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Research Article

Eigenvalue Problems for p-Laplacian Functional Dynamic Equations on Time Scales

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This paper is concerned with the existence and nonexistence of positive solutions of the p-Laplacian functional dynamic equation on a time scale, $[\phi_p(x^{\Delta}(t))]^{\nabla} + \lambda a(t) f(x(t), x(u(t))) = 0$, $t \in (0, T)$, $x_0(t) = \psi(t)$, $t \in [-\tau, 0]$, $x(0) - B_0(x^{\Delta}(0)) = 0$, $x^{\Delta}(T) = 0$. We show that there exists a $\lambda^* > 0$ such that the above boundary value problem has at least two, one, and no positive solutions for $0 < \lambda < \lambda^*$, $\lambda = \lambda^*$ and $\lambda > \lambda^*$, respectively.

1. Introduction

Let $\mathbb{T}$ be a closed nonempty subset of $\mathbb{R}$, and let $\mathbb{T}$ have the subspace topology inherited from the Euclidean topology on $\mathbb{R}$. In some of the current literature, $\mathbb{T}$ is called a time scale (please see [1, 2]). For notation, we will use the convention that, for each interval J of $\mathbb{R}$, J will denote time-scale interval, that is, $J := J \cap \mathbb{T}$.

In this paper, let $\mathbb{T}$ be a time scale such that $-\tau$, 0, $T \in \mathbb{T}$. We are concerned with the existence of positive solutions of the p-Laplacian dynamic equation on a time scale

$$\begin{gathered}\left[\phi_p\left(x^{\Delta}(t)\right)\right]^{\nabla} + \lambda a(t) f\left(x(t), x\left(\mu(t)\right)\right) = 0, \quad t \in (0, T), \\ x_0(t) = \psi(t), \quad t \in [-\tau, 0], \quad x(0) - B_0\left(x^{\Delta}(0)\right) = 0, \quad x^{\Delta}(T) = 0,\end{gathered} \tag{1.1}$$

where $\phi_p(u)$ is the p-Laplacian operator, that is, $\phi_p(u) = |u|^{p-2}u$, $p > 1$, $(\phi_p)^{-1}(u) = \phi_q(u)$, where $1/p + 1/q = 1$.

(H1) The function $f : (\mathbb{R}^+)^2 \to \mathbb{R}^+$ is continuous and nondecreasing about each element; $f(0, 0) \geq c > 0$.

(H2) The function $a : \mathbb{T} \to \mathbb{R}^+$ is left dense continuous (i.e., $a \in C_{\mathrm{ld}}(\mathbb{T}, \mathbb{R}^+)$) and does not vanish identically on any closed subinterval of $[0,T]$. Here $C_{\mathrm{ld}}(\mathbb{T}, \mathbb{R}^+)$ denotes the set of all left dense continuous functions from $\mathbb{T}$ to $\mathbb{R}^+$.

(H3) $\varphi : [-\tau, 0] \to \mathbb{R}^+$ is continuous and $\tau > 0$.

(H4) $\mu : [0,T] \to [-\tau, T]$ is continuous, $\mu(t) \le t$ for all t.

(H5) $B_0 : \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing; $B_0(ks) = kB_0(s)$, $k \in \mathbb{R}^+$ and satisfies that there exist $\beta \ge \delta > 0$ such that

$$\delta s \le B_0(s) \le \beta s \quad \text{for } s \in \mathbb{R}^+. \tag{1.2}$$

(H6) $\lim_{x \to \infty} f(x, \varphi(s))/x^{p-1} = \infty$ uniformly in $s \in [-\tau, 0]$.

p-Laplacian problems with two-, three-, m-point boundary conditions for ordinary differential equations and finite difference equations have been studied extensively, for example, see [1–4] and references therein. However, there are not many concerning the p-Laplacian problems on time scales, especially for p-Laplacian functional dynamic equations on time scales.

The motivations for the present work stems from many recent investigations in [5–10] and references therein. Especially, Kaufmann and Raffoul [7] considered a nonlinear functional dynamic equation on a time scale and obtained sufficient conditions for the existence of positive solutions, Li and Liu [10] studied the eigenvalue problem for second-order nonlinear dynamic equations on time scales. In this paper, our results show that the number of positive solutions of (1.1) is determined by the parameter λ. That is to say, we prove that there exists a $\lambda^* > 0$ such that (1.1) has at least two, one, and no positive solutions for $0 < \lambda < \lambda^*$, $\lambda = \lambda^*$ and $\lambda > \lambda^*$, respectively.

