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*Research Article*

# Infinite Horizon Discrete Time Control Problems for Bounded Processes

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We establish Pontryagin Maximum Principles in the strong form for infinite horizon optimal control problems for bounded processes, for systems governed by difference equations. Results due to Ioffe and Tihomirov are among the tools used to prove our theorems. We write necessary conditions with weakened hypotheses of concavity and without invertibility, and we provide new results on the adjoint variable. We show links between bounded problems and nonbounded ones. We also give sufficient conditions of optimality.



## 1. Introduction

The first works on infinite horizon optimal control problems are due to Pontryagin and his school [1]. They were followed by few others [2–6]. We consider in this paper an infinite horizon Optimal Control problem in the discrete time framework. Such problems are fundamental in the macroeconomics growth theory [7–10] and see references of [11]. Even in the finite horizon case, the discrete time framework presents significant differences from the continuous time one. Boltianski [12] shows that in the discrete time case, a convexity condition is needed to guarantee a strong Pontryagin Principle while this last one can be obtained without such condition in the continuous time setting. We study our problem in the space of bounded sequences $\ell_\infty$, which allows us to use Analysis in Banach spaces instead of using reductions to finite horizon problems as in [5, 6]. According to Chichlinisky [13, 14], the space of bounded sequences was first used in economics by Debreu [15]. It can also be found in [7, 8, 16]. We obtain Pontryagin Maximum Principles in the strong form using weaker convexity hypotheses than the traditional ones and without invertibility [5]. When we study the problem in a general sequence space it turns out that the infinite series will not always converge. Therefore we present other notions of optimality that are currently used, notably in the economic literature, see [3, 4, 9] and we show how our problem can be related to these other problems. We end the paper by establishing sufficient conditions of optimality.

Now we briefly describe the contents of the paper. In Section 2 we introduce the notations and the problem, then we state Theorems 2.1 and 2.2 which give necessary conditions of optimality namely the existence of the adjoint variable in the space $\ell_1$ satisfying the adjoint equation and the strong Pontryagin maximum principle. In Section 3 we prove these theorems through some lemmas and using results due to Ioffe-Tihomirov [17]. In Section 4 we introduce some other notions of optimality for problems in the nonbounded case and we show links with our problem. For example, we show that when the objective function is positive then a bounded solution is a solution among the unbounded processes. Finally we give sufficient conditions of optimality for problems in the bounded and unbounded cases adapting for each case the approprate transversality condition.

## 2. Pontryagin maximum principles for bounded processes

We first precise our notations. Let $\Omega$ be a nonempty open convex subset of $\mathbb{R}^n$ and $U$ a nonempty compact subset of $\mathbb{R}^m$. Let $\Phi : \Omega \times U \to \mathbb{R}$ and, for all $t \in \mathbb{N}$, $f_t : \Omega \times U \to \mathbb{R}^n$. We set $(\underline{x}, \underline{u}) = ((x_t)_t, (u_t)_t)$.

Define dom $J = \{(\underline{x}, \underline{u}) \in \Omega^{\mathbb{N}} \times U^{\mathbb{N}} : \sum_{t=0}^{+\infty} \beta^t \Phi(x_t, u_t)$ converges in $\mathbb{R}\}$.

For every $\underline{x} \in \mathbb{R}^n$ define $C(\underline{x})$ as the closure of the set of terms of the sequence $\underline{x}$. If $\underline{x} \in \ell_\infty(\mathbb{N}, \mathbb{R}^n), C(\underline{x})$ is compact. We set $\mathcal{X} = \{\underline{x} = (x_t)_t \in \ell_\infty(\mathbb{N}, \mathbb{R}^n),$ such that $C(\underline{x}) \subset \Omega\}$. $\mathcal{X}$ is thus the set of the bounded sequences which are in the interior of $\Omega$. Note that $\mathcal{X}$ is a convex open subset of $\ell^\infty(\mathbb{N}, \mathbb{R}^n)$ since $\Omega$ is open and convex. We set $\mathcal{U} = \{\underline{u} = (u_t)_t \in U^{\mathbb{N}}\}$. Define $\mathrm{Adm}(\eta) = \{(\underline{x}, \underline{u}) \in \Omega^{\mathbb{N}} \times \mathcal{U} : x_{t+1} = f_t(x_t, u_t),\ t \in \mathbb{N},$ and $x_0 = \eta\}$; it is the set of admissible processes with respect to the considered dynamical system.

Let $\beta \in (0, 1)$. We consider first the following problem (P1):

$$\begin{gathered} \text{Maximize } J(\underline{x}, \underline{u}) = \sum_{t=0}^{+\infty} \beta^t \Phi(x_t, u_t) \\ x_{t+1} = f_t(x_t, u_t), \quad t \in \mathbb{N} \\ x_0 = \eta \\ (\underline{x}, \underline{u}) \in \mathcal{X} \times \mathcal{U} \end{gathered} \tag{P1}$$

which can be written as follows.

(P1) Maximize $J(\underline{x}, \underline{u})$ when $(\underline{x}, \underline{u}) \in \mathrm{Adm}(\eta) \cap (\mathcal{X} \times \mathcal{U})$.

**Theorem 2.1.** *Let $(\underline{\hat{x}}, \underline{\hat{u}})$ be a solution of* (P1). *Assume the following.*

(i) *For all $u \in U$, the mapping $x \mapsto \Phi(x, u)$ is of class $C^1$ on $\Omega$ and for all $t \in \mathbb{N}$, the mapping $x \mapsto f_t(x, u)$ is Fréchet-differentiable on $\Omega$.*

(ii) *For all $t \in \mathbb{N}$, for all $x_t \in \Omega$, for all $u'_t, u''_t \in U$, for all $\alpha \in [0, 1]$, there exists $u_t \in U$ such that*

$$\begin{aligned} \Phi(x_t, u_t) &\geq \alpha \Phi(x_t, u'_t) + (1 - \alpha) \Phi(x_t, u''_t) \\ f_t(x_t, u_t) &= \alpha f_t(x_t, u'_t) + (1 - \alpha) f_t(x_t, u''_t). \end{aligned} \tag{2.1}$$

(iii) *For any compact set* $C \subset \Omega$, *there exists a constant* $K_C$ *such that for all* $t \in \mathbb{N}$, *for all* $x, x' \in C$, *for all* $u \in U$, $\|f_t(x,u)\| \le K_C$ *and* $\|D_{x_t} f_t(x,u) - D_{x_t} f_t(x',u)\| \le K_C \|x - x'\|$.

(iv) *There exists* $r > 0$ *such that* $B(\underline{\hat{x}}, r) \subset \mathcal{X}$ *and for all* $(x_t, u_t) \in B(\hat{x}_t, r) \times U$,

$$\sup_{t \ge 0} \left\| D_{x_t} f_t(x_t, u_t) \right\| < 1. \tag{2.2}$$

*Then there exists* $(p_{t+1})_t \in \ell_1(\mathbb{N}, \mathbb{R}^n)$ *such that*

(a) $p_t = p_{t+1} \circ D_{x_t} f_t(\hat{x}_t, \hat{u}_t) + \beta^t D_{x_t} \Phi(\hat{x}_t, \hat{u}_t)$, *for all* $t$,

(b) $\beta^t \Phi(\hat{x}_t, \hat{u}_t) + \langle p_{t+1}, f_t(\hat{x}_t, \hat{u}_t) \rangle \ge \beta^t \Phi(\hat{x}_t, u_t) + \langle p_{t+1}, f_t(\hat{x}_t, u_t) \rangle$, *for all* $t$, *for all* $u_t \in U$,

(c) $\lim_{t \to +\infty} p_t = 0$.

*Comments*

For continuous time problems, one does not need conditions to obtain a strong Pontryagin maximum principle, both in the finite horizon case (see, e.g., [18]) and in the infinite horizon case (see, e.g., [5]). But for discrete time problems, strong Pontryagin principles cannot hold without an additional assumption namely a convexity condition, as Boltyanski shows in [12] for the finite horizon framework. Condition (ii) comes from the Ioffe and Tihomirov book [17]. It generalizes the usual convexity condition used to garantee a strong Pontryagin maximum principle. The usual condition is: $U$ convex subset, $\Phi$ concave with respect to $u$ and for every $t$, $f_t$ affine with respect to $u$. It implies condition (ii). In (iii) the condition $\|f_t(x,u)\| \le K_C$ is satisfied when $f_t$ is continuous (since $U$ is compact) and the condition $\|D_{x_t} f_t(x,u) - D_{x_t} f_t(x',u)\| \le K_C \|x - x'\|$ is satisfied when $D^2_{x_t} f_t$ exists and is continuous.

