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Research Article

On the Asymptotic Integration of Nonlinear Dynamic Equations

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The purpose of this paper is to study the existence and asymptotic behavior of solutions to a class of second-order nonlinear dynamic equations on unbounded time scales. Four different results are obtained by using the Banach fixed point theorem, the Boyd and Wong fixed point theorem, the Leray-Schauder nonlinear alternative, and the Schauder fixed point theorem. For each theorem, an illustrative example is presented. The results provide unification and some extensions in the time scale setup of the theory of asymptotic integration of nonlinear equations both in the continuous and discrete cases.

1. Introduction

This work is devoted to the study of the existence and asymptotic behavior of solutions to the nonlinear dynamic equation

$$u^{\Delta\Delta} + f(t,u) = 0, \quad t \in \mathbb{T}, \tag{1.1}$$

where the function $f : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is continuous and $\mathbb{T}$ is a time scale (i.e., a nonempty closed subset of the real numbers; see [1, 2] and Section 2 below) that has a minimal element $t_0 > 0$ and is unbounded above, that is,

$$\lim_{n\to\infty} t_n = \infty \text{ for some set } \{t_n : n \in \mathbb{N}\} \subset \mathbb{T}. \tag{1.2}$$

In this paper, we offer conditions that ensure that for given $a, b \in \mathbb{R}$, there exists a solution u of (1.1) satisfying the asymptotic behavior

$$u(t) = at + b + o(1) \quad \text{as } t \longrightarrow \infty. \tag{1.3}$$

In [3], the general solution of the linear dynamic equation

$$u^{\Delta\Delta} + hu^{\sigma} = 0, \quad t \geq t_0, \tag{1.4}$$

where $u^{\sigma} = u \circ \sigma$, is proved to have the asymptotic representation (1.3) whenever

$$\int_{t_0}^{\infty} \sigma(s)|h(s)|\Delta s < \infty. \tag{1.5}$$

The study is extended in [4] to the investigation of oscillatory solutions for the more general dynamic equation

$$u^{\Delta\Delta} + h(t)u^{\Delta^{\sigma}} + g(t)(f \circ u^{\sigma}) = 0, \tag{1.6}$$

where the coefficients h and g satisfy some integral conditions. The existence of solutions converging to zero is considered in [5] for a linear nonhomogeneous dynamic equation in a self-adjoint form. In [6], the authors considered the nonlinear dynamic equation (1.1) for the time scales $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = k\mathbb{Z}$. They assumed the existence of some positive rd-continuous function h and a positive nondecreasing function g with $g(0) = 0$ and $g(u) > 0$ for $u > 0$ such that

$$|f(t,u)| \leq h(t)g\left(\frac{|u|}{t}\right) \tag{1.7}$$

with

$$\int^{\infty} h(t)\Delta t < \infty. \tag{1.8}$$

Then, they obtained the linear behavior of solutions (see [6, Theorem 4.1])

$$\lim_{t\to\infty} \frac{u(t)}{t} = a, \quad a \neq 0. \tag{1.9}$$

We mention that the dynamic equation (1.1) contains as special cases both differential ($\mathbb{T} = \mathbb{R}$) and difference ($\mathbb{T} = \mathbb{Z}$) equations of the form

$$u'' + f(t,u) = 0, \quad t \in \mathbb{R}, \qquad \Delta^2 u + f(t,u) = 0, \quad t \in \mathbb{Z}. \tag{1.10}$$

[7, Chapter 8] is entirely devoted to the asymptotic behavior of linear difference equations and contains some classical and fundamental results. The mth order nonlinear difference equation

$$\Delta^m u + \alpha_n f(u) = 0, \quad n \in \mathbb{N}, \tag{1.11}$$

where $m \in \mathbb{N}$, $\alpha : \mathbb{N} \to \mathbb{R}$, and $f : \mathbb{R} \to \mathbb{R}$, is studied in [8]. Sufficient conditions which guarantee existence of solutions converging to some limit or having certain types of asymptotic behavior are given. In the particular case of second-order difference equations ($m = 2$), a solution u_n is shown to have the asymptotic representation (see [8, Theorem 2, page 4692])

$$u_n = an + b + o(1), \quad n \geq n_0, \tag{1.12}$$

provided that

$$\sum_{n\geq 1} n|\alpha_n| < \infty \tag{1.13}$$

and both boundedness and uniform continuity of f are assumed. For further related results in the discrete case, we refer the reader to [7–15]. In the continuous case, that is, $\mathbb{T} = \mathbb{R}$, the study of the linear differential equation

$$u'' + h(t)u = 0, \tag{1.14}$$

where

$$\int^{\infty} t|h(t)|dt < \infty, \tag{1.15}$$

goes back to at least the works of Bôcher [16] and Dini [17] published at the beginning of the twentieth century, and it was also adapted by Bellman [18] in 1947, where the limit $\lim_{t\to\infty} u'(t) = a$ is obtained yielding by L'Hospital's rule the limiting behavior $\lim_{t\to\infty} u(t)/t = a$. The nonlinear differential equation

$$u'' + h(t)g(u) = 0 \tag{1.16}$$

has been initially studied by Bihari in 1957 under (1.15) with further growth assumptions upon the nonlinearity g (see [19]). The problem of the existence and extendability of solutions for nonlinear ordinary differential equations has been widely investigated during the last couple of years (see, e.g., [20–22]). Regarding the general theory of asymptotic integration of ODEs, more details and recent developments may be found in the works [23–30] and the references therein. Note also that (1.3) is referred to as Property (L) for the continuous case in [29], and it seems that this notion was introduced first in [31].