For convenience, we list the following well-known definitions which can be found in [11–13] and the references therein.

Definition 1.1. For $t < \sup \mathbb{T}$ and $r > \inf \mathbb{T}$, define the forward jump operator σ and the backward jump operator ρ, respectively, as

$$\sigma(t) = \inf\{\tau \in \mathbb{T} \mid \tau > t\} \in \mathbb{T}, \qquad \rho(r) = \sup\{\tau \in \mathbb{T} \mid \tau < r\} \in \mathbb{T} \quad \forall t, r \in \mathbb{T}. \tag{1.3}$$

If $\sigma(t) > t$, t is said to be right scattered, and if $\rho(r) < r$, r is said to be left scattered. If $\sigma(t) = t$, t is said to be right dense, and if $\rho(r) = r$, r is said to be left dense. If $\mathbb{T}$ has a right-scattered minimum m, define $\mathbb{T}_\kappa = \mathbb{T} - \{m\}$; otherwise set $\mathbb{T}_\kappa = \mathbb{T}$. If $\mathbb{T}$ has a left-scattered maximum M, define $\mathbb{T}^\kappa = \mathbb{T} - \{M\}$; otherwise set $\mathbb{T}^\kappa = \mathbb{T}$.

Definition 1.2. For $x : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$, define the deltaderivative of $x(t)$, $x^\Delta(t)$, to be the number (when it exists), with the property that, for any $\varepsilon > 0$, there is a neighborhood U of t such that

$$\left| [x(\sigma(t)) - x(s)] - x^\Delta(t)[\sigma(t) - s] \right| < \varepsilon |\sigma(t) - s| \quad \forall s \in U. \tag{1.4}$$

For $x : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_\kappa$, define the nabla derivative of $x(t)$, $x^\nabla(t)$, to be the number (when it exists), with the property that, for any $\varepsilon > 0$, there is a neighborhood V of t such that

$$\left| [x(\rho(t)) - x(s)] - x^\nabla(t)[\rho(t) - s] \right| < \varepsilon |\rho(t) - s| \quad \forall s \in V. \tag{1.5}$$

If $\mathbb{T} = \mathbb{R}$, then $x^\Delta(t) = x^\nabla(t) = x'(t)$. If $\mathbb{T} = \mathbb{Z}$, then $x^\Delta(t) = x(t+1) - x(t)$ is forward difference operator while $x^\nabla(t) = x(t) - x(t-1)$ is the backward difference operator.

Definition 1.3. If $F^{\Delta}(t) = f(t)$, then define the delta integral by $\int_a^t f(s)\Delta s = F(t) - F(a)$. If $\Phi^{\nabla}(t) = f(t)$, then define the nabla integral by $\int_a^t f(s)\nabla s = \Phi(t) - \Phi(a)$.

The following lemma is crucial to prove our main results.

Lemma 1.4 ([14]). *Let E be a Banach space and let P be a cone in E. For $r > 0$, define $P_r = \{x \in P : \|x\| < r\}$. Assume that $F : \overline{P}_r \to P$ is completely continuous such that $Fx \neq x$ for $x \in \partial P_r = \{x \in P : \|x\| = r\}$.*

(i) *If $\|Fx\| \geq \|x\|$ for $x \in \partial P_r$, then $i(F, P_r, P) = 0$.*

(ii) *If $\|Fx\| \leq \|x\|$ for $x \in \partial P_r$, then $i(F, P_r, P) = 1$.*

2. Positive solutions

We note that $x(t)$ is a solution of (1.1) if and only if

$$x(t) = \begin{cases} B_0\left(\phi_q\left(\int_0^T \lambda a(r) f(x(r), x(\mu(r)))\nabla r\right)\right) \\ \quad + \int_0^t \phi_q\left(\int_s^T \lambda a(r) f(x(r), x(\mu(r)))\nabla r\right)\Delta s, & t \in [0,T], \\ \psi(t), & t \in [-\tau, 0]. \end{cases} \tag{2.1}$$

Let $E = C_{\mathrm{ld}}([0,T], \mathbb{R})$ be endowed with the norm $\|x\| = \max_{t\in[0,T]}|x(t)|$ and define the cone of E by

$$P = \left\{x \in E : x(t) \geq \frac{\delta}{T+\beta}\|x\| \text{ for } t \in [0,T]\right\}. \tag{2.2}$$

Clearly, E is a Banach space with the norm $\|x\|$. For each $x \in E$, extend $x(t)$ to $[-\tau, T]$ with $x(t) = \psi(t)$ for $t \in [-\tau, 0]$.

