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Research Article

Oscillation for Second-Order Nonlinear Delay Dynamic Equations on Time Scales

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By means of Riccati transformation technique, we establish some new oscillation criteria for the second-order nonlinear delay dynamic equations $(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta} + p(t)f(x(\tau(t))) = 0$ on a time scale $\mathbb{T}$; here $\gamma > 0$ is a quotient of odd positive integers with r and p real-valued positive rd-continuous functions defined on $\mathbb{T}$. Our results not only extend some results established by Hassan in 2008 but also unify the oscillation of the second-order nonlinear delay differential equation and the second-order nonlinear delay difference equation.

1. Introduction

The theory of time scales, which has recently received a lot of attention, was introduced by Hilger in his Ph.D. Thesis in 1988 in order to unify continuous and discrete analysis (see Hilger [1]). Several authors have expounded on various aspects of this new theory; see the survey paper by Agarwal et al. [2] and references cited therein. A book on the subject of time scales, by Bohner and Peterson [3], summarizes and organizes much of the time scale calculus. We refer also to the last book by Bohner and Peterson [4] for advances in dynamic equations on time scales. For the notation used hereinafter we refer to the next section that provides some basic facts on time scales extracted from Bohner and Peterson [3].

A time scale $\mathbb{T}$ is an arbitrary closed subset of the reals, and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential and of difference equations. Many other interesting time scales exist, and they give rise to plenty of applications, among them the study of population dynamic models which are discrete in season (and may follow a difference scheme with variable step-size or often modeled by continuous dynamic systems), die out, say in winter, while their eggs are incubating or dormant, and then in season again, hatching gives rise to a nonoverlapping population (see Bohner and Peterson [3]). Not only does the new theory of the so-called "dynamic equations" unify the theories of differential equations and difference equations but also it extends these classical cases to cases "in between", for example, to the so-called q-difference equations when $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0, q > 1\}$ (which has important applications in quantum theory) and can be applied on different types of time scales like $\mathbb{T} = h\mathbb{N}$, $\mathbb{T} = \mathbb{N}^2$, and $\mathbb{T} = \mathbb{T}_n$ the space of the harmonic numbers.

In recent years, there has been much research activity concerning the oscillation and nonoscillation of solutions of various equations on time scales, and we refer the reader to Bohner and Saker [5], Erbe [6], and Hassan [7]. However, there are few results dealing with the oscillation of the solutions of delay dynamic equations on time scales [8–15].

Following this trend, in this paper, we are concerned with oscillation for the second-order nonlinear delay dynamic equations

$$\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + p(t)f(x(\tau(t))) = 0, \quad t \in \mathbb{T}. \tag{1.1}$$

We assume that $\gamma > 0$ is a quotient of odd positive integers, r and p are positive, real-valued rd-continuous functions defined on $\mathbb{T}$, $\tau : \mathbb{T} \to \mathbb{R}$ is strictly increasing, and $\widetilde{\mathbb{T}} := \tau(\mathbb{T}) \subset \mathbb{T}$ is a time scale, $\tau(t) \le t$ and $\tau(t) \to \infty$ as $t \to \infty$, $f \in C(\mathbb{R}, \mathbb{R})$ which satisfies for some positive constant L, $f(x)/x^{\gamma} \ge L$, for all nonzero x.

For oscillation of the second-order delay dynamic equations, Agarwal et al. [8] considered the second-order delay dynamic equations on time scales

$$x^{\Delta\Delta}(t) + p(t)x(\tau(t)) = 0, \quad t \in \mathbb{T}, \tag{1.2}$$

and established some sufficient conditions for oscillation of (1.2).

Zhang and Shanliang [15] studied the second-order nonlinear delay dynamic equations on time scales

$$x^{\Delta\Delta}(t) + p(t)f(x(t-\tau)) = 0, \quad t \in \mathbb{T}, \tag{1.3}$$

and the second-order nonlinear dynamic equations on time scales

$$x^{\Delta\Delta}(t) + p(t)f(x(\sigma(t))) = 0, \quad t \in \mathbb{T}, \tag{1.4}$$

where $\tau \in \mathbb{R}$ and $t - \tau \in \mathbb{T}$, $f : \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing ($f'(u) > k > 0$), and $uf(u) > 0$ for $u \neq 0$, and they established the equivalence of the oscillation of (1.3) and (1.4), from which obtained some oscillation criteria and comparison theorems for (1.3). However, the results established in [15] are valid only when the graininess function $\mu(t)$ is bounded which is a restrictive condition. Also the restriction $f'(u) > k > 0$ is required.