Inspired and motivated by the results obtained both for difference and differential equations, our aim in this paper is to extend some of these results to nonlinear dynamic equations on time scales. In order to obtain existence of global solutions and their asymptotic behavior at positive infinity, we consider an arbitrary time scale (unbounded above) and we will be interested in the asymptotic behavior (1.3) of a solution u of (1.1). Here a and b are real numbers. Considered in the spirit of the linear asymptotic conditions (1.9) and (1.12), the asymptotic development (1.3) will be used throughout this work. Indeed, (1.1) may be seen as a perturbation of the homogeneous equation $u^{\Delta\Delta} = 0$, the solutions of which are the straight lines $u(t) = at+b$. Taking into account the restrictions (1.5), (1.8), (1.9), (1.12), (1.13), and (1.15), our results will also depend heavily on the growth of the nonlinear function f with respect to the unknown u.

The setup of this paper is as follows. Section 2 contains some preliminary definitions and results from the theory of time scales. In Section 3, we only state the main theorems. These are four distinct results, each of which guarantees the existence of asymptotically linear solutions according to (1.3). For the first two results, Lipschitz-like hypotheses are assumed on the nonlinearity, the third one is concerned with the sublinear growth case, while the fourth and last one generalizes a result from [6]. In the first three theorems, existence of solutions asymptotic to any prescribed line is proved while in the last one, we describe linear behavior of some solution. Section 4 features some examples that illustrate the applicability of the main results. The proofs of the main results are presented in Section 5. They are based on the fixed point theorems of Banach, Boyd and Wong, the Leray-Schauder nonlinear alternative, and Schauder, respectively. We end this paper with some concluding remarks in Section 6.

2. Preliminaries

In this section, we gather some standard definitions, properties, and notations from the time scales calculus (see [1, 2]).

Definition 2.1. Define the *forward and backward jump operators* $\sigma : \mathbb{T} \to \mathbb{T}$ and $\rho : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf\{s > t : s \in \mathbb{T}\}, \qquad \rho(t) := \sup\{s < t : s \in \mathbb{T}\}, \tag{2.1}$$

respectively. A function $g : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* if it is continuous at points $t \in \mathbb{T}$ with $\sigma(t) = t$ and if it has finite left-sided limits at points $t \in \mathbb{T}$ with $\rho(t) = t$.

Definition 2.2. For $t \in \mathbb{T}$ and a function $g : \mathbb{T} \to \mathbb{R}$, define the *delta derivative* $g^{\Delta}(t)$ to be the number (if it exists) with the property that given $\varepsilon > 0$, there is a neighborhood U of t such that

$$|[g(\sigma(t)) - g(s)] - g^{\Delta}(t)[\sigma(t) - s]| < \varepsilon|\sigma(t) - s|, \quad \forall s \in U. \tag{2.2}$$

Define also the *second delta derivative* by $g^{\Delta\Delta} = (g^{\Delta})^{\Delta}$.

Definition 2.3. If G is an *antiderivative* of $g : \mathbb{T} \to \mathbb{R}$, that is, $G^{\Delta} = g$, then the *integral* of g is defined by

$$\int_a^b g(t)\Delta t = G(b) - G(a).z \tag{2.3}$$

Moreover, improper integrals are defined by

$$\int_a^{\infty} g(t)\Delta t = \lim_{T\to\infty}\int_a^T g(t)\Delta t. \tag{2.4}$$

Remark 2.4. A well-known existence theorem [1, Theorem 1.74] says that rd-continuous functions possess antiderivatives.

Remark 2.5. Note that in the case $\mathbb{T} = \mathbb{R}$, we have

$$\begin{gathered} \sigma(t) = \rho(t) = t, \qquad f^{\Delta}(t) = f'(t), \\ f^{\Delta\Delta}(t) = f''(t), \qquad \int_a^b f(t)\Delta t = \int_a^b f(t)dt, \end{gathered} \tag{2.5}$$

and in the case $\mathbb{T} = \mathbb{Z}$, we have

$$\begin{gathered} \sigma(t) = t + 1, \qquad \rho(t) = t - 1, \qquad f^{\Delta}(t) = \Delta f(t) := f(t+1) - f(t), \\ f^{\Delta\Delta}(t) = f(t+2) - 2f(t+1) + f(t), \qquad \int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t) \text{ if } a < b. \end{gathered} \tag{2.6}$$

Remark 2.6. In the theory of orthogonal polynomials and quantum calculus, an appropriate time scale is $\mathbb{T} = \overline{q^{\mathbb{Z}}} = \{q^k : k \in \mathbb{Z}\} \cup \{0\}$, where $q > 1$, and thus we have

$$\sigma(t) = qt, \qquad \rho(t) = (q-1)t, \qquad f^{\Delta}(t) = \frac{f(qt) - f(t)}{(q-1)t},$$
$$f^{\Delta\Delta}(t) = \frac{f(q^2t) - 2f(qt) + f(t)}{(q-1)^2t^2}, \qquad \int_1^t f(s)\Delta s = (q-1)\sum_{i=1}^{\log_q t-1} q^i f(q^i) \text{ if } t > 1, \tag{2.7}$$

(see [32, Lemma 2(ii)]). In this case, (1.1) is called a q-difference equation.