Define $F_\lambda : P \to E$ as

$$\begin{aligned} F_\lambda x(t) = B_0\left(\phi_q\left(\int_0^T \lambda a(r) f(x(r), x(\mu(r)))\nabla r\right)\right) \\ + \int_0^t \phi_q\left(\int_s^T \lambda a(r) f(x(r), x(\mu(r)))\nabla r\right)\Delta s, \quad t \in [0,T]. \end{aligned} \tag{2.3}$$

We seek a fixed point, x_1, of F_λ in the cone P. Define

$$x(t) = \begin{cases} x_1(t), & t \in [0,T], \\ \psi(t), & t \in [-\tau, 0]. \end{cases} \tag{2.4}$$

Then $x(t)$ denotes a positive solution of BVP (1.1).

It follows from (2.3) that the following lemma holds.

Lemma 2.1. *Let F_λ be defined by* (2.3)*. If $x \in P$, then*

(i) $F_\lambda(P) \subset P$.

(ii) $F_\lambda : P \to P$ *is completely continuous.*

The proof of Lemma 2.1 can be found in [15].

We need to define further subsets of $[0,T]$ with respect to the delay μ. Set

$$Y_1 := \{t \in [0,T] : \mu(t) < 0\}; \qquad Y_2 := \{t \in [0,T] : \mu(t) \geq 0\}. \tag{2.5}$$

Throughout this paper, we assume $Y_1 \neq \varnothing$ and $\phi_q(\int_{Y_1} a(r)\nabla r) > 0$.

Lemma 2.2. *Suppose that (H1)–(H5) hold. Then there exists a $\lambda^* > 0$ such that the operator F_λ has a fixed point $x^* \in P \setminus \{\theta\}$ at λ^*, where θ is the zero element of the Banach space E.*

Proof. Set

$$e(t) = B_0\left(\phi_q\left(\int_0^T a(r)\nabla r\right)\right) + \int_0^t \phi_q\left(\int_s^T a(r)\nabla r\right)\Delta s, \quad t \in [0,T]. \tag{2.6}$$

We know that $e \in P$. Let $\lambda^* = M_{f_e}^{-1}$, where

$$M_{f_e} = \max_{r\in[0,T]} f(e(r), e(\mu(r))) \geq c > 0,$$

$$\begin{aligned}(F_{\lambda^*}x)(t) = B_0&\left(\phi_q\left(\int_0^T \lambda^* a(r) f(x(r), x(\mu(r)))\nabla r\right)\right) \\ &+ \int_0^t \phi_q\left(\int_s^T \lambda^* a(r) f(x(r), x(\mu(r)))\nabla r\right)\Delta s, \quad t \in [0,T].\end{aligned} \tag{2.7}$$

From above, we have

$$e(t) \geq (F_{\lambda^*}e)(t). \tag{2.8}$$

Let $x_0(t) = e(t)$ and $x_n(t) = (F_{\lambda^*}x_{n-1})(t)$, $n = 1, 2, \ldots, t \in [0,T]$. Then

$$x_0(t) \geq x_1(t) \geq \cdots \geq x_n(t) \geq \cdots \geq (c\lambda^*)^{q-1} e(t). \tag{2.9}$$

By the Lebesgue dominated convergence theorem [16] together with (H3), it follows that $\{x_n\}_{n=0}^{\infty} = \{F_{\lambda^*}^n x_0\}_{n=0}^{\infty}$ decreases to a fixed point $x^* \in P \setminus \{\theta\}$ of the operator F_{λ^*}. The proof is complete. □

Lemma 2.3. *Suppose that (H1)–(H6) hold and that $\mathbf{I} \subset [b, \infty)$ for some $b > 0$. Then there exists a constant $C_{\mathbf{I}} > 0$ such that for all $\lambda \in \mathbf{I}$ and all possible fixed points x of F_λ at λ, one has $\|x\| < C_{\mathbf{I}}$.*