Sahiner [13] considered the second-order nonlinear delay dynamic equations on time scales

$$x^{\Delta\Delta}(t) + p(t)f(x(\tau(t))) = 0, \quad t \in \mathbb{T}, \tag{1.5}$$

where $f : \mathbb{R} \to \mathbb{R}$ is continuous, $uf(u) > 0$ for $u \neq 0$ and $|f(u)| \geq L|u|$, $\tau : \mathbb{T} \to \mathbb{T}$, $\tau(t) \leq t$ and $\lim_{t\to\infty}\tau(t) = \infty$, and he proved that if there exists a Δ-differentiable function $\delta(t)$ such that

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[p(s)\delta^{\sigma}(s)\frac{\tau(s)}{\sigma(s)} - \frac{\sigma(s)\left(\delta^{\Delta}(s)\right)^2}{4Lk^2\delta^{\sigma}(s)\tau(s)}\right]\Delta s = \infty, \tag{1.6}$$

then every solution of (1.5) oscillates. Now, we observe that the condition (1.6) depends on an additional constant $k \in (0,1)$ which implies that the results are not sharp (see Erbe et al. [10]).

Han et al. [12] investigated the second-order Emden-Fowler delay dynamic equations on time scales

$$x^{\Delta\Delta}(t) + p(t)x^{\gamma}(\tau(t)) = 0, \quad t \in \mathbb{T}, \tag{1.7}$$

established some sufficient conditions for oscillation of (1.7), and extended the results given in [8].

Erbe et al. [10] considered the general nonlinear delay dynamic equations on time scales

$$\left(p(t)x^{\Delta}(t)\right)^{\Delta} + q(t)f(x(\tau(t))) = 0, \quad t \in \mathbb{T}, \tag{1.8}$$

where p and q are positive, real-valued rd-continuous functions defined on $\mathbb{T}$, $\tau : \mathbb{T} \to \mathbb{T}$ is rd-continuous, $\tau(t) \leq t$ and $\tau(t) \to \infty$ as $t \to \infty$, and $f \in C(\mathbb{R},\mathbb{R})$ such that satisfies for some positive constant L, $|f(x)| \geq L|x|$, $xf(x) > 0$ for all nonzero x, and they extended the generalized Riccati transformation techniques in the time scales setting to obtain some new oscillation criteria which improve the results given by Zhang and Shanliang [15] and Sahiner [13].

Clearly, (1.2), (1.3), (1.5), and (1.8) are the special cases of (1.1). In this paper, we consider the second-order nonlinear delay dynamic equation on time scales (1.1).

As we are interested in oscillatory behavior, we assume throughout this paper that the given time scale $\mathbb{T}$ is unbounded above. We assume $t_0 \in \mathbb{T}$, and it is convenient to assume $t_0 > 0$. We define the time scale interval of the form $[t_0,\infty)_{\mathbb{T}}$ by $[t_0,\infty)_{\mathbb{T}} = [t_0,\infty) \cap \mathbb{T}$.

We shall also consider the two cases

$$\int_{t_0}^{\infty}\left(\frac{1}{r(t)}\right)^{1/\gamma}\Delta t = \infty, \tag{1.9}$$

$$\int_{t_0}^{\infty}\left(\frac{1}{r(t)}\right)^{1/\gamma}\Delta t < \infty. \tag{1.10}$$

The paper is organized as follows. In Section 2, we intend to use the Riccati transformation technique, a simple consequence of Keller's chain rule, and an inequality to obtain some sufficient conditions for oscillation of all solutions of (1.1). In Section 3, we give an example in order to illustrate the main results.

2. Main Results

In this section, we give some new oscillation criteria for (1.1). In order to prove our main results, we will use the formula

$$((x(t))^{\gamma})^{\Delta} = \gamma \int_0^1 [hx^{\sigma}(t) + (1-h)x(t)]^{\gamma-1} x^{\Delta}(t)\mathrm{d}h, \tag{2.1}$$

where $x(t)$ is delta differentiable and eventually positive or eventually negative, which is a simple consequence of Keller's chain rule (see Bohner and Peterson [3, Theorem 1.90]). Also, we need the following auxiliary result.