We conclude this section with an auxiliary result that will be needed frequently for the proofs of the main theorems in Section 5.

Lemma 2.7. *Let $g : \mathbb{T} \to [0,\infty)$ be rd-continuous and assume*

$$G^* := \int_{t_0}^{\infty} (\sigma(s) - t_0)g(s)\Delta s < \infty. \tag{2.8}$$

Then,

(i) $\int_t^{\infty} (\sigma(s) - t)g(s)\Delta s \le G^*$, *for all* $t \ge t_0$,

(ii) $\int_t^{\infty} (\sigma(s) - t)g(s)\Delta s \to 0$ *as* $t \to \infty$,

(iii) $\int_{t_0}^{\infty} g(s)\Delta s < \infty$.

Proof. For fixed $T \in \mathbb{T}$ and $t \in (t_0, T)$, [1, Theorem 1.117(ii)] can be used to show that

$$G(t) := \int_t^T (\sigma(s) - t)g(s)\Delta s \text{ implies } G^{\Delta}(t) = -\int_t^T g(s)\Delta s \le 0 \tag{2.9}$$

so that G is nonincreasing and hence $G(t) \le G(t_0) \le G^*$. Now, let $T \to \infty$ to obtain (i). Next,

$$0 \le \int_t^{\infty} (\sigma(s) - t)g(s)\Delta s \le \int_t^{\infty} (\sigma(s) - t_0)g(s)\Delta s \longrightarrow 0 \quad \text{as } t \longrightarrow \infty \tag{2.10}$$

so that (ii) holds. Finally, for sufficiently large $T \in \mathbb{T}$, let $t_* \in (t_0 + 1, T] \cap \mathbb{T}$ so that

$$\begin{aligned}\int_{t_0}^T g(s)\Delta s &= \int_{t_*}^T g(s)\Delta s + \int_{t_0}^{t_*} g(s)\Delta s \\ &\le \int_{t_*}^T (\sigma(s) - t_0)g(s)\Delta s + \int_{t_0}^{t_*} g(s)\Delta s \\ &\le G^* + \int_{t_0}^{t_*} g(s)\Delta s.\end{aligned} \tag{2.11}$$

Letting $T \to \infty$, we derive (iii). □

3. Main results

Throughout this paper, for a given nonnegative rd-continuous function $h:\mathbb{T}\to\mathbb{R}$, we consider (when they exist) the constants

$$H^* := \int_{t_0}^{\infty}(\sigma(s)-t_0)h(s)\Delta s, \qquad H^{**} := \int_{t_0}^{\infty}(\sigma(s)-t_0)sh(s)\Delta s. \tag{3.1}$$

We are now in position to state the four main results of this paper.

Theorem 3.1. *Assume*

$$\exists L>0 \text{ with} \int_{t_0}^{\infty}(\sigma(s)-t_0)|f(s,0)|\Delta s \le L, \tag{3.2}$$

$$|f(t,u_1)-f(t,u_2)| \le h(t)|u_1-u_2|, \quad \forall t\ge t_0,\ u_1,u_2\in\mathbb{R}, \tag{3.3}$$

$$H^*<1, \qquad H^{**}<\infty. \tag{3.4}$$

Then, for all $a,b\in\mathbb{R}$, (1.1) *has a solution* u *on* $[t_0,\infty)$ *satisfying* (1.3).

Theorem 3.2. *Assume* $H^*<\infty$, (3.2), *and*

$$\left|f(t,u_1)-f(t,u_2)\right| \le h(t)\frac{\left|u_1-u_2\right|}{H^*+\left|u_1-u_2\right|}, \quad \forall t\ge t_0,\ u_1,u_2\in\mathbb{R}. \tag{3.5}$$

Then, the conclusion of Theorem 3.1 holds true.

It is clear that (3.5) is stronger than (3.3) of Theorem 3.1. However, the assumption $H^*<\infty$ in Theorem 3.2 is weaker than the restriction $H^*<1$ in (3.4) and no further restriction is made on the second integral H^{**}.

Theorem 3.3. *Assume* (3.4) *and*

$$|f(t,u)| \le h(t)|u|, \quad \forall (t,u)\in\mathbb{T}\times\mathbb{R}. \tag{3.6}$$

Then, the conclusion of Theorem 3.1 holds true.

In the last existence result, we are rather concerned with existence of at least one solution asymptotic to a specified line.

Theorem 3.4. *Assume* $H^*<\infty$ *and suppose there exists a nondecreasing function* $g:[0,\infty)\to[0,\infty)$ *such that*

$$|f(t,u)| \le h(t)g\left(\frac{|u|}{t}\right), \quad \forall (t,u)\in\mathbb{T}\times\mathbb{R}. \tag{3.7}$$

Suppose also that there exist $a,b\in\mathbb{R}$, $K>0$, *and* $t^*\ge t_0$ *such that*

$$H^*\sup\left\{g(s):0\le s\le \frac{K}{t^*}+\frac{|b|}{t^*}+|a|\right\} \le K. \tag{3.8}$$

Then, (1.1) *has a solution* u *on* $[t^*,\infty)$ *satisfying* (1.3).

4. Examples

In this section, we illustrate each of the four theorems given in Section 3 by means of an example.

