{\lambda_{\min}(P_i)\}\sum_{i=1}^{N} e_i^T e_i \le V \le N \max_{1\le i\le N}\{\lambda_{\max}(P_i)\}\sum_{i=1}^{N} e_i^T e_i \le b\sum_{i=1}^{N} e_i^T e_i. \tag{3.5}$$

For any $n \in (N_k, N_{k+1}),\ k \in \mathbb{N}$, we get

$$\begin{aligned}
V_i(n+1) &= e_i(n+1)^T P_i(n+1) e_i(n+1) \\
&= \left[A(n)e_i(n) + \widetilde{\varphi} + \widetilde{g}_i\right]^T P_i(n+1)\left[A(n)e_i(n) + \widetilde{\varphi} + \widetilde{g}_i\right] \\
&= e_i(n)^T A^T(n) P_i(n+1) A(n) e_i(n) + 2e_i(n)^T A^T(n) P_i(n+1)\widetilde{\varphi} + \widetilde{\varphi}^T P_i(n+1)\widetilde{\varphi} \\
&\quad + 2e_i(n)^T A^T(n) P_i(n+1)\widetilde{g}_i + 2\widetilde{\varphi}^T A^T(n) P_i(n+1)\widetilde{g}_i + \widetilde{g}_i^T P_i(n+1)\widetilde{g}_i.
\end{aligned} \tag{3.6}$$

By Lemma 2.8, the terms in (3.6) can be estimated as

$$\begin{aligned}
&e_i(n)^T A^T(n) P_i(n+1)\widetilde{\varphi} \\
&\quad \le \left\| e_i(n)^T A^T(n) P_i^{1/2}(n+1)\right\| \left\| P_i^{1/2}(n+1)\widetilde{\varphi}\right\| \\
&\quad = \sqrt{e_i(n)^T A^T(n) P_i(n+1) A(n) e_i(n)}\sqrt{\widetilde{\varphi}^T P_i(n+1)\widetilde{\varphi}} \\
&\quad \le L_{i_k}\sqrt{\lambda_{\max}\left(P_i^{-1}(n+1) A^T(n) P_i(n+1) A(n)\right)\cdot \lambda_{\max}\left(P_i(n+1)\right)}\cdot\sqrt{e_i(n)^T P_i(n+1) e_i(n)}\sqrt{e_i(n)^T e_i(n)} \\
&\quad \le L_{i_k}\sqrt{\lambda_{\max}\left(P_i^{-1}(n+1) A^T(n) P_i(n+1) A(n)\right)}\sqrt{\frac{\lambda_{\max}\left(P_i(n+1)\right)}{\lambda_{\min}\left(P_i(n+1)\right)}}\cdot e_i(n)^T P_i(n+1) e_i(n),
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
&e_i(n)^T A^T(n) P_i(n+1)\widetilde{g}_i \\
&\quad \le \left\| e_i(n)^T A^T(n) P_i(n+1)\right\| \left\|\widetilde{g}_i\right\| \\
&\quad \le \sqrt{e_i^T(n) A^T(n) P_i(n+1) A(n) e_i(n)}\cdot\sum_{j=1}^{N} r_{ij}\left\| e_j(n)\right\| \\
&\quad \le \sqrt{\lambda_{\max}\left(P_i^{-1}(n+1) A^T(n) P_i(n+1) A(n)\right)}\cdot\sum_{j=1}^{N} r_{ij}\left\| e_i(n)\right\| \left\| e_j(n)\right\| \\
&\quad \le \sqrt{\lambda_{\max}\left(P_i^{-1}(n+1) A^T(n) P_i(n+1) A(n)\right)}\cdot\sum_{j=1}^{N}\frac{r_{ij}}{2}\left(\epsilon e_i^T(n) e_i(n) + \epsilon^{-1} e_j^T(n) e_j(n)\right),
\end{aligned} \tag{3.8}$$

here, Young's inequality is used, $2ab \le \varepsilon a^2 + b^2/\varepsilon$, for any $\varepsilon > 0$,

$$
\begin{aligned}
\widetilde{\varphi}^T P_i(n+1)\widetilde{\varphi} &\le L_{i_k}^2 \lambda_{\max}\big(P_i(n+1)\big) e_i^T(n) e_i(n), \\
\widetilde{g}_i^T P_i(n+1)\widetilde{g}_i &\le \lambda_{\max}\big(P_i(n+1)\big)\widetilde{g}_i^T \widetilde{g}_i \\
&\le \lambda_{\max}\big(P_i(n+1)\big)\left(\sum_{j=1}^{N} r_{ij}\|e_j(n)\|\right)^2 \\
&= \lambda_{\max}\big(P_i(n+1)\big)|e|^T r_i^T r_i |e| \\
&\le \lambda_{\max}\big(P_i(n+1)\big)\lambda_{\max}(r_i^T r_i) e_i^T(n) e_i(n) \\
&= \lambda_{\max}\big(P_i(n+1)\big)\sum_{j=1}^{N} r_{ij}^2 e_i^T(n) e_i(n),
\end{aligned}
\tag{3.9}
$$