Conclusion (a) is the adjoint equation, conclusion (b) is the strong Pontryagin maximum principle and conclusion (c) is a transversality condition at infinity. In our case (c) is immediately obtained since $(p_{t+1})_t$ is in $\ell_1(\mathbb{N}, \mathbb{R}^n)$, but in general (nonbounded cases) it is very delicate to obtain such a conclusion. [9]

In the next theorem we consider the autonomous case. Thus the hypotheses are simpler and easier to manipulate.

**Theorem 2.2.** *Let* $f_t = f$ *for all* $t \in \mathbb{N}$. *Let* $(\underline{\hat{x}}, \underline{\hat{u}})$ *be a solution of* (P1). *Assume that the following conditions are fulfilled.*

(i) *For all* $u \in U$, *the mappings* $x \mapsto \Phi(x,u)$ *and* $x \mapsto f(x,u)$ *are of class* $C^1$ *on* $\Omega$.

(ii) *For all* $t \in \mathbb{N}$, *for all* $x_t \in \Omega$, *for all* $u'_t, u''_t \in U$, *for all* $\alpha \in [0,1]$, *there exists* $u_t \in U$ *such that*

$$\begin{aligned} \Phi(x_t, u_t) &\ge \alpha \Phi(x_t, u'_t) + (1-\alpha) \Phi(x_t, u''_t), \\ f(x_t, u_t) &= \alpha f(x_t, u'_t) + (1-\alpha) f(x_t, u''_t). \end{aligned} \tag{2.3}$$

(iii) $\sup_{t \ge 0} \|D_{x_t} f(\hat{x}_t, \hat{u}_t)\| < 1$.

*Then there exists* $(p_{t+1})_t \in \ell_1(\mathbb{N}, \mathbb{R}^n)$ *such that the assertions (a), (b), and (c) of Theorem 2.1 are satisfied.*

## 3. Proofs of Theorems 2.1 and 2.2

*Proof of Theorem 2.1*

*First part*

The first part of the proof goes through several lemmas.

**Lemma 3.1.** *$J$ is well-defined and under hypothesis (i) of Theorem 2.1, for all $\underline{u}$, the mapping $\underline{x} \mapsto J(\underline{x}, \underline{u})$ is of class $C^1$ and one has, for all $\delta\underline{x} \in \ell_\infty(\mathbb{N}, \mathbb{R}^n)$, $D_{\underline{x}} J(\underline{x}, \underline{u})\delta\underline{x} = \sum_{t=0}^{+\infty} \beta^t D_{x_t}\Phi(x_t, u_t)\delta x_t$.*

For the proof see [19].

We set $F(\underline{x}, \underline{u}) = (f_t(x_t, u_t) - x_{t+1})_{t\geq 0}$ for all $(\underline{x}, \underline{u}) \in \mathcal{X} \times \mathcal{U}$.

**Lemma 3.2.** *Assume that hypothesis (iii) of Theorem 2.1 holds. Then for $(\underline{x}, \underline{u}) \in \ell_\infty(\mathbb{N}, \Omega) \times \mathcal{U}$, one has $F(\underline{x}, \underline{u}) \in \ell_\infty(\mathbb{N}, \mathbb{R}^n)$. Moreover, if in addition hypotheses (i) and (iv) of Theorem 2.1 hold, then for all $\underline{u}$, the mapping $\underline{x} \mapsto F(\underline{x}, \underline{u})$ is of class $C^1$ on the ball $B(\underline{\widehat{x}}, r)$ in $\ell_\infty(\mathbb{N}, \mathbb{R}^n)$ and for all $\delta\underline{x} \in \ell_\infty(\mathbb{N}, \mathbb{R}^n)$, one has $D_{\underline{x}} F(\underline{x}, \underline{u})\delta\underline{x} = (D_{x_t} f_t(x_t, u_t)\delta x_t - \delta x_{t+1})_{t\in\mathbb{N}}$.*

*Proof.* Let $(\underline{x}, \underline{u}) \in \ell_\infty(\mathbb{N}, \Omega) \times \mathcal{U}$. Let $K_C$ be the constant of (iii) with $C = C(\underline{x})$. So for all $t \in \mathbb{N}$, $\|f_t(x_t, u_t) - x_{t+1}\| \leq K_C + \sup_{t\in\mathbb{N}} \|x_t\|$. So one has $F(\underline{x}, \underline{u}) \in \ell_\infty(\mathbb{N}, \mathbb{R}^n)$.

Assume now that hypotheses (i) and (iv) of Theorem 2.1 hold. Let us show that $\underline{x} \mapsto F(\underline{x}, \underline{u})$ is of class $C^0$ on $B(\underline{\widehat{x}}, r)$. Take $(\underline{x}^0, \underline{u}) \in B(\underline{\widehat{x}}, r) \times \mathcal{U}$. Let $\epsilon > 0$ be given. Let $\underline{x} \in B(\underline{\widehat{x}}, r)$ be such that $\|\underline{x} - \underline{x}^0\| < \min\{2r, \epsilon/2\}$. Then, for all $t \in \mathbb{N}$, $\|f_t(x_t, u_t) - x_{t+1} + f_t(x_t^0, u_t) - x_{t+1}^0\| \leq \|f_t(x_t^0, u_t) - f_t(x_t, u_t)\| + \|x_{t+1} - x_{t+1}^0\| \leq \sup_{y_t\in]x_t^0, x_t[}\|D_{x_t} f_t(y_t, u_t)\| \cdot \|x_t - x_t^0\| + \|x_{t+1} - x_{t+1}^0\| < (\epsilon/2)(\sup_{t\geq 0}\|D_{x_t} f_t(y_t, u_t)\| + 1) < \epsilon$ under (iv), which implies that $\|F(\underline{x}, \underline{u}) - F(\underline{x}^0, \underline{u})\|_\infty < \epsilon$.

Let us now show that $\underline{x} \mapsto F(\underline{x}, \underline{u})$ is Fréchet-differentiable on $B(\underline{\widehat{x}}, r)$. Take $(\underline{x}^0, \underline{u}) \in B(\underline{\widehat{x}}, r) \times \mathcal{U}$. Let $\epsilon > 0$ be given. Let $\underline{x} \in B(\underline{\widehat{x}}, r)$ be such that $\|\underline{x} - \underline{x}^0\| < \min\{2r, \epsilon/2\}$. Then, for all $t \in \mathbb{N}$, $\|f_t(x_t, u_t) - x_{t+1} + f_t(x_t^0, u_t) - x_{t+1}^0 + D_{x_t} f_t(x_t^0, u_t)(x_t - x_t^0) - (x_{t+1} - x_{t+1}^0)\| \leq (\sup_{y_t\in]x_t^0, x_t[}\|D_{x_t} f_t(y_t, u_t) - D_{x_t} f_t(x_t^0, u_t)\|)\|(x_t - x_t^0)\| \leq (\sup_{y_t\in]x_t^0, x_t[}(\|D_{x_t} f_t(y_t, u_t)\| + \|D_{x_t} f_t(x_t^0, u_t)\|)\|(x_t - x_t^0)\| < \epsilon$ under (iii). But this implies that $\|F(\underline{x}, \underline{u}) - F(\underline{x}^0, \underline{u}) - D_{x_t} f_t(x_t^0, u_t)(x_t - x_t^0) - ((x_{t+1} - x_{t+1}^0))_{t\in\mathbb{N}}\|_\infty < \epsilon$. Thus $\underline{x} \mapsto F(\underline{x}, \underline{u})$ is Fréchet-differentiable at $\underline{x}^0$ and $D_{\underline{x}} F(\underline{x}^0, \underline{u})\delta\underline{x} = (D_{x_t} f_t(x_t^0, u_t)\delta x_t - \delta x_{t+1})_{t\in\mathbb{N}}$.