Proof. Set

$$S = \{x \in P : F_\lambda x = x, \ \lambda \in \mathbf{I}\}. \tag{2.10}$$

We need to prove that there exists a constant $C_{\mathbf{I}} > 0$ such that $\|x\| < C_{\mathbf{I}}$ for all $x \in S$. If the number of elements of S is finite, then the result is obvious. If not, without loss of generality, we assume that there exists a sequence $\{x_n\}_{n=0}^{\infty}$ such that $\lim_{n\to\infty}\|x_n\| = +\infty$, where $x_n \in P$ is the fixed point of the operator F_λ defined by (2.3) at $\lambda_n \in \mathbf{I}$ $(n = 1, 2, \ldots)$.

Then

$$x_n(t) \geq \frac{\delta}{T+\beta}\|x_n\|, \quad t \in [0,T]. \tag{2.11}$$

We choose $J > 0$ such that

$$\frac{Jb^{q-1}\delta^2}{T+\beta}\phi_q\left(\int_{Y_1} a(r)\nabla r\right) > 1, \tag{2.12}$$

$L > 0$ such that

$$f(x,\varphi(s)) \geq (Jx)^{p-1}, \quad x > L,\ s \in [-\tau, 0]. \tag{2.13}$$

In view of (H6) there exists an N sufficiently large such that $\|x_N\| > L$. For $t \in [0,T]$, we have

$$\begin{aligned}
\|x_N\| &= \|F_{\lambda_N}x_N\| \\
&= (F_{\lambda_N}x_N)(T) \\
&\geq \delta\phi_q\left(\int_0^T \lambda_N a(r) f(x_N(r), x_N(\mu(r)))\nabla r\right) \\
&\geq \delta\phi_q\left(\int_{Y_1} \lambda_N a(r) f(x_N(r), \varphi(\mu(r)))\nabla r\right) \\
&> \delta J b^{q-1}\min_{t\in Y_1}\phi_q\left(\int_{Y_1} a(r) x_N^{p-1}(r)\nabla r\right) \\
&\geq \frac{Jb^{q-1}\delta^2}{T+\beta}\|x_N\|\phi_q\left(\int_{Y_1} a(r)\nabla r\right) \\
&> \|x_N\|,
\end{aligned} \tag{2.14}$$

which is a contradiction. The proof is complete. □

Lemma 2.4. *Suppose that (H1)–(H5) hold and that the operator F_λ has a positive fixed point x in P at $\lambda > 0$. Then for every $\lambda_* \in (0, \lambda)$ the operator F_λ has a fixed point $x_* \in P \setminus \{\theta\}$ at λ_*, and $x_* < x$.*

Proof. Let $x(t)$ be the fixed point of the operator F_λ at λ. Then

$$\begin{aligned}
x(t) &= B_0\left(\phi_q\left(\int_0^T \lambda a(r) f(x(r), x(\mu(r)))\nabla r\right)\right) + \int_0^t \phi_q\left(\int_s^T \lambda a(r) f(x(r), x(\mu(r)))\nabla r\right)\Delta s \\
&> B_0\left(\phi_q\left(\int_0^T \lambda_* a(r) f(x(r), x(\mu(r)))\nabla r\right)\right) + \int_0^t \phi_q\left(\int_s^T \lambda_* a(r) f(x(r), x(\mu(r)))\nabla r\right)\Delta s,
\end{aligned} \tag{2.15}$$

where $0 < \lambda_* < \lambda$. Set

$$(F_{\lambda_*}x)(t) = B_0\left(\phi_q\left(\int_0^T \lambda_* a(r) f(x(r), x(\mu(r)))\nabla r\right)\right) + \int_0^t \phi_q\left(\int_s^T \lambda_* a(r) f(x(r), x(\mu(r)))\nabla r\right)\Delta s, \tag{2.16}$$

$x_0(t) = x(t)$, and $x_n = F_{\lambda_*}x_{n-1} = (F_{\lambda_*}^n x_0)(t)$. Then

$$(c\lambda_*)^{(q-1)} e(t) \le x_{n+1} \le x_n \le \cdots \le x_1(t) \le x_0(t), \tag{2.17}$$

where $e(t)$ is also defined by (2.6), which implies that $\{F_{\lambda_*}^n x\}_{n=0}^{\infty}$ decreases to a fixed point $x_* \in P \setminus \{\theta\}$ of the operator F_{λ_*}, and $x_* < x$. The proof is complete. □