where $r_i = (r_{i1}, r_{i2}, \ldots, r_{iN})$ and $|e| = (\|e_1\|, \|e_2\|, \ldots, \|e_N\|)^T$, and

$$
\begin{aligned}
\widetilde{\varphi}^T P_i(n+1)\widetilde{g}_i &\le \left\|\widetilde{\varphi}^T P_i^{1/2}(n+1)\right\| \left\|P_i^{1/2}(n+1)\widetilde{g}_i\right\| \\
&\le \frac{\nu_i}{2}\widetilde{\varphi}^T P_i(n+1)\widetilde{\varphi} + \frac{\nu_i^{-1}}{2}\widetilde{g}_i^T P_i(n+1)\widetilde{g}_i.
\end{aligned}
\tag{3.10}
$$

Substituting (3.9) into (3.10) and substituting (3.7)–(3.10) into (3.6), we obtain that

$$
\begin{aligned}
V_i(n+1) \le e_i(n)^T \Bigg\{ & A^T(n)P_i(n+1)A(n) + 2L_{i_k}\sqrt{\lambda_{\max}\big(P_i^{-1}(n+1)A^T(n)P_i(n+1)A(n)\big)} \\
& \cdot \sqrt{\frac{\lambda_{\max}\big(P_i(n+1)\big)}{\lambda_{\min}\big(P_i(n+1)\big)}} P_i(n+1) + \epsilon\sqrt{\lambda_{\max}\big(P_i^{-1}(n+1)A^T(n)P_i(n+1)A(n)\big)}\sum_{j=1}^{N} r_{ij} I \\
& + \lambda_{\max}\big(P_i(n+1)\big)\left[(1+\nu_i)L_{i_k}^2 + (1+\nu_i^{-1})\lambda_{\max}(r_i^T r_i)\right] I \Bigg\} e_i(n) \\
& + \epsilon^{-1}\sqrt{\lambda_{\max}\big(P_i^{-1}(n+1)A^T(n)P_i(n+1)A(n)\big)}\sum_{j=1}^{N} r_{ij} e_j^T(n) e_j(n).
\end{aligned}
\tag{3.11}
$$

It follows from (3.1) that for all $n \in (N_k, N_{k+1})$,

$$
V(n+1) = \sum_{i=1}^{N} V_i(n+1) \le \sum_{i=1}^{N} \alpha_{ik}(n) e_i^T(n) P_i(n) e_i(n) \le \overline{\alpha}_k(n)\sum_{i=1}^{N} e_i^T(n) P_i(n) e_i(n) = \overline{\alpha}_k(n) V(n),
\tag{3.12}
$$

where for a fixed n, $\overline{\alpha}_k(n) = \max_{1\le i\le N}\{\alpha_{ik}(n)\}$.

When $n = N_k$, we get

$$\begin{aligned}
V_i(n+1) &= e_i(n+1)^T P_i(n+1) e_i(n+1) \\
&= \left[e_i(n) + B_{i_k} e_i(n)\right]^T P_i(n+1)\left[e_i(n) + B_{i_k} e_i(n)\right] \\
&= e_i(n)^T \left(I + B_{i_k}\right)^T P_i(n+1)\left(I + B_{i_k}\right) e_i(n) \\
&\leq \lambda_{\max}\left(P^{-1}\left(I + B_{i_k}\right)^T P_i(n+1)\left(I + B_{i_k}\right)\right) e_i(n)^T P_i(n+1) e_i(n) \\
&\leq \beta_{ik}(n) e_i(n)^T P_i(n) e_i(n),
\end{aligned} \tag{3.13}$$

which implies that for $n = N_k$,

$$V(n+1) = \sum_{i=1}^{N} V_i(n+1) \leq \sum_{i=1}^{N} \beta_{ik}(n) e_i(n)^T P_i(n) e_i(n) \leq \overline{\beta}_k(n) \sum_{i=1}^{N} e_i(n)^T P_i(n) e_i(n), \tag{3.14}$$

where $\overline{\beta}_k(n) = \max_{1\leq i\leq N}\{\beta_{ik}(n)\}$.