For convenience, we note that

$$\begin{gathered} d_+(t) = \max\{0, d(t)\}, \quad d_-(t) = \max\{0, -d(t)\}, \quad R(t) := r^{1/\gamma}(t) \int_{t_1}^{t} \frac{\Delta s}{r^{1/\gamma}(s)}, \\ \alpha(t) := \frac{R(t)}{R(t)+\mu(t)}, \quad \beta(t) := \alpha(t), \quad 0<\gamma\le 1; \quad \beta(t) := \alpha^{\gamma}(t), \quad \gamma>1. \end{gathered} \tag{2.2}$$

Lemma 2.1. *Assume that* (1.9) *holds and assume further that $x(t)$ is an eventually positive solution of* (1.1)*. Then there exists a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that*

$$x^{\Delta}(t) > 0, \quad x(t) > R(t)x^{\Delta}(t), \quad \left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} < 0, \quad \frac{x(t)}{x^{\sigma}(t)} > \alpha(t), \quad t \in [t_1, \infty)_{\mathbb{T}}. \tag{2.3}$$

The proof is similar to that of Hassan [7, Lemma 2.1], and so is omitted.

Lemma 2.2 (Chain Rule). *Assume that $\tau : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := \tau(\mathbb{T}) \subset \mathbb{T}$ is a time scale, $\tau(\sigma(t)) = \sigma(\tau(t))$. Let $x : \widetilde{\mathbb{T}} \to \mathbb{R}$. If $\tau^{\Delta}(t)$, and let $x^{\Delta}(\tau(t))$ exist for $t \in \mathbb{T}^{\kappa}$, then $(x(\tau(t)))^{\Delta}$ exist, and*

$$(x(\tau(t)))^{\Delta} = x^{\Delta}(\tau(t))\tau^{\Delta}(t). \tag{2.4}$$

Proof. Let $0 < \varepsilon < 1$ be given and define $\varepsilon^* = \varepsilon[1 + |\tau^{\Delta}(t)| + |x^{\Delta}(\tau(t))|]^{-1}$. Note that $0 < \varepsilon^* < 1$. According to the assumptions, there exist neighborhoods $\mathcal{N}_1$ of t and $\mathcal{N}_2$ of $\tau(t)$ such that

$$\begin{gathered} \left|\tau(\sigma(t)) - \tau(s) - (\sigma(t)-s)\tau^{\Delta}(t)\right| \le \varepsilon^*|\sigma(t)-s|, \quad s \in \mathcal{N}_1, \\ \left|x(\sigma(\tau(t))) - x(r) - (\sigma(\tau(t))-r)x^{\Delta}(\tau(t))\right| \le \varepsilon^*|\sigma(\tau(t))-r|, \quad r \in \mathcal{N}_2. \end{gathered} \tag{2.5}$$

Put $\mathcal{N} = \mathcal{N}_1 \bigcap \tau^{-1}\mathcal{N}_2$ and let $s \in \mathcal{N}$. Then $s \in \mathcal{N}_1$ and $\tau(s) \in \mathcal{N}_2$ and

$$
\begin{aligned}
&\left|x(\tau(\sigma(t))) - x(\tau(s)) - (\sigma(t) - s)x^{\Delta}(\tau(t))\tau^{\Delta}(t)\right| \\
&\quad= \Big|x(\tau(\sigma(t))) - x(\tau(s)) - (\sigma(\tau(t)) - \tau(s))x^{\Delta}(\tau(t)) \\
&\qquad+\Big[\sigma(\tau(t)) - \tau(s) - (\sigma(t) - s)\tau^{\Delta}(t)\Big]x^{\Delta}(\tau(t))\Big| \\
&\quad= \Big|x(\sigma(\tau(t))) - x(\tau(s)) - (\sigma(\tau(t)) - \tau(s))x^{\Delta}(\tau(t)) \\
&\qquad+\Big[\tau(\sigma(t)) - \tau(s) - (\sigma(t) - s)\tau^{\Delta}(t)\Big]x^{\Delta}(\tau(t))\Big| \\
&\quad\le\varepsilon^*|\sigma(\tau(t)) - \tau(s)| + \varepsilon^*|\sigma(t) - s|\left|x^{\Delta}(\tau(t))\right| \\
&\quad\le \varepsilon^*\left\{\left|\sigma(\tau(t)) - \tau(s) - (\sigma(t) - s)\tau^{\Delta}(t)\right| + |\sigma(t) - s|\left|\tau^{\Delta}(t)\right| + |\sigma(t) - s|\left|x^{\Delta}(\tau(t))\right|\right\} \\
&\quad= \varepsilon^*\left\{\left|\tau(\sigma(t)) - \tau(s) - (\sigma(t) - s)\tau^{\Delta}(t)\right| + |\sigma(t) - s|\left|\tau^{\Delta}(t)\right| + |\sigma(t) - s|\left|x^{\Delta}(\tau(t))\right|\right\} \\
&\quad\le \varepsilon^*\left\{\varepsilon^*|\sigma(t) - s| + |\sigma(t) - s|\left|\tau^{\Delta}(t)\right| + |\sigma(t) - s|\left|x^{\Delta}(\tau(t))\right|\right\} \\
&\quad= \varepsilon^*|\sigma(t) - s|\left\{\varepsilon^* + \left|\tau^{\Delta}(t)\right| + \left|x^{\Delta}(\tau(t))\right|\right\} \\
&\quad\le \varepsilon^*|\sigma(t) - s|\left\{1 + \left|\tau^{\Delta}(t)\right| + \left|x^{\Delta}(\tau(t))\right|\right\} \\
&\quad= \varepsilon|\sigma(t) - s|.
\end{aligned}
\tag{2.6}
$$