Lemma 2.5. *Suppose that (H1)–(H6) hold. Let $\wedge = \{\lambda > 0 : F_\lambda$ have at least one fixed point at λ in $P\}$. Then $\wedge$ is bounded above.*

Proof. Suppose to the contrary that there exists a fixed point sequence $\{x_n\}_{n=0}^{\infty} \subset P$ of F_λ at λ_n such that $\lim_{n\to\infty} \lambda_n = \infty$. Then we need to consider two cases:

(i) there exists a constant $H > 0$ such that $\|x_n\| \le H$, $n = 0, 1, 2 \ldots$;

(ii) there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} \|x_{n_k}\| = \infty$ which is impossible by Lemma 2.3.

Only (i) is considered. We can choose $M > 0$ such that $f(0,0) > MH$, and further $f(x_n, x_n(\mu)) > MH$. For $t \in [0,T]$, we have

$$x_n(t) = B_0\left(\phi_q\left(\int_0^T \lambda_n a(r) f(x_n(r), x_n(\mu(r)))\nabla r\right)\right) + \int_0^t \phi_q\left(\int_s^T \lambda_n a(r) f(x_n(r), x_n(\mu(r)))\nabla r\right)\Delta s. \tag{2.18}$$

Now we consider (2.18).Assume that the case (i) holds. Then

$$\begin{aligned} H \ge x_n(t) &\ge B_0\left(\phi_q\left(\int_0^T (\lambda_n a(r) MH)\nabla r\right)\right) + \int_0^t \phi_q\left(\int_s^T (\lambda_n a(r) MH)\nabla r\right)\Delta s \\ &= (\lambda_n MH)^{q-1} e(t) \\ &\ge (\lambda_n MH)^{q-1} \frac{\delta}{T+\beta}\|e\| \end{aligned} \tag{2.19}$$

leads to

$$1 \ge (\lambda_n M)^{q-1} H^{q-2} \frac{\delta}{T+\beta}\|e\| \quad \text{for } t \in [0,T], \tag{2.20}$$

which is a contradiction. The proof is complete. □

Lemma 2.6. *Let $\lambda^* = \sup \wedge$. Then $\wedge = (0, \lambda^*]$, where $\wedge$ is defined just as in Lemma 2.5.*

Proof. In view of Lemma 2.4, it follows that $(0,\lambda^*) \subset \wedge$. We only need to prove $\lambda^* \in \wedge$. In fact, by the definition of λ^*, we may choose a distinct nondecreasing sequence $\{\lambda_n\}_{n=1}^{\infty} \subset \wedge$ such that $\lim_{n\to\infty}\lambda_n = \lambda^*$. Let $x_n \in P$ be the positive fixed point of F_λ at λ_n, $n = 1,2,\ldots$. By Lemma 2.3, $\{x_n\}_{n=1}^{\infty}$ is uniformly bounded, so it has a subsequence denoted by $\{x_n\}_{n=1}^{\infty}$, converging to $x_{\lambda^*} \in P$. Note that

$$x_n(t) = B_0\left(\phi_q\left(\int_0^T \lambda_n a(r) f(x_n(r), x_n(\mu(r)))\nabla r\right)\right) + \int_0^t \phi_q\left(\int_s^T \lambda_n a(r) f(x_n(r), x_n(\mu(r)))\nabla r\right)\Delta s. \tag{2.21}$$

Taking the limitation $n\to\infty$ to both sides of (2.21), and using the Lebesgue dominated convergence theorem [16], we have

$$x_{\lambda^*} = B_0\left(\phi_q\left(\int_0^T \lambda^* a(r) f(x_{\lambda^*}(r), x_{\lambda^*}(\mu(r)))\nabla r\right)\right) + \int_0^t \phi_q\left(\int_s^T \lambda^* a(r) f(x_{\lambda^*}(r), x_{\lambda^*}(\mu(r)))\nabla r\right)\Delta s, \tag{2.22}$$

which shows that F_λ has a positive fixed point x_{λ^*} at $\lambda = \lambda^*$. The proof is complete. □