To show the continuity of $\underline{x} \mapsto D_{\underline{x}} F(\underline{x}, \underline{u})$ at $\underline{x}^0$ let $K_B$ be the constant of hypothesis (iii) corresponding to $B(\underline{\widehat{x}}, r)$. Let $\epsilon > 0$ be given and let $\underline{x} \in B(\underline{\widehat{x}}, r)$ be such that $\|\underline{x} - \underline{x}^0\| < \min\{2r, \epsilon/K_B\} \cdot \|D_{\underline{x}} F(\underline{x}, \underline{u}) - D_{\underline{x}} F(\underline{x}^0, \underline{u})\|_\infty \leq \sup_{t\in\mathbb{N}}\|D_{x_t} f_t(x_t^0, u_t) - D_{x_t} f_t(x_t, u_t)\| \leq \sup_{t\in\mathbb{N}} K_B\|x_t^0 - x_t\| < \epsilon$. So $F$ is of class $C^1$. □

**Lemma 3.3.** *Under hypothesis (ii) of Theorem 2.1, for all $\underline{x} \in \mathcal{X}$, for all $\underline{u}', \underline{u}'' \in \mathcal{U}$, for all $\alpha \in [0, 1]$, there exists $u \in \mathcal{U}$ such that*

$$\begin{aligned} J(\underline{x}, \underline{u}) &\geq \alpha J(\underline{x}, \underline{u}') + (1 - \alpha) J(\underline{x}, \underline{u}'') \\ F(\underline{x}, \underline{u}) &= \alpha F(\underline{x}, \underline{u}') + (1 - \alpha) F(\underline{x}, \underline{u}''). \end{aligned} \tag{3.1}$$

*Proof.* Let $\underline{x} = (x_t)_t \in \mathcal{X}$, $\underline{u}' = (u_t')_t \in \mathcal{U}$, $\underline{u}'' = (u_t'')_t \in \mathcal{U}$ and $\alpha \in [0, 1]$. Hypothesis (ii) of Theorem 2.1 implies for all $t \in \mathbb{N}$ the existence of $u_t \in U$ such that

$$\begin{aligned} \Phi(x_t, u_t) &\geq \alpha\Phi(x_t, u'_t) + (1-\alpha)\Phi(x_t, u''_t) \\ f_t(x_t, u_t) &= \alpha f_t(x_t, u'_t) + (1-\alpha) f_t(x_t, u''_t). \end{aligned} \tag{3.2}$$

Therefore we obtain

$$\begin{gathered} \sum_{t=0}^{+\infty} \beta^t \Phi(x_t, u_t) \geq \alpha \sum_{t=0}^{+\infty} \beta^t \Phi(x_t, u'_t) + (1-\alpha) \sum_{t=0}^{+\infty} \beta^t \Phi(x_t, u''_t) \\ \left(f_t(x_t, u_t) - x_{t+1}\right)_t = \alpha\left(f_t(x_t, u'_t) - x_{t+1}\right)_t + (1-\alpha)\left(f_t(x_t, u''_t) - x_{t+1}\right)_t. \end{gathered} \tag{3.3}$$

Set $\underline{u} = (u_t)_t$, so $\underline{u} \in \mathcal{U}$ and satisfies the required relations. □

**Lemma 3.4.** *Under hypotheses (i) and (iv) of Theorem 2.1,* $\operatorname{Im} D_{\underline{x}} F(\underline{\widehat{x}}, \underline{\widehat{u}}) = \ell_\infty(\mathbb{N}, \mathbb{R}^n)$.

*Proof.* Since $D_{\underline{x}} F(\underline{x}, \underline{u})\delta\underline{x} = (D_{x_t} f_t(x_t, u_t)\delta x_t - \delta x_{t+1})_{t\in\mathbb{N}}, \delta x_0 = 0$, the problem is a problem of bounded solutions of first-order linear difference equations.

Let $(M_t)_{t\geq 0} \in \ell_\infty(\mathbb{N}, (\mathbb{R}^n, \mathbb{R}^n))$. Assume that $\sup_{t\geq 1}\|M_t\| < 1$. Then for all $(b_t)_{t\geq 0} \in \ell_\infty(\mathbb{N}, \mathbb{R}^n)$ there exists a unique $(h_t)_{t\geq 0} \in \ell_\infty(\mathbb{N}, \mathbb{R}^n)$ such that for all $t \geq 0$,

$$h_{t+1} - M_t h_t = b_t, \tag{3.4}$$

where $h_0 = 0$.

Consider the operator $\mathcal{T} : \ell_\infty(\mathbb{N}^*, \mathbb{R}^n) \to \ell_\infty(\mathbb{N}^*, \mathbb{R}^n)$ such that for all $\underline{h} \in \ell_\infty(\mathbb{N}^*, \mathbb{R}^n)$,

$$\mathcal{T}(\underline{h}) = \left(h_t - M_{t-1} h_{t-1}\right)_{t\geq 1} \tag{3.5}$$

$\mathcal{T} = I + T$ where

$$\begin{gathered} I = \text{identity of } \ell_\infty(\mathbb{N}^*, \mathbb{R}^n) \\ T(\underline{h}) := \left(0, -M_1 h_1, -M_2 h_2, \ldots, -M_t h_t, \ldots\right). \end{gathered} \tag{3.6}$$

Recall that the $\|\cdot\|_{\ell_\infty}$ norm of $\underline{z} \in (\mathbb{R}^n)^{\mathbb{N}^*}$ is defined by $\|\underline{z}\|_{\ell_\infty} = \sup_{t\geq 1}\|z_t\|$ and that the norm of a linear operator $S$ between normed spaces is defined by $\|S\|_{\mathcal{L}} = \sup_{\|\underline{z}\|\leq 1}\|S(\underline{z})\|$.

So $\|T(\underline{h})\|_{\ell_\infty} = \sup_{t\geq 1}\| - M_t h_t\| \leq (\sup_{t\geq 1}\|M_t\|)\|\underline{h}\|_\infty$. So $\|T\|_{\mathcal{L}} \leq \sup_{t\geq 1}\|M_t\| < 1$. Since $\mathcal{T} = I + T$ and $\|T\|_{\mathcal{L}} < 1, \mathcal{T}$ is invertible so it is surjective.

Set $M_t = D_{x_t} f_t(\widehat{x}_t, \widehat{u}_t)$. Then under (iv) one has $\sup_{t\geq 1}\|D_{x_t} f_t(\widehat{x}_t, \widehat{u}_t)\| < 1$. So $\mathcal{T}$ is surjective that is $\operatorname{Im} D_{\underline{x}} F(\underline{\widehat{x}}, \underline{\widehat{u}}) = \ell_\infty(\mathbb{N}, \mathbb{R}^n)$. □

Recall that $\ell_\infty^*(\mathbb{N}, \mathbb{R}) = \ell_1(\mathbb{N}, \mathbb{R}) \oplus \ell_1^d(\mathbb{N}, \mathbb{R})$ where $\ell_1^d(\mathbb{N}, \mathbb{R})$ consists of all singular functionals, see Aliprantis and Border [20]. In fact it consists (up to scalar multiples) of all extensions of the "limit functional" to $\ell_\infty$.

If $\theta \in \ell_1^d(\mathbb{N}, \mathbb{R})$, then there exists $k \in \mathbb{R}$ such that for all $\underline{x} \in c = c(\mathbb{N}, \mathbb{R})$, $\theta(\underline{x}) = k\cdot\lim_{t\to\infty} x_t$. ($c$ being the space of convergent sequences having a limit in $\mathbb{R}$.)

**Lemma 3.5** ($\ell_\infty^*(\mathbb{N},\mathbb{R}^n) = \ell_1(\mathbb{N},\mathbb{R}^n) \oplus \ell_1^d(\mathbb{N},\mathbb{R}^n)$). *If* $\theta \in \ell_1^d(\mathbb{N},\mathbb{R}^n)$ *then* $\theta = (\theta^1,\theta^2,\ldots,\theta^n)$ *where* $\theta^i \in \ell_1^d(\mathbb{N},\mathbb{R})$ *for every* $i = 1,\ldots,n$. *So there exists* $k = (k_1,\ldots,k_n) \in \mathbb{R}^n$ *such that for all* $\underline{x} \in c(\mathbb{N},\mathbb{R}^n)$, $\theta(\underline{x}) = \langle k, \lim_{t\to\infty} x_t\rangle$.