The proof is completed. □

Theorem 2.3. *Assume that* (1.9) *holds and* $\tau \in C^1_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{T}), \tau(\sigma(t)) = \sigma(\tau(t))$. *Furthermore, assume that there exists a positive function* $\delta \in C^1_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ *and for all sufficiently large* t_1, *one has*

$$
\limsup_{t\to\infty}\int_{t_1}^{t}\left(Lp(s)\alpha^{\gamma}(\tau(s))\delta^{\sigma}(s) - \frac{r(\tau(s))\left(\left(\delta^{\Delta}(s)\right)_+\right)^{\gamma+1}}{(\gamma+1)^{\gamma+1}\left(\delta^{\sigma}(s)\beta(\tau(s))\tau^{\Delta}(s)\right)^{\gamma}}\right)\Delta s = \infty. \tag{2.7}
$$

Then (1.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Proof. Suppose that (1.1) has a nonoscillatory solution $x(t)$ on $[t_0,\infty)_{\mathbb{T}}$. We may assume without loss of generality that $x(t) > 0$ and $x(\tau(t)) > 0$ for all $t \in [t_1,\infty)_{\mathbb{T}}, t_1 \in [t_0,\infty)_{\mathbb{T}}$. We shall consider only this case, since the proof when $x(t)$ is eventually negative is similar. In view of Lemma 2.1, we get (2.3). Define the function $\omega(t)$ by

$$
\omega(t) = \delta(t)\frac{r(t)\left(x^{\Delta}(t)\right)^{\gamma}}{(x(\tau(t)))^{\gamma}}, \quad t \in [t_1,\infty)_{\mathbb{T}}. \tag{2.8}
$$

Then $\omega(t) > 0$. In view of (1.1) and (2.8) we get

$$\omega^{\Delta}(t) \leq -Lp(t)\delta^{\sigma}(t)\frac{(x(\tau(t)))^{\gamma}}{(x(\tau^{\sigma}(t)))^{\gamma}} + \frac{\delta^{\Delta}(t)}{\delta(t)}\omega(t) - \frac{\delta^{\sigma}(t)r(t)\left(x^{\Delta}(t)\right)^{\gamma}\left((x(\tau(t)))^{\gamma}\right)^{\Delta}}{(x(\tau(t)))^{\gamma}(x(\tau^{\sigma}(t)))^{\gamma}}. \tag{2.9}$$

When $0 < \gamma \leq 1$, using (2.1) and (2.4), we have

$$\begin{aligned}
\left((x(\tau(t)))^{\gamma}\right)^{\Delta} &= \gamma\int_0^1 \left[h(x(\tau(t)))^{\sigma} + (1-h)x(\tau(t))\right]^{\gamma-1}(x(\tau(t)))^{\Delta}\mathrm{d}h \\
&\geq \gamma\int_0^1 \left[h(x(\tau^{\sigma}(t))) + (1-h)x(\tau^{\sigma}(t))\right]^{\gamma-1}(x(\tau(t)))^{\Delta}\mathrm{d}h \\
&= \gamma(x(\tau^{\sigma}(t)))^{\gamma-1}x^{\Delta}(\tau(t))\tau^{\Delta}(t).
\end{aligned} \tag{2.10}$$