Example 4.1 (application of Theorem 3.1). Let $\gamma : \mathbb{R} \to \mathbb{R}$ satisfy

$$\gamma(0) = 1, \qquad \left|\gamma(u_1) - \gamma(u_2)\right| \le \left|u_1 - u_2\right|, \quad \forall u_1, u_2 \in \mathbb{R}, \tag{4.1}$$

for example, $\gamma(u) = \arctan(u) + 1$. By Theorem 3.1, for any $a, b \in \mathbb{R}$, the difference equation

$$\Delta^2 u + \frac{\gamma(u)}{4t^2 2^t} = 0, \quad t \in \mathbb{T} = \mathbb{N}, \tag{4.2}$$

has at least one solution u on $\mathbb{N}$ satisfying (1.3). Indeed, let

$$h(t) = \frac{1}{4t^2 2^t}, \qquad f(t,u) = h(t)\gamma(u), \qquad L = \frac{1}{4}. \tag{4.3}$$

According to (4.1), clearly (3.3) is satisfied as well as (3.2):

$$\int_1^\infty (\sigma(s) - 1)|f(s,0)|\Delta s = \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1}{2}\right)^n \le \frac{1}{4}\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{4} = L. \tag{4.4}$$

Moreover,

$$0 < H^* \le H^{**} = \int_1^\infty (\sigma(s) - 1)sh(s)\Delta s = \sum_{n=1}^{\infty} n^2 \frac{1}{4n^2}\left(\frac{1}{2}\right)^n = \frac{1}{4} < 1 \tag{4.5}$$

so that (3.4) also holds.

Example 4.2 (application of Theorem 3.2). Let $\mathbb{T}$ be any time scale which is unbounded above such that its graininess is bounded above. Suppose also $1 \in \mathbb{T}$. Let $p > 1$. By [2, Example 5.72],

$$M := \int_1^\infty \frac{\Delta s}{s^p} < \infty. \tag{4.6}$$

Let $\gamma : \mathbb{R} \to \mathbb{R}$ be such that

$$\gamma(0) = 0, \qquad \left|\gamma(u_1) - \gamma(u_2)\right| \le \frac{\left|u_1 - u_2\right|}{M + \left|u_1 - u_2\right|}, \quad \forall u_1, u_2 \in \mathbb{R}, \tag{4.7}$$

for example, $\gamma(u) = |u|/(M + |u|)$:

$$\begin{aligned} |\gamma(u_1) - \gamma(u_2)| &= \frac{M\left|\left|u_1\right| - \left|u_2\right|\right|}{\left(M + \left|u_1\right|\right)\left(M + \left|u_2\right|\right)} \\ &\le \frac{M\left|u_1 - u_2\right|}{M^2 + M\left(\left|u_1\right| + \left|u_2\right|\right)} \\ &\le \frac{\left|u_1 - u_2\right|}{M + \left|u_1 - u_2\right|} \\ &= \gamma\left(u_1 - u_2\right). \end{aligned} \tag{4.8}$$

By Theorem 3.2, for any $a, b \in \mathbb{R}$, the dynamic equation

$$u^{\Delta\Delta} + \frac{\gamma(u)}{t^p \sigma(t)} = 0, \quad t \geq 1, \ t \in \mathbb{T}, \tag{4.9}$$

has at least one solution u on $\mathbb{T}$ satisfying (1.3). To prove this, let

$$h(t) = \frac{1}{t^p \sigma(t)}, \qquad f(t, u) = h(t)\gamma(u), \qquad L = 1. \tag{4.10}$$

Now note that

$$H^* = \int_1^\infty (\sigma(s) - 1)h(s)\Delta s \leq \int_1^\infty \sigma(s)h(s)\Delta s = \int_1^\infty \frac{\Delta s}{s^p} = M < \infty. \tag{4.11}$$

Thus, together with (4.7), we clearly have (3.5) and (3.2).

Example 4.3 (application of Theorem 3.3). For any $a, b \in \mathbb{R}$, the dynamic equation

$$u^{\Delta\Delta} + h(t)\ln(1 + |u|) = 0, \quad t \geq t_0, \ t \in \mathbb{T}, \tag{4.12}$$

has a solution satisfying the asymptotic representation (1.3) provided (3.4) is fulfilled, for example, $h(t) = 1/(4t^2 2^t)$. This follows directly from Theorem 3.3.

Example 4.4 (application of Theorem 3.4). Let $q > 1$. By Theorem 3.4, the q-difference equation

$$u^{\Delta\Delta} + \frac{e^{|u|/t}}{(t \log_q t)^2} = 0, \quad t \in q^{\mathbb{N}}, \tag{4.13}$$

has at least one solution u on $q^{\mathbb{N}}$ which behaves as $u(t) = t$ when $t \to \infty$. In fact, setting

$$h(t) = \frac{1}{(t \log_q t)^2}, \qquad g(s) = e^s, \qquad f(t, u) = h(t)g\left(\frac{|u|}{t}\right), \tag{4.14}$$

we can first see (refer to Remark 2.6) that the integral

$$H^* = \int_q^\infty (\sigma(s) - q)h(s)\Delta s \leq q\int_q^\infty \frac{\Delta s}{s(\log_q s)^2}\Delta s = q(q-1)\sum_{n=1}^\infty \frac{1}{n^2} \tag{4.15}$$

converges. Moreover, (3.7) is clearly satisfied. Let $b = 0$ and $K = H^* e^{2+|a|}$. Since $\mathbb{T}$ is unbounded above, there exists $\alpha \geq 1$ such that $t^* = \alpha K \in \mathbb{T}$. Then,

$$\begin{aligned} H^* \sup\left\{g(s) : 0 \leq s \leq \frac{K}{t^*} + \frac{|b|}{t^*} + |a|\right\} &= H^* \sup\left\{e^s : 0 \leq s \leq 1 + |a| + \frac{1}{\alpha}\right\} \\ &= H^* e^{1+|a|+1/\alpha} \leq H^* e^{2+|a|} = K \end{aligned} \tag{4.16}$$

so that (3.8) holds true as well.