*Second part*

Our optimal control problem can be written as the following abstract static optimisation problem in a Banach space:

$$\begin{gathered}\text{Maximize } J(\underline{x},\underline{u})\\ F(\underline{x},\underline{u}) = 0\\ (\underline{x},\underline{u}) \in \mathcal{X}\times\mathcal{U}\end{gathered} \tag{3.7}$$

that satisfies all conditions of Theorem 4.3, Ioffe-Tihomirov [17]. So we can apply this theorem and obtain the existence of $\lambda_0 \in \mathbb{R}$, $P \in \ell_\infty^*(\mathbb{N},\mathbb{R}^n)$, not all zero, $\lambda_0 \geq 0$, such that:

$$\begin{aligned}&\text{(AE)}\quad \lambda_0 D_{\underline{x}} J(\underline{\widehat{x}},\underline{\widehat{u}}) + D_{\underline{x}} F^*(\underline{\widehat{x}},\underline{\widehat{u}})P = 0,\\ &\text{(PMP)}\quad (\lambda_0 J + \langle P,F\rangle)(\underline{\widehat{x}},\underline{\widehat{u}}) \geq (\lambda_0 J + \langle P,F\rangle)(\underline{\widehat{x}},\underline{u}),\quad \forall \underline{u}\in\mathcal{U}.\end{aligned} \tag{3.8}$$

(AE) denotes the adjoint equation of this problem and (PMP) the Pontryagin maximum principle. They can be written, respectively:

$$\begin{aligned}&\lambda_0\sum_{t=0}^{+\infty}\beta^t D_{x_t}\Phi(\widehat{x}_t,\widehat{u}_t)\delta x_t + \left\langle P,\left(D_{x_t}f_t(\widehat{x}_t,\widehat{u}_t)\delta x_t - \delta x_{t+1}\right)_{t\geq 0}\right\rangle = 0,\\ &\qquad\qquad \forall \underline{\delta x}\in\ell_\infty(\mathbb{N},\mathbb{R}^n) \text{ such that } \delta x_0 = 0,\\ &\lambda_0\sum_{t=0}^{+\infty}\beta^t\Phi(\widehat{x}_t,\widehat{u}_t) + \left\langle P,\left(f_t(\widehat{x}_t,\widehat{u}_t) - \widehat{x}_{t+1}\right)_{t\geq 0}\right\rangle\\ &\qquad \geq \lambda_0\sum_{t=0}^{+\infty}\beta^t\Phi(\widehat{x}_t,u_t) + \left\langle P,\left(f_t(\widehat{x}_t,u_t) - \widehat{x}_{t+1}\right)_{t\geq 0}\right\rangle,\quad \forall \underline{u}\in\mathcal{U}\end{aligned} \tag{3.9}$$

Set $P = \underline{p} + \theta$ where $\underline{p} \in \ell_1(\mathbb{N},\mathbb{R}^n)$ and $\theta \in \ell_1^d(\mathbb{N},\mathbb{R}^n)$.

(AE) becomes:

$$\begin{aligned}&\lambda_0\sum_{t=0}^{+\infty}\beta^t D_{x_t}\Phi(\widehat{x}_t,\widehat{u}_t)\delta x_t + \sum_{t=0}^{+\infty}\langle p_{t+1}, D_{x_t}f_t(\widehat{x}_t,\widehat{u}_t)\delta x_t\rangle - \sum_{t=0}^{+\infty}\langle p_{t+1},\delta x_{t+1}\rangle\\ &\qquad + \left\langle\theta,\left(D_{x_t}f_t(\widehat{x}_t,\widehat{u}_t)\delta x_t - \delta x_{t+1}\right)_{t\geq 0}\right\rangle = 0,\quad \forall\underline{\delta x}\in\ell_\infty(\mathbb{N},\mathbb{R}^n).\end{aligned} \tag{3.10}$$

So we get

$$\sum_{t=0}^{+\infty}\left\langle \lambda_0\beta^t D_{x_t}\Phi(\widehat{x}_t,\widehat{u}_t) + D_{x_t} f_t^*(\widehat{x}_t,\widehat{u}_t)p_{t+1} - p_t, \delta x_t\right\rangle$$
$$= -\left\langle \theta, \left(D_{x_t} f_t(\widehat{x}_t,\widehat{u}_t)\delta x_t - \delta x_{t+1}\right)_{t\geq 0}\right\rangle, \quad \forall \underline{\delta x} \in \ell_\infty(\mathbb{N},\mathbb{R}^n) \text{ with } \delta x_0 = 0. \tag{3.11}$$

Let $z$ be arbitrarily chosen in $\mathbb{R}^n$ and let $t \geq 1$ be in $\mathbb{N}$. Consider the sequence $(\delta x_s)_s$ defined as follows:

$$\delta x_s = \begin{cases} z, & \text{if } s = t \\ 0, & \text{if } s \neq t. \end{cases} \tag{3.12}$$

So one has $D_{x_s} f_s(\widehat{x}_s,\widehat{u}_s)\delta x_s - \delta x_{s+1} = 0$ if $s \geq t+1$, hence $(D_{x_s} f_s(\widehat{x}_s,\widehat{u}_s)\delta x_s - \delta x_{s+1})_s \in c_0 \subset c$.

Thus, it holds that $\langle \theta, (D_{x_s} f_s(\widehat{x}_s,\widehat{u}_s)\delta x_s - \delta x_{s+1})_s\rangle = \langle k, \lim_{s\to\infty}(D_{x_s} f_s(\widehat{x}_s,\widehat{u}_s)\delta x_s - \delta x_{s+1})\rangle = \langle k, 0\rangle = 0$.

Now $\sum_{s=1}^{+\infty}\langle \lambda_0\beta^s D_{x_s}\Phi(\widehat{x}_s,\widehat{u}_s) + D_{x_s} f_s^*(\widehat{x}_s,\widehat{u}_s)p_{s+1} - p_s, \delta x_s\rangle = \langle \lambda_0\beta^t D_{x_t}\Phi(\widehat{x}_t,\widehat{u}_t) + D_{x_t} f_t^*(\widehat{x}_t,\widehat{u}_t)p_{t+1} - p_t, z\rangle$.

Therefore, for all $t \geq 1$ and for all $z \in \mathbb{R}^n$ one has

$$\left\langle \lambda_0\beta^t D_{x_t}\Phi(\widehat{x}_t,\widehat{u}_t) + D_{x_t} f_t^*(\widehat{x}_t,\widehat{u}_t)p_{t+1} - p_t, z\right\rangle = 0 \tag{3.13}$$

which implies

$$\lambda_0\beta^t D_{x_t}\Phi(\widehat{x}_t,\widehat{u}_t) + D_{x_t} f_t^*(\widehat{x}_t,\widehat{u}_t)p_{t+1} - p_t = 0, \quad \forall t \geq 1, \tag{3.14}$$

that is,

$$p_t = p_{t+1} \circ D_{x_t} f_t(\widehat{x}_t,\widehat{u}_t) + \lambda_0\beta^t D_{x_t}\Phi(\widehat{x}_t,\widehat{u}_t), \quad \forall t \geq 1, \tag{3.15}$$

(PMP) becomes:

$$\lambda_0\sum_{t=0}^{+\infty}\beta^t\left(\Phi(\widehat{x}_t,\widehat{u}_t) - \Phi(\widehat{x}_t,u_t)\right) + \left\langle P, \left(f_t(\widehat{x}_t,\widehat{u}_t) - f_t(\widehat{x}_t,u_t)\right)_{t\geq 0}\right\rangle \geq 0, \quad \forall \underline{u} \in \mathcal{U}. \tag{3.16}$$

So $\lambda_0\sum_{t=0}^{+\infty}\beta^t(\Phi(\widehat{x}_t,\widehat{u}_t)-\Phi(\widehat{x}_t,u_t))+\langle \underline{p}, (f_t(\widehat{x}_t,\widehat{u}_t)-f_t(\widehat{x}_t,u_t))_{t\geq 0}\rangle+\langle \theta, (f_t(\widehat{x}_t,\widehat{u}_t)-f_t(\widehat{x}_t,u_t))_{t\geq 0}\rangle \geq 0$ for all $\underline{u} \in \mathcal{U}$.

Consider, for all $u_t \in U$, the sequences $(\underline{u}^t)_{s\in\mathbb{N}}$ defined as follows:

$$u_s^t = \begin{cases} \widehat{u}_t, & \text{if } s \neq t \\ u_t, & \text{if } s = t. \end{cases} \tag{3.17}$$

Since the inequality holds for every $\underline{u} \in \mathcal{U}$, we obtain

$$\lambda_0 \beta^t \left(\Phi(\widehat{x}_t, \widehat{u}_t) - \Phi(\widehat{x}_t, u_t)\right) + \left\langle p_{t+1}, f_t(\widehat{x}_t, \widehat{u}_t) - f_t(\widehat{x}_t, u_t)\right\rangle \geq 0, \quad \forall u_t \in U \tag{3.18}$$

using $\langle \theta, (f_t(\widehat{x}_t, \widehat{u}_t) - f_t(\widehat{x}_t, u_t))_{t\geq 0}\rangle = 0$ as $(f_t(\widehat{x}_t, \widehat{u}_t) - f_t(\widehat{x}_t, u_t))_{t\geq 0}$ is of finite support.