So, by (2.9)

$$\omega^{\Delta}(t) \leq -Lp(t)\delta^{\sigma}(t)\frac{(x(\tau(t)))^{\gamma}}{(x(\tau^{\sigma}(t)))^{\gamma}} + \frac{\delta^{\Delta}(t)}{\delta(t)}\omega(t) - \frac{\gamma\delta^{\sigma}(t)r(t)\left(x^{\Delta}(t)\right)^{\gamma}x^{\Delta}(\tau(t))\tau^{\Delta}(t)}{(x(\tau(t)))^{\gamma}x(\tau^{\sigma}(t))}, \tag{2.11}$$

hence, we get

$$\begin{aligned}
\omega^{\Delta}(t) \leq{}& -Lp(t)\delta^{\sigma}(t)\frac{(x(\tau(t)))^{\gamma}}{(x(\tau^{\sigma}(t)))^{\gamma}} + \frac{\delta^{\Delta}(t)}{\delta(t)}\omega(t) \\
&- \frac{\gamma\delta^{\sigma}(t)r(t)\left(x^{\Delta}(t)\right)^{\gamma+1}\tau^{\Delta}(t)}{(x(\tau(t)))^{\gamma+1}}\frac{x(\tau(t))}{x(\tau^{\sigma}(t))}\frac{x^{\Delta}(\tau(t))}{x^{\Delta}(t)},
\end{aligned} \tag{2.12}$$

and using Lemma 2.1, by $\left(r(t)(x^{\Delta}(t))^{\gamma}\right)^{\Delta} < 0$, we find $x^{\Delta}(\tau(t))/x^{\Delta}(t) \geq (r(t)/r(\tau(t)))^{1/\gamma}$, by $x(t)/x^{\sigma}(t) > \alpha(t)$, we get $x(\tau(t))/x(\tau^{\sigma}(t)) > \alpha(\tau(t))$, and so we obtain

$$\begin{aligned}
\omega^{\Delta}(t) &\leq -Lp(t)\delta^{\sigma}(t)\alpha^{\gamma}(\tau(t)) + \frac{\delta^{\Delta}(t)}{\delta(t)}\omega(t) \\
&\quad - \gamma\delta^{\sigma}(t)\frac{x(\tau(t))}{x(\tau^{\sigma}(t))}\frac{\tau^{\Delta}(t)}{r^{1/\gamma}(\tau(t))(\delta(t))^{\gamma+1/\gamma}}(\omega(t))^{\gamma+1/\gamma} \\
&\leq -Lp(t)\delta^{\sigma}(t)\alpha^{\gamma}(\tau(t)) + \frac{\delta^{\Delta}(t)}{\delta(t)}\omega(t) \\
&\quad - \gamma\delta^{\sigma}(t)\alpha(\tau(t))\frac{\tau^{\Delta}(t)}{r^{1/\gamma}(\tau(t))(\delta(t))^{\gamma+1/\gamma}}(\omega(t))^{\gamma+1/\gamma}.
\end{aligned} \tag{2.13}$$

When $\gamma > 1$, using (2.1) and (2.4), we have

$$\begin{aligned}
\left((x(\tau(t)))^{\gamma}\right)^{\Delta} &= \gamma\int_{0}^{1}\left[h(x(\tau(t)))^{\sigma} + (1-h)x(\tau(t))\right]^{\gamma-1}(x(\tau(t)))^{\Delta}\mathrm{d}h \\
&\geq \gamma\int_{0}^{1}\left[h(x(\tau(t))) + (1-h)x(\tau(t))\right]^{\gamma-1}(x(\tau(t)))^{\Delta}\mathrm{d}h \\
&= \gamma(x(\tau(t)))^{\gamma-1}x^{\Delta}(\tau(t))\tau^{\Delta}(t).
\end{aligned} \tag{2.14}$$

So, we get

$$\begin{aligned}
\omega^{\Delta}(t) &\leq -Lp(t)\delta^{\sigma}(t)\frac{(x(\tau(t)))^{\gamma}}{(x(\tau^{\sigma}(t)))^{\gamma}} + \frac{\delta^{\Delta}(t)}{\delta(t)}\omega(t) \\
&\quad - \gamma\delta^{\sigma}(t)\frac{r(t)\left(x^{\Delta}(t)\right)^{\gamma+1}}{(x(\tau(t)))^{\gamma+1}}\frac{x^{\gamma}(\tau(t))}{x^{\gamma}(\tau^{\sigma}(t))}\frac{x^{\Delta}(\tau(t))}{x^{\Delta}(t)}\tau^{\Delta}(t) \\
&\leq -Lp(t)\delta^{\sigma}(t)\alpha^{\gamma}(\tau(t)) + \frac{\delta^{\Delta}(t)}{\delta(t)}\omega(t) \\
&\quad - \gamma\delta^{\sigma}(t)\alpha^{\gamma}(\tau(t))\frac{\tau^{\Delta}(t)}{r^{1/\gamma}(\tau(t))(\delta(t))^{\gamma+1/\gamma}}(\omega(t))^{\gamma+1/\gamma}.
\end{aligned} \tag{2.15}$$