Hence, for all $k \in \mathbb{N}$,

$$V(N_k + 1) \leq \overline{\beta}_k(N_k) V(N_k). \tag{3.15}$$

Since

$$\gamma_j = \begin{cases} \sqrt{\overline{\alpha}_k(j)}, & \text{if } j \in (N_k, N_{k+1}), \\ \sqrt{\overline{\beta}_k(j)}, & \text{if } j = N_k,\ k \in \mathbb{N}, \end{cases} \tag{3.16}$$

and $\gamma_0 = 1$, then from (3.12)–(3.15), for any $n \in (N_k, N_{k+1}]$, we obtain that

$$V(n) \leq \left(\prod_{j=0}^{n-1} \gamma_j^2\right) V(0) = e^{2\sum_{j=0}^{n-1} \ln \gamma_j} V(0). \tag{3.17}$$

Denote $e(n) = (e_1^T(n), e_2^T(n), \ldots, e_N^T(n))^T$. By (3.5), we get

$$\|e(n)\| \leq \sqrt{\frac{b}{a}} e^{\sum_{j=0}^{n-1} \ln \gamma_j} \|e(0)\|, \quad n \in \mathbb{N}. \tag{3.18}$$

Hence, if $\sum_{j=0}^{\infty} \ln \gamma_j = -\infty$, then for any $e_i(0) \in \mathbb{R}^{n\times n}$, by (3.14), $\lim_{n\to\infty} \|e_i(n)\| = 0$. Thus, the error system (2.4) is asymptotically stable. Therefore, the uncertain dynamical network (2.1) is robust synchronization with system (2.2) by the impulsive controllers $\{N_k, B_{i_k}\}$. The proof is complete. □

Corollary 3.2. *Suppose that Assumptions 2.4 and 2.5 hold, and assume that there exist positive constants $\nu_i > 0$, $i \in \mathbb{N}$, such that the following condition is satisfied:*

$$\sum_{j=0}^{\infty} \ln \gamma_j = -\infty, \tag{3.19}$$

where

$$\gamma_j = \begin{cases} \sqrt{\alpha_{i_k}(j)}, & \text{if } j \in (N_k, N_{k+1}), \\ \|I + B_j\|, & \text{if } j = N_k,\ k \in \mathbb{N}, \end{cases}$$
$$\alpha_{i_k}(n) = \left(\|A(n)\| + L_{i_k}\right)^2 + \left(\sum_{j=1}^{N}(r_{ij} + r_{ji})\right)\|A(n)\| + \nu_i L_{i_k}^2 + \left(1 + \nu_i^{-1}\right)\sum_{j=1}^{N} r_{ij}^2. \tag{3.20}$$

Then, for any initial conditions $x_i(0) = x_{i0}$, $y(0) = y_0 \in \mathbb{U}$, the uncertain discrete dynamical network (2.1) *is robust impulsive synchronization with system* (2.2) *by the impulsive controllers $\{N_k, B_{i_k}\}$.*

Proof. By the similar proof of Theorem 3.1, with $P_i(n) = I$, $\epsilon = 1$, $i = 1, 2, \ldots, N$, we obtain that the result holds. The details are omitted here. □

Remark 3.3. (i) By Corollary 3.2, if there does not exist coupling in the network, that is, $r_{ij} = 0$, $i, j, = 1, 2, \ldots, N$, then the sufficient condition for the robust synchronization of the network simplifies to

$$\sum_{j=0}^{\infty} \ln \gamma_j = -\infty, \quad \text{where } \gamma_j = \begin{cases} \|A(j)\| + L_{j_k}, & \text{if } j \in (N_k, N_{k+1}), \\ \|I + B_j\|, & \text{if } j = N_k,\ k \in \mathbb{N}. \end{cases} \tag{3.21}$$

Hence, Corollary 3.2 is the generalization of the results established in [20].

(ii) If $\mathbb{U} = \mathbb{R}^n$, then the error system (2.4) is globally asymptotically stable; that is, the robust impulsive synchronization can be achieved globally.

In the following, we consider the case in which the parameters r_{ij} are not all known, but there exist positive constants $K_{1i} > 0$, $K_{2i} > 0$, $K_{3i} > 0$, $i = 1, 2, \ldots, N$, such that

$$\sum_{j=1}^{N} r_{ij} \le K_{1i}, \quad \sum_{j=1}^{N} r_{ji} \le K_{2i}, \quad \sum_{j=1}^{N} r_{ij}^2 \le K_{3i}, \quad i = 1, 2, \ldots, N. \tag{3.22}$$

Theorem 3.4. *Assume that Assumptions 2.4-2.5 and conditions* (ii)-(iii) *of Theorem 3.1 hold, while condition* (i) *of Theorem 3.1 is changed into the following one:*

(i) *for all $n \in (N_k, N_{k+1}]$, $k \in \mathbb{N}$, the following inequalities hold:*

$$\begin{aligned} A^T(n)P_i(n+1)A(n) &+ 2L_{i_k}\sqrt{\lambda_{\max}\left(P_i^{-1}(n+1)A^T(n)P_i(n+1)A(n)\right)} \cdot \sqrt{\frac{\lambda_{\max}\left(P_i(n+1)\right)}{\lambda_{\min}\left(P_i(n+1)\right)}}P_i(n+1) \\ &+ \lambda_{\max}\left(P_i(n+1)\right)\left[(1+\nu_i)L_{i_k}^2 + \left(1+\nu_i^{-1}\right)K_{i3}\right]I \\ &+ \sqrt{\lambda_{\max}\left(P_i^{-1}(n+1)A^T(n)P_i(n+1)A(n)\right)}\left(\epsilon K_{i1} + \epsilon^{-1}K_{i2}\right)I \\ &\le \alpha_{ik}(n)P_i(n). \end{aligned} \tag{3.23}$$