5. Proofs

For any $a, b \in \mathbb{R}$, consider the transformation $v(t) = u(t) - at - b$. Then, u is a solution of (1.1) if and only if v is a solution of

$$v^{\Delta\Delta}(t) + f(t, v(t) + at + b) = 0, \quad t \geq t_0. \tag{5.1}$$

Consider the space

$$C_0 := C_0([t_0, \infty)_{\mathbb{T}}) = \{v \in C([t_0, \infty)_{\mathbb{T}}, \mathbb{R}) : \lim_{t\to\infty} v(t) = 0\}, \tag{5.2}$$

where $[t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$. Endowed with the supnorm $\|v\| = \sup\{|v(t)| : t \geq t_0\}$, C_0 is a Banach space. Define the mapping A for $v \in C_0$ (if the improper integral exists) by

$$Av(t) = \int_t^\infty (t - \sigma(s)) f(s, v(s) + as + b)\Delta s. \tag{5.3}$$

It is clear that fixed points of the operator A are solutions of (5.1). Observe also that Av is continuous if v is continuous, since then (note that f is assumed to be continuous and that σ is rd-continuous) an rd-continuous function is integrated (note that this is possible due to Remark 2.4) which yields a delta differentiable and hence a continuous function; see [1, Theorem 1.16(i)]. Now, we are ready to prove the four results giving not only asymptotic behavior but also existence of global solutions.

5.1. Result based on the Banach fixed point theorem

The proof of Theorem 3.1 relies on the Banach fixed point theorem, which we recall here for completeness.

Theorem 5.1 (the Banach fixed point theorem). *Let X be a Banach space and let $A : X \to X$ be a contraction. Then, A has a unique fixed point in X.*

Proof of Theorem 3.1. Let $v \in C_0$. First, we use (3.3) to find

$$|f(s, v(s) + as + b) - f(s, 0)| \leq h(s)|v(s) + as + b| \leq h(s)(\|v\| + |a|s + |b|) \tag{5.4}$$

so that

$$\begin{aligned} |Av(t)| &\leq \int_t^\infty (\sigma(s) - t)|f(s, v(s) + as + b) - f(s, 0)|\Delta s + \int_t^\infty (\sigma(s) - t)|f(s, 0)|\Delta s \\ &\leq (\|v\| + |b|)\int_t^\infty (\sigma(s) - t)h(s)\Delta s + |a|\int_t^\infty (\sigma(s) - t)sh(s)\Delta s + \int_t^\infty (\sigma(s) - t)|f(s, 0)|\Delta s, \end{aligned} \tag{5.5}$$

which tends to 0 as $t \to \infty$ when applying (3.2) and (3.4) together with Lemma 2.7(ii) three times. Hence, $Av \in C_0$ and therefore $A : C_0 \to C_0$. Moreover, passing to the supremum above and using Lemma 2.7(i) three times, we also find that A is indeed well defined and that

$$\|Av\| \leq \left(\|v\| + |b|\right)H^{*} + |a|H^{**} + L. \tag{5.6}$$

Next, let $v_1, v_2 \in C_0$. With (3.3), we get

$$\left|f\left(s, v_1(s) + as + b\right) - f\left(s, v_2(s) + as + b\right)\right| \leq h(s)\left\|v_1 - v_2\right\| \tag{5.7}$$

so that

$$|Av_1(t) - Av_2(t)| \leq \left\|v_1 - v_2\right\| \int_t^{\infty} (\sigma(s) - t)h(s)\Delta s \leq H^{*}\left\|v_1 - v_2\right\|, \tag{5.8}$$

where we used the first part of (3.4) together with Lemma 2.7(i). Passing to the supremum, we get

$$\left\|Av_1 - Av_2\right\| \leq H^{*}\left\|v_1 - v_2\right\|, \tag{5.9}$$

and due to the first part of (3.4), A is a contraction. According to Theorem 5.1, A has a fixed point in C_0. □

5.2. Result based on the Boyd and Wong fixed point theorem

To prove Theorem 3.2, we employ the Boyd and Wong fixed point theorem from [33], which extends Theorem 5.1 and is recalled here (together with a pertinent definition) for completeness.