So by the definition of $\beta(t)$, we obtain, for $\gamma > 0$,

$$\omega^{\Delta}(t) \leq -Lp(t)\delta^{\sigma}(t)\alpha^{\gamma}(\tau(t)) + \frac{\left(\delta^{\Delta}(t)\right)_{+}}{\delta(t)}\omega(t) - \gamma\delta^{\sigma}(t)\beta(\tau(t))\frac{\tau^{\Delta}(t)}{r^{\lambda-1}(\tau(t))\delta^{\lambda}(t)}\omega^{\lambda}(t), \tag{2.16}$$

where $\lambda =: \gamma + 1/\gamma$. Define $A \geq 0$ and $B \geq 0$ by

$$A^{\lambda} := \frac{\gamma\delta^{\sigma}(t)\beta(\tau(t))\tau^{\Delta}(t)}{(\delta(t))^{\lambda}r^{\lambda-1}(\tau(t))}\omega^{\lambda}(t), \qquad B^{\lambda-1} := \frac{r^{\lambda-1/\lambda}(\tau(t))\left(\delta^{\Delta}(t)\right)_{+}}{\lambda\left(\gamma\delta^{\sigma}(t)\beta(\tau(t))\tau^{\Delta}(t)\right)^{1/\lambda}}. \tag{2.17}$$

Then, using the inequality

$$\lambda AB^{\lambda-1} - A^{\lambda} \leq (\lambda-1)B^{\lambda}, \quad \lambda \geq 1 \tag{2.18}$$

yields

$$\frac{\left(\delta^{\Delta}(t)\right)_{+}}{\delta(t)}\omega(t) - \gamma\delta^{\sigma}(t)\beta(\tau(t))\frac{\tau^{\Delta}(t)}{r^{\lambda-1}(\tau(t))\delta^{\lambda}(t)}\omega^{\lambda}(t) \leq \frac{r(\tau(t))\left(\left(\delta^{\Delta}(t)\right)_{+}\right)^{\gamma+1}}{(\gamma+1)^{\gamma+1}\left(\delta^{\sigma}(t)\beta(\tau(t))\tau^{\Delta}(t)\right)^{\gamma}}. \tag{2.19}$$

From (2.16), we find

$$\omega^{\Delta}(t) \leq -Lp(t)\delta^{\sigma}(t)\alpha^{\gamma}(\tau(t)) + \frac{r(\tau(t))\left(\left(\delta^{\Delta}(t)\right)_{+}\right)^{\gamma+1}}{(\gamma+1)^{\gamma+1}\left(\delta^{\sigma}(t)\beta(\tau(t))\tau^{\Delta}(t)\right)^{\gamma}}. \tag{2.20}$$

Integrating the inequality (2.20) from t_1 to t, we obtain

$$\int_{t_1}^{t}\left(Lp(s)\alpha^{\gamma}(\tau(s))\delta^{\sigma}(s) - \frac{r(\tau(s))\left(\left(\delta^{\Delta}(s)\right)_{+}\right)^{\gamma+1}}{(\gamma+1)^{\gamma+1}\left(\delta^{\sigma}(s)\beta(\tau(s))\tau^{\Delta}(s)\right)^{\gamma}}\right)\Delta s \leq \omega(t_1) - \omega(t) \leq \omega(t_1), \tag{2.21}$$

which contradicts (2.7). The proof is completed. □

Remark 2.4. From Theorem 2.3, we can obtain different conditions for oscillation of all solutions of (1.1) with different choices of $\delta(t)$.

Theorem 2.5. *Assume that* (1.9) *holds and* $\tau \in C_{rd}^{1}([t_0,\infty)_{\mathbb{T}},\mathbb{T}), \tau(\sigma(t)) = \sigma(\tau(t))$. *Furthermore, assume that there exist functions* $H, h \in C_{rd}(\mathbb{D},\mathbb{R})$, *where* $\mathbb{D} \equiv \{(t,s) : t \geq s \geq t_0\}$ *such that*

$$H(t,t) = 0, \quad t \geq t_0, \qquad H(t,s) > 0, \quad t > s \geq t_0, \tag{2.22}$$

and H *has a nonpositive continuous* Δ*-partial derivation* $H^{\Delta_s}(t,s)$ *with respect to the second variable and satisfies*