Then, for any initial conditions $x_i(0) = x_{i0}$, $y(0) = y_0 \in \mathbb{U}$, the uncertain dynamical network (2.1) *is robust impulsive synchronization with system* (2.2) *by the impulsive controllers $\{N_k, B_{i_k}\}$.*

Proof. By the similar proof of Theorem 3.1, we obtain that the result of this theorem holds. The details are omitted here. □

4. Examples and simulations

In this section, three representative examples are given for illustration.

Example 4.1. Consider the entire discrete dynamical network in form of (2.1), where $x_i = (x_{i1}, x_{i2}, x_{i3})^T$, and the functions f, g_i, $i = 1, 2, \ldots, N$, satisfy

$$f(k, x_i) = \begin{pmatrix} -3x_{i1} + x_{i2} + \sin k^2 \\ -x_{i1} + 2x_{i2} - \sin x_{i2} - \cos k \\ x_{i3} + \sin x_{i3} + 2\sin(k-1) \end{pmatrix},$$
$$g_j(x) = \begin{pmatrix} x_{j1} - 2x_{j+1,1} + x_{j+2,1} \\ 0 \\ -x_{j3} + 2x_{j+1,3} - x_{j+2,3} \end{pmatrix}, \tag{4.1}$$

where $j = 1, 2, \ldots, N-2$, and $g_{N-1}(x_1, x_2, \ldots, x_N) = g_N(x_1, x_2, \ldots, x_N) = 0$.

Let $\tilde{f}(k, x_i, y) = f(k, x_i) - f(k, y) = A(k)e_i + \tilde{\varphi}(k, e_i)$, where $y = (y_1, y_2, y_3)^T$, $A(k) = \begin{pmatrix} -3 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and $\tilde{\varphi}(k, e_i) = \begin{pmatrix} 0 \\ -\sin x_{i2} + \sin y_2 \\ \sin x_{i3} - \sin y_3 \end{pmatrix}$.

It is easy to show that $\|A(k)\| = 3.6180$, $\|\tilde{\varphi}(k, e_i)\| \le \|e_i\|$, that is, $L_{i_k} = 1$, for any $x_i, y \in \mathbb{R}^3$, and

$$\|\tilde{g}_i(x, y)\| = \|g_i(x_1, x_2, \ldots, x_N) - g_i(y, y, \ldots, y)\| \le \sqrt{2}\|e_i\| + 2\sqrt{2}\|e_{i+1}\| + \sqrt{2}\|e_{i+2}\|, \tag{4.2}$$

where $i = 1, 2, \ldots, N-2$, and

$$\|\tilde{g}_i(x, y)\| = \|g_i(x_1, x_2, \ldots, x_N) - g_i(y, y, \ldots, y)\| = 0, \quad i = N-1, N. \tag{4.3}$$

Let $N = 10$, then we obtain that $\alpha_{i_k}(n) \le 169.1249$. By Corollary 3.2, we can choose many impulsive control laws $\{N_k, B_{N_k},\ k \in \mathbb{N},\}$ such that the error system is asymptotically stable. In the following, we take $N_k = 3k$ and $B_{N_k} = \begin{pmatrix} -0.995 & 0 & 0 \\ 0 & -0.995 & 0 \\ 0 & 0 & -0.995 \end{pmatrix}$, then

$$\gamma_j = \begin{cases} \sqrt{\alpha_{i_k}(j)} \le 13.0048, & \text{if } j \ne N_k, \\ \|I + B_j\| = 0.005, & \text{if } j = N_k,\ k \in \mathbb{N}. \end{cases} \tag{4.4}$$

Let $S_n = \sum_{j=1}^{n} \ln \gamma_j$, then for $k \in \mathbb{N}$,

$$S_n \le \begin{cases} k(2\ln 13.0048 + \ln 0.005) = -0.1677k, & \text{if } n = 3k, \\ k(2\ln 13.0048 + \ln 0.005) + \ln 13.0048 = -0.1677k + 2.5653, & \text{if } n = 3k+1, \\ k(2\ln 13.0048 + \ln 0.005) + 2\ln 13.0048 = -0.1677k + 5.1306, & \text{if } n = 3k+2, \end{cases} \tag{4.5}$$

which leads to $\sum_{j=1}^{\infty} \ln \gamma_j = \lim_{n\to\infty} S_n = -\infty$. Then, by Corollary 3.2, we obtain that the impulsive controllers $\{N_k, B_{i_k}\}$ designed as above can achieve the robust synchronization for this uncertain discrete dynamical network.