Definition 5.2. Let X be a Banach space and let $A : X \to X$ be a mapping. A is said to be a *nonlinear contraction* if there exists a continuous nondecreasing function $\psi : [0,\infty) \to [0,\infty)$ such that $\psi(0) = 0$ and $\psi(x) < x$ for all $x > 0$ with the property

$$\|Au - Av\| \leq \psi\left(\|u - v\|\right), \quad \forall u, v \in X. \tag{5.10}$$

Theorem 5.3 (the Boyd and Wong fixed point theorem). *Let X be a Banach space and let $A : X \to X$ be a nonlinear contraction. Then, A has a unique fixed point in X.*

Proof of Theorem 3.2. Let $v \in C_0$. From (3.5), we infer the estimates

$$\left|f\left(s, v(s) + as + b\right) - f(s, 0)\right| \leq h(s)\frac{|v(s) + as + b|}{H^{*} + |v(s) + as + b|} \leq h(s) \tag{5.11}$$

so that

$$\begin{aligned} |Av(t)| &\leq \int_t^{\infty} (\sigma(s) - t)\left|f\left(s, v(s) + as + b\right) - f(s, 0)\right|\Delta s + \int_t^{\infty} (\sigma(s) - t)\left|f(s, 0)\right|\Delta s \\ &\leq \int_t^{\infty} (\sigma(s) - t)h(s)\Delta s + \int_t^{\infty} (\sigma(s) - t)\left|f(s, 0)\right|\Delta s, \end{aligned} \tag{5.12}$$

which tends to 0 as $t \to \infty$ when applying (3.2) and $H^* < \infty$ together with Lemma 2.7(ii) twice. Hence, $Av \in C_0$ and therefore $A : C_0 \to C_0$. Furthermore, using Lemma 2.7(i) twice, we find that A is well defined and that

$$\|Av\| \leq H^* + L. \tag{5.13}$$

We introduce a continuous nondecreasing function $\psi : [0, \infty) \to [0, \infty)$ satisfying $\psi(0) = 0$ and $\psi(x) < x$, for all $x > 0$ by

$$\psi(x) = \frac{H^* x}{H^* + x}, \quad \forall x \geq 0. \tag{5.14}$$

Let $v_1, v_2 \in C_0$. Assumption (3.5) yields that

$$\left| f(s, v_1(s) + as + b) - f(s, v_2(s) + as + b) \right| \leq \frac{h(s)}{H^*} \psi(\|v_1 - v_2\|) \tag{5.15}$$

so that

$$\left| Av_1(t) - Av_2(t) \right| \leq \psi(\|v_1 - v_2\|) \int_t^\infty (\sigma(s) - t) \frac{h(s)}{H^*} \Delta s \leq \psi(\|v_1 - v_2\|), \tag{5.16}$$

where we used $0 < H^* < \infty$ together with Lemma 2.7(i). Passing to the supremum, we get

$$\|Av_1 - Av_2\| \leq \psi(\|v_1 - v_2\|), \tag{5.17}$$

and by Definition 5.2, A is a nonlinear contraction. According to Theorem 5.3, A has a fixed point in C_0. □

5.3. Result based on the Leray-Schauder nonlinear alternative

The celebrated Leray-Schauder nonlinear alternative (see, e.g., [34]) is fundamental in the proof of Theorem 3.3. Recall that an operator is said to be *completely continuous* if it is continuous and maps bounded sets into relatively compact sets.

Theorem 5.4 (the Leray-Schauder nonlinear alternative). *Let X be a Banach space, $\Omega \subset X$ bounded and open, $0 \in \Omega$, and $A : \overline{\Omega} \to X$ a completely continuous operator. Then, either there exist $u \in \partial\Omega$ and $\lambda > 1$ such that $Au = \lambda u$ or A has a fixed point in $\overline{\Omega}$.*

We need the time scales version of the compactness criterion for subsets of C_0 which is due to Avramescu for the case $\mathbb{T} = \mathbb{R}$ (see [20, 35]).

Proposition 5.5. *Assume that the subset $B \subset C_0$ has the following properties:*

(i) *B is uniformly bounded, that is, there exists a constant $N > 0$ with*

$$|u(t)| \leq N, \quad \forall t \geq t_0, \; u \in B, \tag{5.18}$$

(ii) *B is equicontinuous, that is, for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ with*

$$\left|u(t_1) - u(t_2)\right| < \varepsilon, \quad \forall t_1, t_2 \geq t_0, \quad |t_1 - t_2| < \delta(\varepsilon), \quad u \in B, \tag{5.19}$$

(iii) *B is equiconvergent, that is, for every $\varepsilon > 0$ there exists $t_*(\varepsilon) > t_0$ with*

$$|u(t)| < \varepsilon, \quad \forall t \geq t_*(\varepsilon),\ u \in B. \tag{5.20}$$

Then, B is relatively compact.

Proof. Following [36, proof of Proposition 2.2], consider an interval $[\alpha, \beta]_{\mathbb{T}} = [\alpha, \beta] \cap \mathbb{T}$, $\alpha < \beta$, and $C = C([\alpha, \beta]_{\mathbb{T}}, \mathbb{R})$. The spaces C_0 and C are isomorphic by the mapping Φ defined by

$$(\Phi x)(t) = \begin{cases} x(\varphi(t)), & \text{if } t \in [\alpha, \beta)_{\mathbb{T}}, \\ x(\infty), & \text{if } t = \beta, \end{cases} \tag{5.21}$$

where $\varphi : [\alpha, \beta)_{\mathbb{T}} \to \mathbb{T}$ is a continuous, strictly nondecreasing function with $\lim_{t \to \beta^-} \varphi(t) = \infty$. From (ii) and (iii), B is equicontinuous in C_0. Then, $\Phi(B)$ is equicontinuous and uniformly bounded in C. By the Arzelà-Ascoli theorem for time scales [37, Lemma 2.6], we conclude that $\Phi(B)$ is relatively compact in C, which completes the proof. □