**Lemma 3.6.** *($\lambda_0 \neq 0$).*

*Proof.* Recall we obtained the existence of $\lambda_0 \in \mathbb{R}$, $P \in \ell_\infty^*(\mathbb{N}, \mathbb{R}^n)$, not all zero, $\lambda_0 \geq 0$, such that:

$$\lambda_0 D_{\underline{x}} J(\underline{\widehat{x}}, \underline{\widehat{u}}) + D_{\underline{x}} F^*(\underline{\widehat{x}}, \underline{\widehat{u}}) P = 0. \tag{3.19}$$

If $\lambda_0 = 0$, then $P = 0$ since $\operatorname{Im} D_{\underline{x}} F(\underline{\widehat{x}}, \underline{\widehat{u}}) = \ell_\infty(\mathbb{N}, \mathbb{R}^n)$.

Hence $\lambda_0 \neq 0$. We can set it equal to one. □

From Lemma 3.6 and the previous results, conclusions (a) and (b) are satisfied. Conclusion (c) is a straightforward consequence of the belonging of $(p_{t+1})_t$ to $\ell_1(\mathbb{N}, \mathbb{R}^n)$.

**Lemma 3.7.** *($\theta = 0$).*

*Proof.* Indeed we obtained $\langle \theta, (D_{x_t} f_t(\widehat{x}_t, \widehat{u}_t)\delta x_t - \delta x_{t+1})_{t\geq 0}\rangle = 0$, for all $\underline{\delta x} \in \ell_\infty(\mathbb{N}, \mathbb{R}^n)$. Using $\operatorname{Im} D_{\underline{x}} F(\underline{\widehat{x}}, \underline{\widehat{u}}) = \ell_\infty(\mathbb{N}, \mathbb{R}^n)$ one has $\langle \theta, \underline{h}\rangle = 0$, for all $\underline{h} \in \ell_\infty(\mathbb{N}, \mathbb{R}^n)$. Thus $\theta = 0$. □

*Proof of Theorem 2.2.* Define $F$ on $\mathcal{X} \times \mathcal{U}$ such that $F(\underline{x}, \underline{u}) = (f(x_t, u_t) - x_{t+1})_{t\geq 0}$. Under hypothesis (i) of Theorem 2.2, for all $\underline{u} \in \mathcal{U}$, the mappings $\underline{x} \mapsto J(\underline{x}, \underline{u})$ and $\underline{x} \mapsto F(\underline{x}, \underline{u})$ are of class $C^1$ on $\mathcal{X}$. The proof can be found in [19].

We consider the proof of Lemma 3.4 and we set $M_t = D_{x_t} f(\widehat{x}_t, \widehat{u}_t)$. Then the proof goes like that of Theorem 2.1. □

## 4. Results for unbounded problems

We study now problems of maximization over admissible processes which are not necessarily bounded when the optimal solution is bounded. So consider the following problems.

(P2) Maximize $J(\underline{x}, \underline{u})$ on $\operatorname{dom} J \cap \operatorname{Adm}(\eta)$.

(P3) Find $(\underline{\widehat{x}}, \underline{\widehat{u}}) \in \operatorname{dom} J \cap \operatorname{Adm}(\eta)$ such that, for all $(\underline{x}, \underline{u}) \in \operatorname{Adm}(\eta)$, $J(\underline{\widehat{x}}, \underline{\widehat{u}}) \geq \limsup_{T\to\infty} \sum_{t=0}^{T} \beta^t \Phi(x_t, u_t))$.

(P4) Find $(\underline{\widehat{x}}, \underline{\widehat{u}}) \in \operatorname{Adm}(\eta)$ such that, for all $(\underline{x}, \underline{u}) \in \operatorname{Adm}(\eta)$, $\liminf_{T\to\infty} (\sum_{t=0}^{T} \beta^t \Phi(\widehat{x}_t, \widehat{u}_t) - \sum_{t=0}^{T} \beta^t \Phi(x_t, u_t)) \geq 0$.

(P5) Find $(\underline{\widehat{x}}, \underline{\widehat{u}}) \in \operatorname{Adm}(\eta)$ such that, for all $(\underline{x}, \underline{u}) \in \operatorname{Adm}(\eta)$, $\limsup_{T\to\infty} (\sum_{t=0}^{T} \beta^t \Phi(\widehat{x}_t, \widehat{u}_t) - \sum_{t=0}^{T} \beta^t \Phi(x_t, u_t)) \geq 0$.

The optimality notion of (P3) is called "the strong optimality," that of (P4) is called "the overtaking optimality" and that of (P5) the "weak overtaking optimality" in [3] (in the continuous-time framework). Many existence results of overtaking optimal solutions and weakly overtaking optimal solutions are obtained in [3, 4]. In [4] there are also results in the discrete-time framework.

*Remark 4.1.* Notice that $(\underline{\widehat{x}}, \underline{\widehat{u}})$ is an optimal solution of (P3) implies $(\underline{\widehat{x}}, \underline{\widehat{u}})$ is an optimal solution of (P4) which implies $(\underline{\widehat{x}}, \underline{\widehat{u}})$ is an optimal solution of (P5).

Moreover if $(\underline{\widehat{x}}, \underline{\widehat{u}})$ is a bounded optimal solution of (P4) then (P3) and (P4) reduce to the same problem.

**Lemma 4.2.** *The two following assertions hold.*

(a) *If $(\underline{\widehat{x}}, \underline{\widehat{u}})$ is an optimal solution of problem (P2), (P3), (P4) or (P5) and $(\underline{\widehat{x}}, \underline{\widehat{u}}) \in \mathcal{X} \times \mathcal{U}$ then $(\underline{\widehat{x}}, \underline{\widehat{u}})$ is an optimal solution of problem* (P1). *Therefore Theorem 2.1 applies.*

(b) *Assume $\Phi \geq 0$ on $\Omega \times U$. If $(\underline{\widehat{x}}, \underline{\widehat{u}})$ is an optimal solution of problem (P3) or (P4) and $(\underline{\widehat{x}}, \underline{\widehat{u}}) \in \mathcal{X} \times \mathcal{U}$ then* $\operatorname{dom} J = \operatorname{Adm}(\eta)$.

*Proof.* (a) Since $\mathcal{X} \times \mathcal{U} \cap \operatorname{Adm}(\eta) \subset \operatorname{dom} J \cap \operatorname{Adm}(\eta) \subset \operatorname{Adm}(\eta)$, a bounded optimal solution of (P2) or (P3) is an optimal solution of (P1). Suppose now that $(\underline{\widehat{x}}, \underline{\widehat{u}})$ is a bounded optimal solution of (P4) that is $\liminf_{T \to \infty}(\sum_{t=0}^{T} \beta^t \Phi(\widehat{x}_t, \widehat{u}_t) - \sum_{t=0}^{T} \beta^t \Phi(x_t, u_t)) \geq 0$, for all $(\underline{x}, \underline{u}) \in \operatorname{Adm}(\eta)$. Since $(\underline{\widehat{x}}, \underline{\widehat{u}}) \in \mathcal{X} \times \mathcal{U}$ this can be written $J(\underline{\widehat{x}}, \underline{\widehat{u}}) \geq \limsup_{T \to \infty} \sum_{t=0}^{T} \beta^t \Phi(x_t, u_t)$, for all $(\underline{x}, \underline{u}) \in \operatorname{Adm}_\eta$ and so in particular for all $(\underline{x}, \underline{u}) \in \mathcal{X} \times \mathcal{U}$.

In that case $\limsup_{T \to \infty} \sum_{t=0}^{T} \beta^t \Phi(x_t, u_t) = J(\underline{x}, \underline{u})$. The proof is analogous for (P5).

(b) If $(\underline{\widehat{x}}, \underline{\widehat{u}})$ is an optimal solution of problem (P3) and $(\underline{\widehat{x}}, \underline{\widehat{u}}) \in \mathcal{X} \times \mathcal{U}$, one has $+\infty > J(\underline{\widehat{x}}, \underline{\widehat{u}}) \geq \limsup_{T \to \infty} \sum_{t=0}^{T} \beta^t \Phi(x_t, u_t)$, for all $(\underline{x}, \underline{u}) \in \operatorname{Adm}_\eta$. Since $\Phi \geq 0$, the sequence $(\sum_{t=0}^{T} \beta^t \Phi(x_t, u_t))_T$ is increasing and since it is also upper bounded it converges in $\mathbb{R}_+$.