Theorem 2.7. *Suppose that (H1)–(H6) hold. Then there exists a* $\lambda^* > 0$ *such that* (1.1) *has at least two, one, and no positive solutions for* $0 < \lambda < \lambda^*$, $\lambda = \lambda^*$ *and* $\lambda > \lambda^*$, *respectively.*

Proof. Assume that (H1)–(H5) hold. Then there exists a $\lambda^* > 0$ such that F_λ has a fixed point $x_{\lambda^*} \in P \setminus \{\theta\}$ at $\lambda = \lambda^*$. In view of Lemma 2.4, F_λ also has a fixed point $x_{\underline{\lambda}} < x_{\lambda^*}$, $x_{\underline{\lambda}} \in P \setminus \{\theta\}$ and $0 < \underline{\lambda} < \lambda^*$. Note that f is continuous on $(\mathbb{R}^+)^2$. For $0 < \underline{\lambda} < \lambda^*$, there exists a $\delta_0 > 0$ such that

$$f(x_{\lambda^*}(r)+\delta, x_{\lambda^*}(\mu(r))+\delta) - f(x_{\lambda^*}(r), x_{\lambda^*}(\mu(r))) \le f(0,0)\left(\frac{\lambda^*}{\underline{\lambda}} - 1\right) \quad \text{for } r \in [0,T],\ 0 < \delta \le \delta_0. \tag{2.23}$$

Hence,

$$\begin{aligned} &\underline{\lambda} a(r) f(x_{\lambda^*}(r) + \delta, x_{\lambda^*}(\mu(r)) + \delta) - \lambda^* a(r) f(x_{\lambda^*}(r), x_{\lambda^*}(\mu(r))) \\ &\quad = \underline{\lambda} a(r)\left[f(x_{\lambda^*}(r) + \delta, x_{\lambda^*}(\mu(r)) + \delta) - f(x_{\lambda^*}(r), x_{\lambda^*}(\mu(r)))\right] \\ &\qquad - (\lambda^* - \underline{\lambda}) a(r) f(x_{\lambda^*}(r), x_{\lambda^*}(\mu(r))) \\ &\quad \le (\lambda^* - \underline{\lambda}) a(r) f(0,0) - (\lambda^* - \underline{\lambda}) f(x_{\lambda^*}(r), x_{\lambda^*}(\mu(r))) \\ &\quad = (\lambda^* - \underline{\lambda}) a(r)\left[f(0,0) - f(x_{\lambda^*}(r), x_{\lambda^*}(\mu(r)))\right] \\ &\quad \le 0, \quad \forall r \in [0,T]. \end{aligned} \tag{2.24}$$

From above, we have

$$F_{\underline{\lambda}}(x_{\lambda^*} + \delta) \le F_{\lambda^*}(x_{\lambda^*}) = x_{\lambda^*} < x_{\lambda^*} + \delta. \tag{2.25}$$

Set $R_1 = \|x_{\lambda^*}(t) + \delta\|$ for $t \in [0,T]$ and $P_{R_1} = \{x \in P : \|x\| < R_1\}$. We have $F_{\underline{\lambda}} x \neq x$ for $x \in \partial R_1$. By Lemma 2.1, $i(F_{\underline{\lambda}}, P_{R_1}, P) = 1$. In view of (H6), we can choose $L > R_1 > 0$ such that

$$f(x, \psi(s)) \geq (Jx)^{p-1},$$
$$\frac{J\underline{\lambda}^{q-1}\delta^2}{T+\beta}\phi_q\left(\int_{Y_1} a(r)\nabla r\right) > 1 \quad \text{for } x > L,\ s \in [-\tau, 0]. \tag{2.26}$$

Set

$$R_2 = \frac{T+\beta}{\delta}(L+1), \qquad P_{R_2} = \{x \in P : \|x\| < R_2\}. \tag{2.27}$$