$$H^{\Delta_s}(\sigma(t),s) + H(\sigma(t),\sigma(s))\frac{\delta^{\Delta}(s)}{\delta(s)} = -\frac{h(t,s)}{\delta(s)}(H(\sigma(t),\sigma(s)))^{\gamma/(\gamma+1)}, \tag{2.23}$$

and for all sufficiently large t_1,

$$\limsup_{t\to\infty}\frac{1}{H(\sigma(t),t_1)}\int_{t_1}^{\sigma(t)}K(t,s)\Delta s = \infty, \tag{2.24}$$

where $\delta(t)$ *is a positive* Δ*-differentiable function and*

$$K(t,s) = LH(\sigma(t),\sigma(s))\alpha^{\gamma}(\tau(s))\delta^{\sigma}(s)p(s) - \frac{r(\tau(s))(h_{-}(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\left(\delta^{\sigma}(s)\beta(\tau(s))\tau^{\Delta}(s)\right)^{\gamma}}. \tag{2.25}$$

Then (1.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Proof. Suppose that (1.1) has a nonoscillatory solution $x(t)$ on $[t_0,\infty)_{\mathbb{T}}$. We may assume without loss of generality that $x(t) > 0$ and $x(\tau(t)) > 0$ for all $t \in [t_1,\infty)_{\mathbb{T}}$, $t_1 \in [t_0,\infty)_{\mathbb{T}}$.

We proceed as in the proof of Theorem 2.3, and we get (2.16). Then from (2.16) with $(\delta^{\Delta}(t))_+$ replaced by $\delta^{\Delta}(t)$, we have

$$Lp(t)\delta^{\sigma}(t)\alpha^{\gamma}(\tau(t)) \le -\omega^{\Delta}(t) + \frac{\delta^{\Delta}(t)}{\delta(t)}\omega(t) - \gamma\delta^{\sigma}(t)\beta(\tau(t))\frac{\tau^{\Delta}(t)}{r^{\lambda-1}(\tau(t))\delta^{\lambda}(t)}\omega^{\lambda}(t). \tag{2.26}$$

Multiplying both sides of (2.26), with t replaced by s, by $H(\sigma(t),\sigma(s))$, integrating with respect to s from t_1 to $\sigma(t)$, we get

$$\begin{aligned}
&\int_{t_1}^{\sigma(t)} H(\sigma(t),\sigma(s))Lp(s)\delta^{\sigma}(s)\alpha^{\gamma}(\tau(s))\Delta s \\
&\quad\le -\int_{t_1}^{\sigma(t)} H(\sigma(t),\sigma(s))\omega^{\Delta}(s)\Delta s + \int_{t_1}^{\sigma(t)} H(\sigma(t),\sigma(s))\frac{\delta^{\Delta}(s)}{\delta(s)}\omega(s)\Delta s \\
&\qquad - \int_{t_1}^{\sigma(t)} H(\sigma(t),\sigma(s))\gamma\delta^{\sigma}(s)\beta(\tau(s))\frac{\tau^{\Delta}(s)}{r^{\lambda-1}(\tau(s))\delta^{\lambda}(s)}\omega^{\lambda}(s)\Delta s.
\end{aligned} \tag{2.27}$$

Integrating by parts and using (2.22) and (2.23), we obtain

$$\begin{aligned}
&\int_{t_1}^{\sigma(t)} H(\sigma(t),\sigma(s))Lp(s)\delta^{\sigma}(s)\alpha^{\gamma}(\tau(s))\Delta s \\
&\quad\le H(\sigma(t),t_1)\omega(t_1) + \int_{t_1}^{\sigma(t)} H^{\Delta_s}(\sigma(t),s)\omega(s)\Delta s + \int_{t_1}^{\sigma(t)} H(\sigma(t),\sigma(s))\frac{\delta^{\Delta}(s)}{\delta(s)}\omega(s)\Delta s \\
&\qquad - \int_{t_1}^{\sigma(t)} H(\sigma(t),\sigma(s))\gamma\delta^{\sigma}(s)\beta(\tau(s))\frac{\tau^{\Delta}(s)}{r^{\lambda-1}(\tau(s))\delta^{\lambda}(s)}\omega^{\lambda}(s)\Delta s \\
&\quad\le H(\sigma(t),t_1)\omega(t_1) \\
&\qquad + \int_{t_1}^{\sigma(t)} \Bigg[-\frac{h(t,s)}{\delta(s)}(H(\sigma(t),\sigma(s)))^{1/\lambda}\omega(s) \\
&\qquad\qquad - H(\sigma(t),\sigma(s))\gamma\delta^{\sigma}(s)\beta(\tau(s))\frac{\tau^{\Delta}(s)}{r^{\lambda-1}(\tau(s))\delta^{\lambda}(s)}\omega^{\lambda}(s)\Bigg]\Delta s \\
&\quad\le H(\sigma(t),t_1)\omega(t_1) \\
&\qquad + \int_{t_1}^{\sigma(t)} \Bigg[\frac{h_-(t,s)}{\delta(s)}(H(\sigma(t),\sigma(s)))^{1/\lambda}\omega(s) \\
&\qquad\qquad - H(\sigma(t),\sigma(s))\gamma\delta^{\sigma}(s)\beta(\tau(s))\frac{\tau^{\Delta}(s)}{r^{\lambda-1}(\tau(s))\delta^{\lambda}(s)}\omega^{\lambda}(s)\Bigg]\Delta s.
\end{aligned} \tag{2.28}$$