So $\limsup_{T \to \infty} \sum_{t=0}^{T} \beta^t \Phi(x_t, u_t) = \lim_{T \to \infty} \sum_{t=0}^{T} \beta^t \Phi(x_t, u_t) = J(\underline{x}, \underline{u})$ and $(\underline{x}, \underline{u}) \in \operatorname{dom} J$. □

**Theorem 4.3.** *Let $f_t = f$ for all $t \in \mathbb{N}$. One assumes the following conditions fulfilled:*

(i) $\Phi \geq 0$ *on* $\Omega \times U$.

(ii) *For all $x \in \Omega$, there exists $v \in U$ such that $x = f(x, v)$.*

*Then one has*

(a) $\sup_{(\underline{x},\underline{u}) \in \operatorname{dom} J \cap \operatorname{Adm}(\eta)} J(\underline{x}, \underline{u}) = \sup_{(\underline{x},\underline{u}) \in (\mathcal{X} \times \mathcal{U}) \cap \operatorname{Adm}(\eta)} J(\underline{x}, \underline{u})$.

(b) *If $(\underline{\widehat{x}}, \underline{\widehat{u}})$ is an optimal solution of problem* (P1), *then it is an optimal solution of problems (P3), (P4), and (P5) which all reduce to the same problem.*

*Remark 4.4.* (b) shows that under a nonnegativity assumption, solving the problem in the space of bounded processes provides solutions for problems in spaces of admissible processes which are not necessarily bounded. This type of results is in the spirit of Blot and Cartigny [21] where problems are studied in the continuous-time case.

*Proof.* (a) It is clear that the following inequality holds:

$$\sup_{(\underline{x},\underline{u}) \in \operatorname{dom} J \cap \operatorname{Adm}(\eta)} J(\underline{x}, \underline{u}) \geq \sup_{(\underline{x},\underline{u}) \in (\mathcal{X} \times \mathcal{U}) \cap \operatorname{Adm}(\eta)} J(\underline{x}, \underline{u}). \tag{4.1}$$

Let $(\underline{\tilde{x}}, \underline{\tilde{u}}) \in \operatorname{dom} J \cap \operatorname{Adm}(\eta)$. Let $\epsilon > 0$ be given and let $T = T_\epsilon$ be such that $0 \leq \sum_{t>T_\epsilon} \beta^t \Phi(\tilde{x}_t, \tilde{u}_t) \leq \epsilon$. Set

$$x'_t = \begin{cases} \tilde{x}_t, & \text{if } t \leq T \\ \tilde{x}_T, & \text{if } t > T, \end{cases}$$
$$u'_t = \begin{cases} \tilde{u}_t, & \text{if } t \leq T \\ u'_T, & \text{if } t > T, \end{cases} \tag{4.2}$$

where $u'_T$ is such that $x'_T = f(x'_T, u'_T)$.$\underline{x}' = (x'_t)_t$ and $\underline{u}' = (u'_t)_t$ are bounded and $(\underline{x}', \underline{u}') \in (\mathcal{X} \times \mathcal{U}) \cap \operatorname{Adm}(\eta)$.

Since $\Phi \geq 0$ on $\Omega \times U$ one has

$$\begin{aligned} J(\underline{x}', \underline{x}') &= \sum_{t=0}^{+\infty} \beta^t \Phi(x'_t, u'_t) \\ &= \sum_{t=0}^{T} \beta^t \Phi(\tilde{x}_t, \tilde{u}_t) + \sum_{t=T+1}^{+\infty} \beta^t \Phi(\tilde{x}_T, u'_T) \\ &\geq \sum_{t=0}^{T} \beta^t \Phi(\tilde{x}_t, \tilde{u}_t) \end{aligned} \tag{4.3}$$

$$\sup_{(\underline{x},\underline{u}) \in (\mathcal{X}\times\mathcal{U}) \cap \operatorname{Adm}(\eta)} J(\underline{x}, \underline{u}) \geq J(\underline{x}', \underline{u}') \geq \sum_{t=0}^{T} \beta^t \Phi(\tilde{x}_t, \tilde{u}_t)$$

so we obtain

$$\epsilon + \sup_{(\underline{x},\underline{u}) \in (\mathcal{X}\times\mathcal{U}) \cap \operatorname{Adm}(\eta)} J(\underline{x}, \underline{u}) \geq J(\underline{\tilde{x}}, \underline{\tilde{u}}). \tag{4.4}$$

Since this is true fo all $\epsilon > 0$, letting $\epsilon \to 0$, we obtain

$$\sup_{(\underline{x},\underline{u}) \in (\mathcal{X}\times\mathcal{U}) \cap \operatorname{Adm}(\eta)} J(\underline{x}, \underline{u}) \geq \sup_{(\underline{x},\underline{u}) \in \operatorname{dom} J \cap \operatorname{Adm}(\eta)} J(\underline{x}, \underline{u}). \tag{4.5}$$

(b) Since $\Phi \geq 0$, for all $(\underline{x}, \underline{u}) \in \operatorname{Adm}(\eta)$, the sequence $(\sum_{t=0}^{T} \beta^t \Phi(x_t, u_t))_T$ is nonnegative and nondecreasing so it converges in $[0, \infty]$.

So $\limsup_{T\to\infty} \sum_{t=0}^{T} \beta^t \Phi(x_t, u_t) = \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t \Phi(x_t, u_t)$. Hence (P3), (P4) reduce to the same problem. Similarly (P5) reduces to it. Let $(\underline{\hat{x}}, \underline{\hat{u}})$ be an optimal solution of problem (P1) and suppose it is not an optimal solution of problem (P3). So there exists $(\underline{x}, \underline{u}) \in \operatorname{Adm}(\eta)$ such that $\lim_{T\to\infty} \sum_{t=0}^{T} \beta^t \Phi(x_t, u_t) = +\infty$ that is

$$\forall R \in \mathbb{R}, \quad \exists T_R \in \mathbb{N}^*, \quad \forall T \geq T_R, \quad \sum_{t=0}^{T} \beta^t \Phi(x_t, u_t) > R. \tag{4.6}$$

Let $R \in \mathbb{R}$ and $T = T_R$. Construct $\underline{x}' = (x'_t)_t$ and $\underline{u}' = (u'_t)_t$ as in (a). Thus $\sum_{t=0}^{+\infty} \beta^t \Phi(x'_t, u'_t) = \sum_{t=0}^{T} \beta^t \Phi(x_t, u_t) + \sum_{t=T+1}^{+\infty} \beta^t \Phi(x_T, u_T) \geq R$.

Hence we obtain $\sup_{(\underline{x},\underline{u}) \in (\mathcal{X} \times \mathcal{U}) \cap \mathrm{Adm}(\eta)} J(\underline{x}, \underline{u}) \geq R$, so $\sup_{(\underline{x},\underline{u}) \in (\mathcal{X} \times \mathcal{U}) \cap \mathrm{Adm}(\eta)} J(\underline{x}, \underline{u}) = +\infty$ which contradicts the hypothesis so $(\underline{\widehat{x}}, \underline{\widehat{u}})$ is an optimal solution of problem (P3). □

Following Michel, [22], for all $t \in \mathbf{N}$ and for all $(x_t, x_{t+1}) \in \Omega \times \Omega$, we define $A_t(x_t, x_{t+1})$ as the set of the $(\nu_t, \zeta_t) \in \mathbf{R} \times \mathbf{R}^n$ for which there exists $u_t \in U$ satisfying $\nu_t \leq \beta^t \Phi(x_t, u_t)$, $\zeta_t = f(x_t, u_t) - x_{t+1}$. We also define $B_t(x_t, x_{t+1})$ as the set of the $(\nu_t, \zeta_t) \in \mathbf{R} \times \mathbf{R}^n$ for which there exists $(u_t, \alpha_t) \in U \times \mathbf{R}^n$ satisfying $\nu_t \leq \beta^t \Phi(x_t, u_t)$, $\alpha^k \zeta_t^k = f^k(x_t, u_t) - x_{t+1}^k$ for all $k = 1, \ldots, n$.

**Theorem 4.5.** *Let $f_t = f$ for all $t \in \mathbb{N}$. Let $(\underline{\widehat{x}}, \underline{\widehat{u}})$ be an optimal solution of problem* (P1). *One assumes the following conditions fulfilled.*

(i) $\Phi \geq 0$ *on* $\Omega \times U$.