Similar to Lemma 2.3, it is easy to obtain that

$$\begin{aligned}
\|F_{\underline{\lambda}} x\| &= (F_{\underline{\lambda}} x)(T) \\
&\geq \delta\phi_q\left(\int_0^T \underline{\lambda} a(r) f(x(r), x(\mu(r)))\nabla r\right) \\
&\geq \delta\phi_q\left(\int_{Y_1} \underline{\lambda} a(r) f(x(r), \psi(\mu(r)))\nabla r\right) \\
&> \delta J\underline{\lambda}^{q-1}\min_{t\in Y_1}\{x(t)\}\phi_q\left(\int_{Y_1} a(r)\nabla r\right) \\
&\geq \frac{J\underline{\lambda}^{q-1}\delta^2}{T+\beta}\|x\|\phi_q\left(\int_{Y_1} a(r)\nabla r\right) \\
&> \|x\| \quad \text{for } x \in \partial P_{R_2}.
\end{aligned} \tag{2.28}$$

In view of Lemma 2.1, $i(F_{\underline{\lambda}}, P_{R_2}, P) = 0$. By the additivity of fixed point index,

$$i(F_{\underline{\lambda}}, P_{R_2} \setminus \overline{P}_{R_1}, P) = i(F_{\underline{\lambda}}, P_{R_2}, P) - i(F_{\underline{\lambda}}, P_{R_1}, P) = -1. \tag{2.29}$$

So, $F_{\underline{\lambda}}$ has at least two fixed points in P. The proof is complete. □

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References

[1] R. Avery and J. Henderson, "Existence of three positive pseudo-symmetric solutions for a one-dimensional p-Laplacian," *Journal of Mathematical Analysis and Applications*, vol. 277, no. 2, pp. 395–404, 2003.

[2] Y. Liu and W. Ge, "Twin positive solutions of boundary value problems for finite difference equations with p-Laplacian operator," *Journal of Mathematical Analysis and Applications*, vol. 278, no. 2, pp. 551–561, 2003.

[3] A. Cabada, "Extremal solutions for the difference ϕ-Laplacian problem with nonlinear functional boundary conditions," *Computers & Mathematics with Applications*, vol. 42, no. 3–5, pp. 593–601, 2001.
[4] F.-H. Wong, "Existence of positive solutions for m-Laplacian boundary value problems," *Applied Mathematics Letters*, vol. 12, no. 3, pp. 11–17, 1999.
[5] C. Song, "Positive solutions of functional difference equations with p-Laplacian operator," *Advances in Difference Equations*, vol. 2006, Article ID 82784, 9 pages, 2006.
[6] H.-R. Sun and W.-T. Li, "Positive solutions for nonlinear three-point boundary value problems on time scales," *Journal of Mathematical Analysis and Applications*, vol. 299, no. 2, pp. 508–524, 2004.
[7] E. R. Kaufmann and Y. N. Raffoul, "Positive solutions for a nonlinear functional dynamic equation on a time scale," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 62, no. 7, pp. 1267–1276, 2005.
[8] H. Luo and Q. Ma, "Positive solutions to a generalized second-order three-point boundary-value problem on time scales," *Electronic Journal of Differential Equations*, vol. 2005, no. 17, pp. 1–14, 2005.
[9] Z. He, "Double positive solutions of three-point boundary value problems for p-Laplacian dynamic equations on time scales," *Journal of Computational and Applied Mathematics*, vol. 182, no. 2, pp. 304–315, 2005.
[10] W.-T. Li and X.-L. Liu, "Eigenvalue problems for second-order nonlinear dynamic equations on time scales," *Journal of Mathematical Analysis and Applications*, vol. 318, no. 2, pp. 578–592, 2006.
[11] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales. An Introduction with Application*, Birkhäuser, Boston, Mass, USA, 2001.
[12] R. P. Agarwal and M. Bohner, "Basic calculus on time scales and some of its applications," *Results in Mathematics*, vol. 35, no. 1-2, pp. 3–22, 1999.
[13] S. Hilger, "Analysis on measure chains—a unified approach to continuous and discrete calculus," *Results in Mathematics*, vol. 18, no. 1-2, pp. 18–56, 1990.
[14] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, vol. 5 of *Notes and Reports in Mathematics in Science and Engineering*, Academic Press, Boston, Mass, USA, 1988.
[15] C. Song, "Existence of solutions for p-Laplacian functional dynamic equations on time scales," *Electronic Journal of Differential Equations*, vol. 2006, no. 113, pp. 1–8, 2006.
[16] B. Aulbach and L. Neidhart, "Integration on measure chains," in *Proceedings of the 6th International Conference on Difference Equations*, pp. 239–252, CRC, Augsburg, Germany, July-August 2004.