Proof of Theorem 3.3. Define

$$\beta := |a|H^{**} + |b|H^*, \qquad m := \begin{cases} \beta(1 - H^*)^{-1}, & \text{if } \beta \neq 0, \\ 1, & \text{if } \beta = 0. \end{cases} \tag{5.22}$$

We also introduce

$$\Omega := \left\{ v \in C_0 : \|v\| < m \right\} \tag{5.23}$$

and note that $\Omega \subset C_0$ is open and by (3.4) satisfies $0 \in \Omega$ since $m > 0$. Let $v \in \overline{\Omega}$. From (3.6), we get

$$\left|f(s, v(s) + as + b)\right| \leq h(s)\left|v(s) + as + b\right| \leq h(s)\left(m + |b| + |a|s\right) \tag{5.24}$$

so that

$$\left|Av(t)\right| \leq \left(m + |b|\right)\int_t^\infty (\sigma(s) - t)h(s)\Delta s + |a|\int_t^\infty (\sigma(s) - t)sh(s)\Delta s, \tag{5.25}$$

which tends to 0 as $t \to \infty$ when applying (3.4) together with Lemma 2.7(ii) twice. This means that $A(\overline{\Omega})$ is equiconvergent (observe Proposition 5.5(iii)) and that $Av \in C_0$ and therefore $A : \overline{\Omega} \to C_0$. By Lemma 2.7(i), A is well defined, and passing to the supremum in (5.25), we get

$$\|Av\| \leq \left(m + |b|\right)H^* + |a|H^{**} = mH^* + \beta. \tag{5.26}$$

We conclude that $A(\overline{\Omega})$ is uniformly bounded (observe Proposition 5.5(i)). Now use (5.24) again to deduce

$$\left|(Av)^{\Delta}(t)\right| = \left|\int_t^\infty f(s,v(s)+as+b)\Delta s\right| \le (m+|b|)\int_{t_0}^\infty h(s)\Delta s + |a|\int_{t_0}^\infty sh(s)\Delta s. \tag{5.27}$$

Using (3.4) and Lemma 2.7(iii) twice, we find that the right-hand side above is equal to a finite constant, say R. Thus,

$$\left|Av(t_2) - Av(t_1)\right| \le R\left|t_2 - t_1\right| \longrightarrow 0 \quad \text{as } t_2 \longrightarrow t_1 \tag{5.28}$$

and so $A(\overline{\Omega})$ is equicontinuous (observe Proposition 5.5(ii)). Altogether, by Proposition 5.5, $A(\overline{\Omega})$ is relatively compact.

It remains to prove that A is continuous. Let $v \in \overline{\Omega}$ and let $(v_n) \subset \overline{\Omega}$ be a sequence converging strongly to the limit v, that is, $\|v_n - v\| \to 0$ as $n \to \infty$. By (5.24), we have the estimate

$$\left|f(s,v_n(s)+as+b)\right| \le (m+|b|+|a|s)h(s). \tag{5.29}$$

Since $H^* < \infty$ and $H^{**} < \infty$ and because of Lemma 2.7(i), we infer from the Lebesgue dominated convergence theorem (see [38] and [37, Theorem 10.1]) that

$$\lim_{n\to\infty}\int_t^\infty (\sigma(s)-t)f(s,v_n(s)+as+b)\Delta s = \int_t^\infty (\sigma(s)-t)f(s,v(s)+as+b)\Delta s. \tag{5.30}$$

Then, $Av_n \to Av$ pointwise as $n \to \infty$. In addition, $A(\overline{\Omega})$ is relatively compact. Then, there exists a subsequence (Av_{n_k}) of (Av_n) converging strongly to a certain $w \in C_0$. As the strong convergence implies the pointwise convergence keeping the limit function, we find that $w = Av$. Now, Av_n converges strongly to Av as $n \to \infty$ and thus the mapping A is continuous.

Altogether, $A : \overline{\Omega} \to C_0$ is completely continuous. Let $v \in \partial\Omega$ and $\lambda > 1$ be such that $Av = \lambda v$. Then, using (5.26),

$$\lambda m = \lambda\|v\| = \|Av\| \le H^* m + \beta \tag{5.31}$$

so that

$$1 < \lambda \le H^* + \frac{\beta}{m} = \begin{Bmatrix} H^*, & \text{if } \beta = 0 \\ 1, & \text{if } \beta \ne 0 \end{Bmatrix} \le 1, \tag{5.32}$$

a contradiction. Hence, by Theorem 5.4, A has a fixed point in $\overline{\Omega}$. □

5.4. Result based on the Schauder fixed point theorem

To prove Theorem 3.4, we appeal to the Schauder fixed point theorem (see, e.g., [34]).