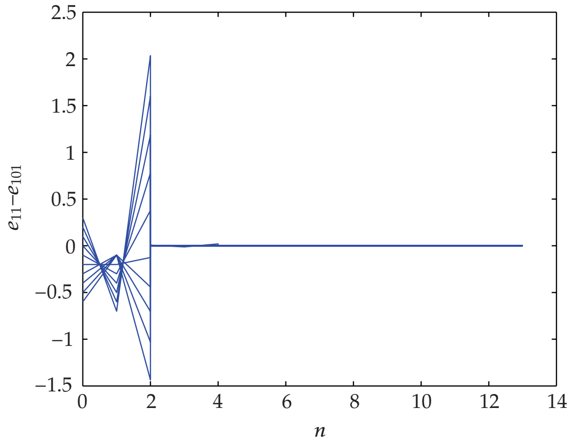

Figure 2: Synchronization errors of e_{k1}, $k = 1, 2, \ldots, 10$.

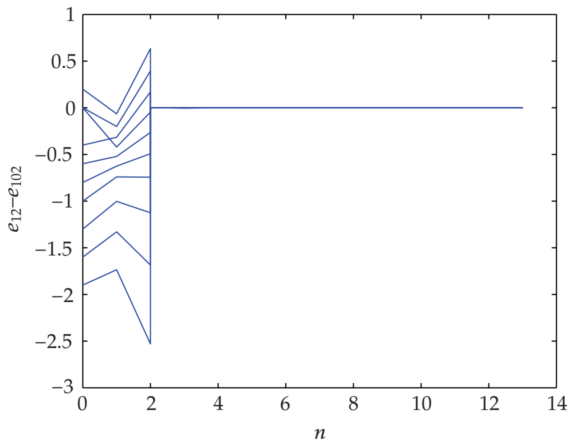

Figure 3: Synchronization errors of e_{k2}, $k = 1, 2, \ldots, 10$.

The numerical simulation is given in Figures 2–4. Here, the initial data are given as $y_0 = (0.1\ 0.5\ 0.4)^T$, $x_{1_0} = (0.4\ 0.7\ 0.6)^T$, $x_{2_0} = (0.3\ 0.5\ 0.4)^T$, $x_{3_0} = (0.2\ 0.3\ 0.2)^T$, $x_{4_0} = (0.1\ 0.1\ 0)^T$, $x_{5_0} = (0\ -0.1\ -0.2)^T$, $x_{6_0} = (-0.1\ -0.3\ -0.4)^T$, $x_{7_0} = (-0.2\ -0.5\ -0.6)^T$, $x_{8_0} = (-0.3\ -0.8\ -0.8)^T$, $x_{9_0} = (-0.4\ -1.1\ -1)^T$, and $x_{10_0} = (-0.5\ -1.4\ -1.2)^T$.

In Figures 2–4, one can see that all the trajectories of the error system for this dynamical network asymptotically approach the origin with the designed robust impulsive controller, where $e_k = (e_{k1}\ e_{k2}\ e_{k3})^T$, $k = 1, 2, \ldots, 10$.

Example 4.2. Here we consider taking the fold chaotic system as nodes of the discretedynamical network. A single fold chaotic system is in form of

$$y(n+1) = Ay(n) + \varphi(y(n)), \quad n \in \mathbb{N}, \tag{4.6}$$

where $y(n) = \begin{pmatrix} y_1(n) \\ y_2(n) \end{pmatrix}$, $A = \begin{pmatrix} -0.1 & 1 \\ 0 & 0 \end{pmatrix}$, $\varphi(y(n)) = \begin{pmatrix} 0 \\ y_1(n)^2 - 1.7 \end{pmatrix}$.

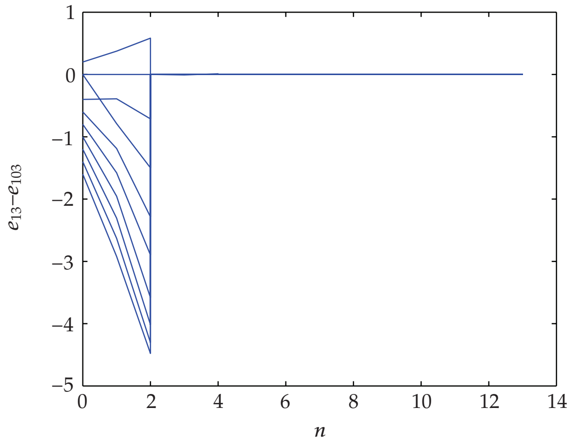

Figure 4: Synchronization errors of e_{k3}, $k = 1, 2, \ldots, 10$.