(ii) *For all $x \in \Omega$, there exists $v \in U$ such that $x = f(x, v)$.*

(iii) *For all $t \in \mathbb{N}$, the mappings $x \mapsto \Phi(x, \widehat{u}_t)$ and $x \mapsto f(x, \widehat{u}_t)$ are Fréchet-differentiable at $\widehat{x}_t$.*

(iv) *For all $t \in \mathbb{N}$, for all $(x_t, x_{t+1}) \in \Omega \times \Omega$,* $\mathrm{co} A_t(x_t, x_{t+1}) \subset B_t(x_t, x_{t+1})$ *where* co *denotes the convex hull.*

(v) *For all $t \in \mathbb{N}$, $D_{x_t} f(\widehat{x}_t, \widehat{u}_t)$ is invertible.*

*Then there exists $\lambda_0 \in \mathbb{R}_+$, $(p_{t+1})_t \in (\mathbb{R}^n)^{\mathbb{N}}$ such that $(\lambda_0, p_1) \neq 0$ and*

(a) $p_t = p_{t+1} \circ D_{x_t} f(\widehat{x}_t, \widehat{u}_t) + \lambda_0 \beta^t D_{x_t} \Phi(\widehat{x}_t, \widehat{u}_t)$, *for all $t$,*

(b) $\lambda_0 \beta^t \Phi(\widehat{x}_t, \widehat{u}_t) + \langle p_{t+1}, f(\widehat{x}_t, \widehat{u}_t) \rangle \geq \lambda_0 \beta^t \Phi(\widehat{x}_t, u_t) + \langle p_{t+1}, f(\widehat{x}_t, u_t) \rangle$, *for all $t$, for all $u_t \in U$.*

*Remark 4.6.* Notice that condition (iv) is a convexity condition and that condition (ii) of Theorem 2.2 implies this condition (iv). Condition (ii) of Theorem 2.2 is equivalent to the following condition: for all $t$, the set $A_t(x_t, x_{t+1})$ is convex.

*Proof.* Use Theorem 4.3 of this paper and apply Theorem 3 in Blot-Chebbi [5]. □

## 5. Sufficient conditions of optimality

Let $H_t(x_t, u_t, p_{t+1}) = \beta^t \Phi(x_t, u_t) + \langle p_{t+1}, f_t(x_t, u_t) \rangle$, for all $t \in \mathbb{N}$.

**Theorem 5.1.** *Let $(\underline{\widehat{x}}, \underline{\widehat{u}}) \in \mathcal{X} \times \mathcal{U} \cap \mathrm{Adm}(\eta)$ where $U$ is convex. One assumes that there exists $(p_{t+1})_t \in \ell_1(\mathbb{N}, \mathbb{R}^n)$ and that the following conditions are fulfilled.*

(i) *The mappings $(x, u) \mapsto \Phi(x, u)$ and for all $t \in \mathbb{N}$, $(x, u) \mapsto f_t(x, u)$ are of class $C^1$ on $\Omega \times U$.*

(ii) $p_t = p_{t+1} \circ D_{x_t} f_t(\widehat{x}_t, \widehat{u}_t) + \beta^t D_{x_t} \Phi(\widehat{x}_t, \widehat{u}_t)$, *for all $t \geq 1$.*

(iii) $\beta^t \Phi(\widehat{x}_t, \widehat{u}_t) + \langle p_{t+1}, f_t(\widehat{x}_t, \widehat{u}_t) \rangle \geq \beta^t \Phi(\widehat{x}_t, u_t) + \langle p_{t+1}, f_t(\widehat{x}_t, u_t) \rangle$, *for all $t$, for all $u_t \in U$.*

(iv) *The mapping $H_t$ is concave with respect to $(x_t, u_t)$, for all $t$.*

*Then $(\underline{\widehat{x}}, \underline{\widehat{u}})$ is an optimal solution of* (P1).

*Proof.* Notice that from (ii), $D_{x_t}H_t(\widehat{x}_t, \widehat{u}_t, p_{t+1}) = p_t$, for all $t \geq 1$.

Let $(\underline{x}, \underline{u}) \in \mathcal{X} \times \mathcal{U} \cap \mathrm{Adm}(\eta)$. For all $t$, one has

$$\begin{aligned}
&\beta^t\Phi(\widehat{x}_t, \widehat{u}_t) - \beta^t\Phi(x_t, u_t) \\
&\quad = H_t(\widehat{x}_t, \widehat{u}_t, p_{t+1}) - \langle p_{t+1}, f_t(\widehat{x}_t, u_t)\rangle - H_t(x_t, u_t, p_{t+1}) + \langle p_{t+1}, f_t(x_t, u_t)\rangle \\
&\quad = H_t(\widehat{x}_t, \widehat{u}_t, p_{t+1}) - H_t(x_t, u_t, p_{t+1}) - \langle D_{x_{t+1}}H_{t+1}(\widehat{x}_{t+1}, \widehat{u}_{t+1}, p_{t+2}), \widehat{x}_{t+1} - x_{t+1}\rangle \\
&\qquad - \langle D_{u_t}H_t(\widehat{x}_t, \widehat{u}_t, p_{t+1}), \widehat{u}_t - u_t\rangle + \langle D_{u_t}H_t(\widehat{x}_t, \widehat{u}_t, p_{t+1}), \widehat{u}_t - u_t\rangle,
\end{aligned} \tag{5.1}$$

therefore, we obtain

$$\begin{aligned}
&\sum_{t=0}^{T}\left(\beta^t\Phi(\widehat{x}_t, \widehat{u}_t) - \beta^t\Phi(x_t, u_t)\right) \\
&\quad = \sum_{t=0}^{T}\Big(H_t(\widehat{x}_t, \widehat{u}_t, p_{t+1}) - H_t(x_t, u_t, p_{t+1}) - \langle D_{x_{t+1}}H_{t+1}(\widehat{x}_{t+1}, \widehat{u}_{t+1}, p_{t+2}), \widehat{x}_{t+1} - x_{t+1}\rangle \\
&\qquad\qquad - \langle D_{u_t}H_t(\widehat{x}_t, \widehat{u}_t, p_{t+1}), \widehat{u}_t - u_t\rangle + \langle D_{u_t}H_t(\widehat{x}_t, \widehat{u}_t, p_{t+1}), \widehat{u}_t - u_t\rangle\Big) \\
&\quad = H_0(\widehat{x}_0, \widehat{u}_0, p_1) - H_0(x_0, u_0, p_1) \\
&\qquad + \sum_{t=1}^{T}\Big(H_t(\widehat{x}_t, \widehat{u}_t, p_{t+1}) - H_t(x_t, u_t, p_{t+1}) \\
&\qquad\qquad - \langle D_{x_t}H_t(\widehat{x}_t, \widehat{u}_t, p_{t+1}), \widehat{x}_t - x_t\rangle - \langle D_{u_t}H_t(\widehat{x}_t, \widehat{u}_t, p_{t+1}), \widehat{u}_t - u_t\rangle\Big) \\
&\qquad - \langle p_{T+1}, \widehat{x}_{T+1} - x_{T+1}\rangle + \sum_{t=1}^{T}\langle D_{u_t}H_t(\widehat{x}_t, \widehat{u}_t, p_{t+1}), \widehat{u}_t - u_t\rangle
\end{aligned} \tag{5.2}$$

Since for all $t \in \mathbb{N}, H_t$ is concave with respect to $x_t$ and $u_t$, one has $H_t(\widehat{x}_t, \widehat{u}_t, p_{t+1}) - H_t(x_t, u_t, p_{t+1}) - \langle D_{x_t}H_t(\widehat{x}_t, \widehat{u}_t, p_{t+1}), \widehat{x}_t - x_t\rangle - \langle D_{u_t}H_t(\widehat{x}_t, \widehat{u}_t, p_{t+1}), \widehat{u}_t - u_t\rangle \geq 0$. Using hypothesis (iii) with $t = 0$ gives $H_0(\widehat{x}_0, \widehat{u}_0, p_1) - H_0(x_0, u_0, p_1) \geq 0$ and using hypothesis (iii), the first order necessary condition for the optimality of $\widehat{u}_t$ is $\langle D_{u_t}H_t(\widehat{x}_t, \widehat{u}_t, p_{t+1}), \widehat{u}_t - u_t\rangle \geq 0$. Thus one has $\sum_{t=0}^{T}(\beta^t\Phi(\widehat{x}_t, \widehat{u}_t) - \beta^t\Phi(x_t, u_t)) \geq \langle p_{T+1}, x_{T+1} - \widehat{x}_{T+1}\rangle$.