Theorem 5.6 (the Schauder fixed point theorem). *Let X be a Banach space and let $B \subset X$ be nonempty, bounded, closed, and convex. Let $A : B \to B$ be a completely continuous operator. Then, A has a fixed point in B.*

Proof of Theorem 3.4. Consider the closed ball $B := \{v \in C_0 : \|v\| \le K\}$ and define

$$\theta := \sup\left\{g(u) : 0 \le u \le \frac{K}{t^*} + \frac{|b|}{t^*} + |a|\right\}. \tag{5.33}$$

Let $v \in B$. By (3.7), we find

$$\left|f(s, v(s) + as + b)\right| \le h(s) g\left(\frac{|v(s) + as + b|}{s}\right) \le \theta h(s), \quad s \ge t^*, \tag{5.34}$$

since for $s \ge t^*$ we have

$$0 \le \frac{\left|v(s) + as + b\right|}{s} \le \frac{\|v\| + |b|}{s} + |a| \le \frac{K + |b|}{s} + |a| \le \frac{K + |b|}{t^*} + |a|. \tag{5.35}$$

By (5.34), for $t \ge t^*$,

$$\left|Av(t)\right| \le \int_t^\infty (\sigma(s) - t)\left|f(s, v(s) + as + b)\right|\Delta s \le \theta \int_t^\infty (\sigma(s) - t) h(s) \Delta s, \tag{5.36}$$

which tends to 0 as $t \to \infty$ when applying $H^* < \infty$ together with Lemma 2.7(ii). This means that $A(B)$ is equiconvergent (observe Proposition 5.5(iii)) and that $Av \in C_0$ and therefore $A : B \to C_0$. Thanks to Lemma 2.7(i), we also get that A is well defined and, passing to the supremum above, we have

$$\|Av\| \le \theta H^* \le K, \tag{5.37}$$

where we used (3.8). We conclude that $A : B \to B$ and that $A(B)$ is uniformly bounded (observe Proposition 5.5(i)). Next, let $t_1, t_2 \in \mathbb{T}$ be such that $t_2 \ge t_1 \ge t^*$. Then,

$$\begin{aligned}
&Av(t_2) - Av(t_1) \\
&= \int_{t_2}^\infty (t_2 - \sigma(s)) f(s, v(s) + as + b)\Delta s - \int_{t_1}^\infty (t_1 - \sigma(s)) f(s, v(s) + as + b)\Delta s \\
&= \int_{t_2}^\infty \{(t_2 - \sigma(s)) - (t_1 - \sigma(s))\} f(s, v(s) + as + b)\Delta s - \int_{t_1}^{t_2} (t_1 - \sigma(s)) f(s, v(s) + as + b)\Delta s \\
&= (t_2 - t_1)\int_{t_2}^\infty f(s, v(s) + as + b)\Delta s + \int_{t_1}^{t_2} (\sigma(s) - t_1) f(s, v(s) + as + b)\Delta s.
\end{aligned} \tag{5.38}$$

Using again (5.34), we find

$$\left|Av(t_2) - Av(t_1)\right| \le (t_2 - t_1)\theta\int_{t_0}^{\infty} h(s)\Delta s + \theta\int_{t_1}^{t_2} \sigma(s)h(s)\Delta s, \tag{5.39}$$

which tends to zero as $t_2 \to t_1$ due to $H^* < \infty$ and Lemma 2.7(iii) (for the second integral, use that σh is rd-continuous and hence has an antiderivative Q by Remark 2.4, and thus this integral equals to $Q(t_2) - Q(t_1)$ and Q is continuous). Therefore, $A(B)$ is equicontinuous (observe Proposition 5.5(ii)). Altogether, by Proposition 5.5, $A(B)$ is relatively compact. As in the proof of Theorem 3.3, we may check that A is continuous. Thus, $A : B \to B$ is completely continuous. According to Theorem 5.6, A has a fixed point in B. □

6. Concluding remarks

In this work, specific results regarding the asymptotic behavior of the nonlinear dynamic equation (1.1) have been obtained, extending some known results in the theories of difference and differential equations, for example to q-difference equations (see Remark 2.6) and to other cases of arbitrary time scales. Not only did our work extend the continuous and the discrete, but it also unified those two important cases and illuminated the common grounds of the corresponding differential and difference equations. As a fundamental contribution to the now well-established theory of time scales, it is hoped that our results will advance the area and stimulate future research on this and related topics. For example, the more general case of delta-derivative depending nonlinearity $f = f(t, u, u^{\Delta})$ may be treated in an analogous manner yielding the asymptotic behavior (1.3). For this purpose, additional restrictions on the growth of f with respect to the derivative u^{Δ} need to be assumed. The space C_0 introduced in Section 5 is then extended to a space involving also the limit at infinity of the delta derivative; accordingly, a new compactness criterion is required. Also notice that the most informative condition is (3.7) which shows how the nonlinearity grows in terms of the ratio u/t.

Apart from Theorem 3.4, Theorems 3.1, 3.2, and 3.3 are concerned with what is usually called the inverse problem of seeking a solution asymptotic to a given line (see [28, 29]). We point out that further to the asymptotic behavior, these theorems also provide existence of solutions to initial value problems for the dynamic equation (1.1). Moreover, the existence of solutions with behavior described by (1.3) does not mean that all solutions behave in the same manner as shown in the nonlinear ordinary differential equation $u'' = (3/t^5)u^2$, $t \ge 1$ (see also [28, Section 5]). Indeed, this equation admits by Theorem 3.1 a solution having Property (L) while the solution $u(t) = 2t^3$ has not.

Finally, we mention that similar Bihari-type existence results of solutions which can be expanded asymptotically as $u(t) = P(t) + o(t)$ near positive infinity may also be obtained for the nonhomogeneous dynamic equation

$$u^{\Delta\Delta} + f(t, u) = p(t), \quad t \in \mathbb{T}, \tag{6.1}$$

where $P^{\Delta\Delta} = p$ and p is a polynomial. For the case $\mathbb{T} = \mathbb{R}$, we refer to [24, Section 8, Theorem 18] (see also [30]).

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