The entire network is given by

$$x_i(n+1) = Ax_i(n) + \varphi(x_i(n)) + g_i(x_1(n), x_2(n), \ldots, x_N(n)), \quad i = 1, 2, \ldots, N, \tag{4.7}$$

where $x_i = (x_{i1}, x_{i2})^T$, and the coupling functions g_i, $i = 1, 2, \ldots, N$, satisfy

$$g_i(x) = \begin{pmatrix} -\epsilon_1 x_{i1}^2 + \epsilon_1 x_{i+1,1}^2 \\ \epsilon_2 x_{i2}^2 - \epsilon_2 x_{i+1,2}^2 \end{pmatrix}, \ \epsilon_1\big| \le 1, \ \big|\epsilon_2\big| \le 1, \ i = 1, 2, \ldots, N-1, \tag{4.8}$$

and $g_N(x_1, x_2, \ldots, x_N) = 0$.

Let $\widetilde{f}(k, x_i, y) = Ae_i + \widetilde{\varphi}(k, e_i)$, where $y = (y_1, y_2)^T$, $A = \begin{pmatrix} -0.1 & 1 \\ 0 & 0 \end{pmatrix}$ and $\widetilde{\varphi}(k, e_i) = \begin{pmatrix} 0 \\ x_{i1}^2 - y_1^2 \end{pmatrix}$.

Let $x(0) = (-1.5, 0.9)^T$, $y(0) = (-1.5, 0.5)^T$. By simulation, we can estimate the attractive domain $\mathbb{U}$ of isolated node: $\mathbb{U} = \{y \in \mathbb{R}^2 : \|y\| \le 1.5\}$. Thus, for any initial conditions $x_{i0}, y_0 \in \mathbb{U}$, it is easy to show that $\|A\| = 1.0050$, $\|\widetilde{\varphi}(k, e_i)\| \le 3\|e_i\|$, that is, $L_{i_k} = 3$, and

$$\big\|\widetilde{g}_i(x, y)\big\| = \big\|g_i(x_1, x_2, \ldots, x_N) - g_i(y, y, \ldots, y)\big\| \le 4\sqrt{1.5}\big(\big\|e_i\big\| + \big\|e_{i+1}\big\|\big), \tag{4.9}$$

where $i = 1, 2, \ldots, N-1$, and

$$\big\|\widetilde{g}_N(x, y)\big\| = \big\|g_N(x_1, x_2, \ldots, x_N) - g_N(y, y, \ldots, y)\big\| = 0. \tag{4.10}$$

Let $N = 10$. By Corollary 3.2, we obtain that $\alpha_{i_k}(n) \le 179.8278$. We choose impulsive control law $\{N_k, B_{N_k}, \ k \in \mathbb{N}, \}$ such that the error system is asymptotically stable. In the following, we take $N_k = 3k$, $B_{N_k} = \begin{pmatrix} -0.995 & 0 \\ 0 & -0.995 \end{pmatrix}$, then

$$\gamma_j = \begin{cases} \sqrt{\alpha_{i_k}(j)} \le 13.4100, & \text{if } j \ne N_k, \\ \big\|I + B_j\big\| = 0.005, & \text{if } j = N_k, \ k \in \mathbb{N}. \end{cases} \tag{4.11}$$

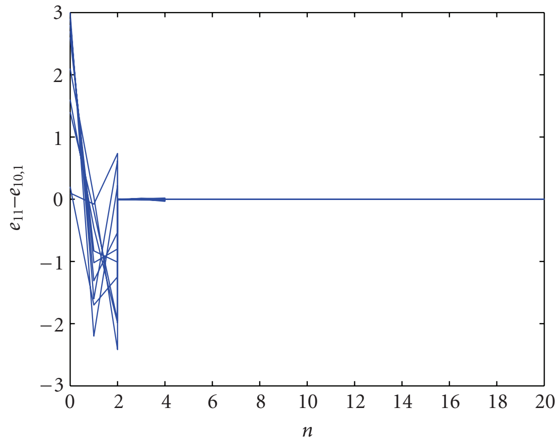

Figure 5: Synchronization errors of e_{k1}, $k = 1, 2, \ldots, 10$.

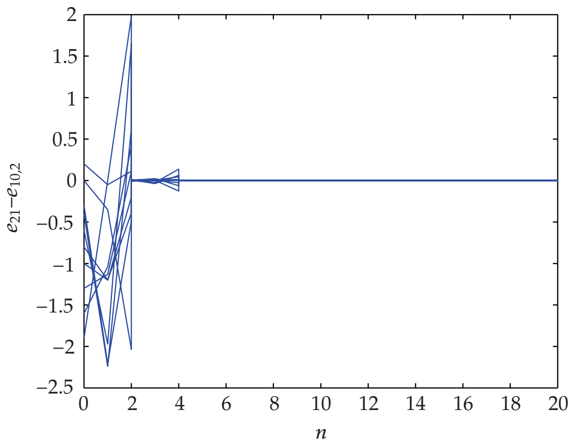

Figure 6: Synchronization errors of e_{k2}, $k = 1, 2, \ldots, 10$.