The hypothesis $(p_{t+1})_t \in \ell_1(\mathbb{N}, \mathbb{R}^n)$ implies $\lim_{t\to+\infty} p_t = 0$ and since $\widehat{\underline{x}}$ and $\underline{x}$ belong to $\ell_\infty(\mathbb{N}, \mathbb{R}^n)$ one has $\|\underline{x} - \widehat{\underline{x}}\| \leq \|\underline{x}\| + \|\widehat{\underline{x}}\| < \infty$. Hence we obtain $\lim_{T\to+\infty}\langle p_{T+1}, x_{T+1} - \widehat{x}_{T+1}\rangle = 0$ so $\lim_{T\to+\infty}\sum_{t=0}^{T}(\beta^t\Phi(\widehat{x}_t, \widehat{u}_t) - \beta^t\Phi(x_t, u_t)) \geq 0$. That is $J(\widehat{\underline{x}}, \widehat{\underline{u}}) - J(\underline{x}, \underline{u}) \geq 0$. □

**Corollary 5.2.** *Let $(\widehat{\underline{x}}, \widehat{\underline{u}}) \in \mathrm{dom}\, J \cap \mathrm{Adm}(\eta)$ (resp., $\mathrm{Adm}(\eta)$). If the hypotheses of the previous theorem are satisfied except that $(p_{t+1})_t \in \ell_1(\mathbb{N}, \mathbb{R}^n)$ is replaced by $(p_{t+1})_t \in (\mathbb{R}^n)^{\mathbb{N}}$ and if the following hypothesis is also satisfied:*

(v) $\liminf_{T\to+\infty}\langle p_{T+1}, x_{T+1} - \widehat{x}_{T+1}\rangle = 0$,

*then $(\widehat{\underline{x}}, \widehat{\underline{u}})$ is a solution of (P3) (resp., (P4)).*

Notice that if $(\widehat{\underline{x}}, \widehat{\underline{u}}) \in \mathrm{Adm}(\eta)$ with $\limsup_{T\to+\infty}\langle p_{T+1}, x_{T+1} - \widehat{x}_{T+1}\rangle = 0$ we obtain that $(\widehat{\underline{x}}, \widehat{\underline{u}})$ is a solution of (P5).

One can weaken the hypothesis of concavity of $H_t$ with respect to $x_t$ and $u_t$ and replace it by the concavity of $H_t^*$ with respect to $x_t$ as the following theorem shows. (See [23] for a quick survey of sufficient conditions.)

Let $H_t^*(x_t, p_{t+1}) = \max_{u_t \in U} H_t(x_t, u_t, p_{t+1})$.

The maximum is attained since $U$ is compact.

**Theorem 5.3.** *Let $(\underline{\widehat{x}}, \underline{\widehat{u}}) \in \mathcal{X} \times \mathcal{U} \cap \mathrm{Adm}(\eta)$. One assumes that there exists $(p_{t+1})_t \in \ell_1(\mathbb{N}, \mathbb{R}^n)$ and that the following hypotheses are fulfilled.*

(i) *For all $u \in U$, the mappings $x \mapsto \Phi(x, u)$ and for all $t \in \mathbb{N}$, $x \mapsto f_t(x, u)$ are of class $C^1$ on $\Omega$.*

(ii) *Also* (iii) *of the previous theorem.*

(iv) *The mapping $H_t^*$ is concave with respect to $x_t$, for all $t$.*

*Then $(\underline{\widehat{x}}, \underline{\widehat{u}})$ is an optimal solution of* (P1).

*Proof.* Let $(\underline{\widehat{x}}, \underline{\widehat{u}}) \in \mathcal{X} \times \mathcal{U} \cap \mathrm{Adm}(\eta)$ and let $(\underline{x}, \underline{u}) \in \mathcal{X} \times \mathcal{U} \cap \mathrm{Adm}(\eta)$. For all $t$, one has

$$\begin{aligned} \beta^t \Phi(\widehat{x}_t, \widehat{u}_t) - \beta^t \Phi(x_t, u_t) &= H_t(\widehat{x}_t, \widehat{u}_t, p_{t+1}) - H_t(x_t, u_t, p_{t+1}) - \langle p_{t+1}, f_t(\widehat{x}_t, u_t) - f_t(x_t, u_t) \rangle \\ &\geq H_t^*(\widehat{x}_t, p_{t+1}) - H_t^*(x_t, p_{t+1}) - \langle p_{t+1}, \widehat{x}_{t+1} - x_{t+1} \rangle \end{aligned} \tag{5.3}$$

by the definition of $H_t^*$ and noticing that $H_t(\widehat{x}_t, \widehat{u}_t, p_{t+1}) = H_t^*(\widehat{x}_t, p_{t+1})$. So we obtain

$$\begin{aligned} &\sum_{t=0}^{T} \left( \beta^t \Phi(\widehat{x}_t, \widehat{u}_t) - \beta^t \Phi(x_t, u_t) \right) \\ &\quad \geq H_0^*(\widehat{x}_0, p_1) - H_0^*(x_0, p_1) + \sum_{t=1}^{T} \left( H_t^*(\widehat{x}_t, p_{t+1}) - H_t^*(x_t, p_{t+1}) - \langle p_t, \widehat{x}_t - x_t \rangle \right) \\ &\quad\quad - \langle p_{T+1}, \widehat{x}_{T+1} - x_{T+1} \rangle \\ &\quad = \sum_{t=1}^{T} \left( H_t^*(\widehat{x}_t, p_{t+1}) - H_t^*(x_t, p_{t+1}) - \langle D_{x_t} H_t(\widehat{x}_t, \widehat{u}_t, p_{t+1}), \widehat{x}_t - x_t \rangle \right) - \langle p_{T+1}, \widehat{x}_{T+1} - x_{T+1} \rangle. \end{aligned} \tag{5.4}$$

(Notice that $H_0^*(\widehat{x}_0, p_1) = H_0^*(x_0, p_1)$.) Now using $D_{x_t} H_t^*(\widehat{x}_t, p_{t+1}) = D_{x_t} H_t(\widehat{x}_t, \widehat{u}_t, p_{t+1})$ (see Seierstad and Sydsaeter [24, page 390]) we obtain $\sum_{t=0}^{T} (\beta^t \Phi(\widehat{x}_t, \widehat{u}_t) - \beta^t \Phi(x_t, u_t)) \geq H_0^*(\widehat{x}_0, p_1) - H_0^*(x_0, p_1) + \sum_{t=1}^{T} (H_t^*(\widehat{x}_t, p_{t+1}) - H_t^*(x_t, p_{t+1}) - \langle D_{x_t} H_t^*(\widehat{x}_t, p_{t+1}), \widehat{x}_t - x_t \rangle) + \langle p_{T+1}, x_{T+1} - \widehat{x}_{T+1} \rangle$. The concavity of $H_t^*$ with respect to $x_t$ gives $\sum_{t=0}^{T} (\beta^t \Phi(\widehat{x}_t, \widehat{u}_t) - \beta^t \Phi(x_t, u_t)) \geq \langle p_{T+1}, x_{T+1} - \widehat{x}_{T+1} \rangle$. Finally $J(\underline{\widehat{x}}, \underline{\widehat{u}}) - J(\underline{x}, \underline{u}) \geq 0$ follows as in the proof of the previous theorem. □

**Corollary 5.4.** *Let $(\underline{\widehat{x}}, \underline{\widehat{u}}) \in \mathrm{dom}\, J \cap \mathrm{Adm}(\eta)$ (resp., $Adm(\eta)$). If the hypotheses of the previous theorem are satisfied except that $(p_{t+1})_t \in \ell_1(\mathbb{N}, \mathbb{R}^n)$ is replaced by $(p_{t+1})_t \in (\mathbb{R}^{\mathbb{N}})^n$ and if the following hypothesis is also satisfied:*

(v) $\liminf_{T \to +\infty} \langle p_{T+1}, x_{T+1} - \widehat{x}_{T+1} \rangle = 0$,

*then $(\underline{\widehat{x}}, \underline{\widehat{u}})$ is a solution of (P3) (resp., (P4)).*

Notice that if $(\underline{\widehat{x}}, \underline{\widehat{u}}) \in \mathrm{Adm}(\eta)$ with $\limsup_{T \to +\infty} \langle p_{T+1}, x_{T+1} - \widehat{x}_{T+1} \rangle = 0$ we obtain that $(\underline{\widehat{x}}, \underline{\widehat{u}})$ is a solution of (P5).

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