Let $S_n = \sum_{j=1}^{n} \ln \gamma_j$, then for $k \in \mathbb{N}$,

$$S_n \le \begin{cases} k(2\ln 13.4100 + \ln 0.005) = -0.1063k, & \text{if } n = 3k, \\ k(2\ln 13.4100 + \ln 0.005) + \ln 13.4100 = -0.1063k + 2.5960, & \text{if } n = 3k+1, \\ k(2\ln 13.4100 + \ln 0.005) + 2\ln 13.4100 = -0.1063k + 5.1920, & \text{if } n = 3k+2, \end{cases} \tag{4.12}$$

which leads to $\sum_{j=1}^{\infty} \ln \gamma_j = \lim_{n\to\infty} S_n = -\infty$. Then, by Corollary 3.2, we obtain that the impulsive controllers $\{N_k, B_{i_k}\}$ designed as above can achieve the robust synchronization for this uncertain discrete dynamical network.

The numerical simulation is given in Figures 5-6. Here, the initial data are given as $y_0 = (-1.5\ 0.5)^T$, $x_{1_0} = (-1.4\ 0.7)^T$, $x_{2_0} = (-1.3\ 0.5)^T$, $x_{3_0} = (0.1\ 0.2)^T$, $x_{4_0} = (-0.1\ 0.1)^T$, $x_{5_0} = (0.6\ -0.1)^T$, $x_{6_0} = (1.1\ -0.3)^T$, $x_{7_0} = (1.2\ -0.5)^T$, $x_{8_0} = (1.3\ -0.8)^T$, $x_{9_0} = (1.4\ -1.1)^T$, and $x_{10_0} = (1.5\ -1.4)^T$. In Figures 5-6, one can see that all the trajectories of the error system for this dynamical network asymptotically approach the origin with the designed robust impulsive controller, where $e_k = (e_{k1}\ e_{k2})^T$, $k = 1, 2, \ldots, 10$.

Example 4.3. Here we consider taking the chaotic Hénon map as nodes of the discrete dynamical network. A single chaotic Hénon map is in form of

$$y(n+1) = Ay(n) + \varphi(y(n)), \quad n \in \mathbb{N}, \tag{4.13}$$

where $y(n) = \begin{pmatrix} y_1(n) \\ y_2(n) \end{pmatrix}$, $A = \begin{pmatrix} 0 & 1 \\ 0.3 & 0 \end{pmatrix}$, and $\varphi(y(n)) = \begin{pmatrix} 1-1.4y_1^2 \\ 0 \end{pmatrix}$.

The entire network is given by

$$x_i(n+1) = Ax_i(n) + \varphi(x_i(n)) + g_i(x_1(n), x_2(n), \ldots, x_N(n)), \quad i = 1, 2, \ldots, N, \tag{4.14}$$

where $x_i = (x_{i1}, x_{i2})^T$, and the coupling functions g_i, $i = 1, 2, \ldots, N$, satisfy

$$g_i(x) = \begin{pmatrix} -\epsilon x_{i1}^2 + \epsilon_1 x_{i+1,1}^2 \\ \epsilon_2 x_{i2}^2 - \epsilon_2 x_{i+1,2}^2 \end{pmatrix}, \ |\epsilon_1| \le 1,\ |\epsilon_2| \le 1,\ i = 1, 2, \ldots, N-1, \tag{4.15}$$

and $g_N(x_1, x_2, \ldots, x_N) = 0$.

Let $\widetilde{f}(k, x_i, y) = Ae_i + \widetilde{\varphi}(k, e_i)$, where $y = (y_1, y_2)^T$, and $\widetilde{\varphi}(k, e_i) = \begin{pmatrix} 1.4(y_1^2 - x_{i1}^2) \\ 0 \end{pmatrix}$.

Let $x(0) = (0.3, -0.6)^T$, $y(0) = (0.3, -0.1)^T$. By simulation, we can estimate the attractive domain $\mathbb{U}$ of isolated node: $\mathbb{U} = \{y \in \mathbb{R}^2 : \|y\| \le 3\}$. Thus, for any initial conditions $x_{i0}, y_0 \in \mathbb{U}$, it is easy to show that $\|A\| = 1.0000$, $\|\widetilde{\varphi}(k, e_i)\| \le 8.4\|e_i\|$, that is, $L_{i_k} = 4.2$, and

$$\|\widetilde{g}_i(x, y)\| = \|g_i(x_1, x_2, \ldots, x_N) - g_i(y, y, \ldots, y)\| \le 8\sqrt{2.1}(\|e_i\| + \|e_{i+1}\|), \tag{4.16}$$

where $i = 1, 2, \ldots, N-1$, and

$$\|\widetilde{g}_N(x, y)\| = \|g_N(x_1, x_2, \ldots, x_N) - g_N(y, y, \ldots, y)\| = 0. \tag{4.17}$$

Let $N = 10$. By Corollary 3.2, we obtain that $\alpha_{i_k}(n) \le 248.6386$. We choose impulsive control law $\{N_k, B_{N_k},\ k \in \mathbb{N},\}$ such that the error system is asymptotically stable. In the following, we take $N_k = 3k$, $B_{N_k} = \begin{pmatrix} -0.996 & 0 \\ 0 & -0.996 \end{pmatrix}$, then

$$\gamma_j = \begin{cases} \sqrt{\alpha_{i_k}(j)} \le 15.7683, & \text{if } j \ne N_k, \\ \|I + B_j\| = 0.004, & \text{if } j = N_k,\ k \in \mathbb{N}. \end{cases} \tag{4.18}$$

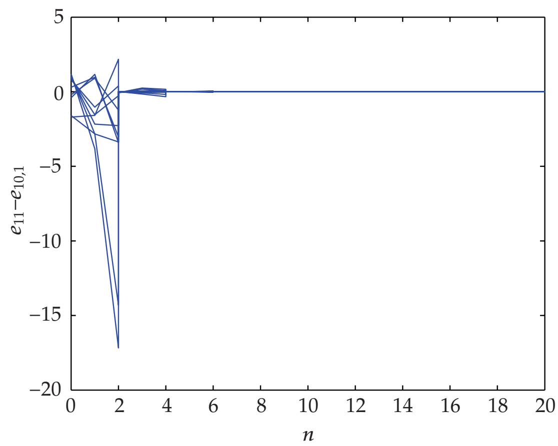

Figure 7: Synchronization errors of e_{k1}, $k = 1, 2, \ldots, 10$.

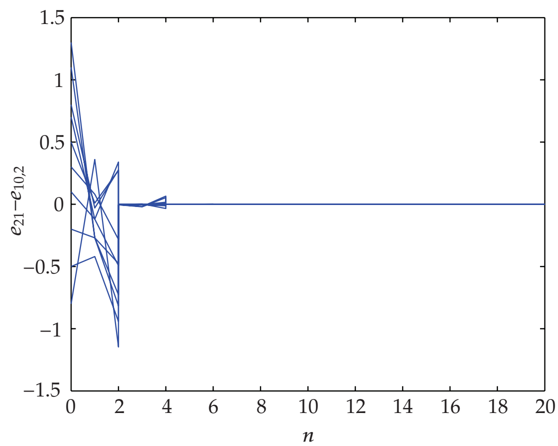

Figure 8: Synchronization errors of e_{k2}, $k = 1, 2, \ldots, 10$.

Let $S_n = \sum_{j=1}^{n} \ln \gamma_j$, then for $k \in \mathbb{N}$,

$$S_n \leq \begin{cases} k(2\ln 15.7683 + \ln 0.004) = -0.0055k, & \text{if } n = 3k, \\ k(2\ln 15.7683 + \ln 0.004) + \ln 15.7683 = -0.0055k + 2.7580, & \text{if } n = 3k+1, \\ k(2\ln 15.7683 + \ln 0.004) + 2\ln 15.7683 = -0.0055k + 5.5160, & \text{if } n = 3k+2, \end{cases} \tag{4.19}$$

which leads to $\sum_{j=1}^{\infty} \ln \gamma_j = \lim_{n\to\infty} S_n = -\infty$. Then, by Corollary 3.2, we obtain that the impulsive controllers $\{N_k, B_{i_k}\}$ designed as above can achieve the robust synchronization for this uncertain discrete dynamical network.

The numerical simulation is given in Figures 7-8. Here, the initial data are given as $y_0 = (0.3 \ \ -0.6)^T$, and x_{k_0}, $k = 1, 2, \ldots, 10$, are the same as in Example 4.2.

In Figures 7-8, one can see that all the trajectories of the error system for this dynamical network asymptotically approach the origin with the designed robust impulsive controller, where $e_k = (e_{k1}\ e_{k2})^T$, $k = 1, 2, \ldots, 10$.

5. Conclusions

In this paper, a robust impulsive control method for synchronization of an uncertain discrete dynamical network has been introduced. The controller so designed is robust to uncertain network coupling. From the aspect of controller structure and robustness to uncertain network coupling, the developed synchronization scheme is more efficient than those reported in the literature to date. Some simple and effective criteria for achieving robust impulsive synchronization have been derived. Because a chaotic system has complex dynamical behaviors and possesses some special features which make the chaotic synchronization very useful to secure communication, it is significative to take discrete chaotic system as nodes in a discrete dynamical network. Three examples demonstrate the effectiveness of the theoretical results obtained in this paper.

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