Again, let $\lambda := \gamma + 1/\gamma$, define $A \geq 0$ and $B \geq 0$ by

$$A^{\lambda} := H(\sigma(t),\sigma(s))\gamma\delta^{\sigma}(s)\beta(\tau(s))\frac{\tau^{\Delta}(s)}{r^{\lambda-1}(\tau(s))(\delta(s))^{\lambda}}\omega^{\lambda}(s),$$
$$B^{\lambda-1} := \frac{h_{-}(t,s)r^{\lambda-1/\lambda}(\tau(s))}{\lambda\left(\gamma\delta^{\sigma}(s)\beta(\tau(s))\tau^{\Delta}(s)\right)^{1/\lambda}}, \tag{2.29}$$

and using the inequality

$$\lambda AB^{\lambda-1} - A^{\lambda} \leq (\lambda-1)B^{\lambda}, \quad \lambda \geq 1, \tag{2.30}$$

we find

$$\frac{h_{-}(t,s)}{\delta(s)}(H(\sigma(t),\sigma(s)))^{1/\lambda}\omega(s) - H(\sigma(t),\sigma(s))\gamma\delta^{\sigma}(s)\beta(\tau(s))\frac{\tau^{\Delta}(s)}{r^{\lambda-1}(\tau(s))\delta^{\lambda}(s)}\omega^{\lambda}(s)$$
$$\leq \frac{r(\tau(s))(h_{-}(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\left(\delta^{\sigma}(s)\beta(\tau(s))\tau^{\Delta}(s)\right)^{\gamma}}. \tag{2.31}$$

Therefore, from the definition of $K(t,s)$, we obtain

$$\int_{t_1}^{\sigma(t)} K(t,s)\Delta s \leq H(\sigma(t),t_1)\omega(t_1), \tag{2.32}$$

and this implies that

$$\frac{1}{H(\sigma(t),t_1)}\int_{t_1}^{\sigma(t)} K(t,s)\Delta s \leq \omega(t_1), \tag{2.33}$$

which contradicts (2.24). This completes the proof. □

Now, we give some sufficient conditions when (1.10) holds, which guarantee that every solution of (1.1) oscillates or converges to zero on $[t_0,\infty)_{\mathbb{T}}$.

Theorem 2.6. *Assume* (1.10) *holds and* $\tau \in C_{rd}^{1}([t_0,\infty)_{\mathbb{T}},\mathbb{T}), \tau(\sigma(t)) = \sigma(\tau(t))$. *Furthermore, assume that there exists a positive function* $\delta \in C_{rd}^{1}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ *, for all sufficiently large* t_1*, such that* (2.7) *or* (2.22), (2.23), *and* (2.24) *hold. If there exists a positive function* $\eta \in C_{rd}^{1}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, $\eta^{\Delta}(t) \geq 0$, *such that*

$$\int_{t_0}^{\infty}\left[\frac{1}{\eta(t)r(t)}\int_{t_0}^{t}\eta^{\sigma}(s)p(s)\Delta s\right]^{1/\gamma}\Delta t = \infty, \tag{2.34}$$

then every solution of (1.1) *is either oscillatory or converges to zero on* $[t_0,\infty)_{\mathbb{T}}$.

Proof. We proceed as in Theorem 2.3 or Theorem 2.5, and we assume that (1.1) has a nonoscillatory solution such that $x(t) > 0$, and $x(\tau(t)) > 0$, for all $t \in [t_1, \infty)_{\mathbb{T}}$.

From the proof of Lemma 2.1, we see that there exist two possible cases for the sign of $x^{\Delta}(t)$. The proof when $x^{\Delta}(t)$ is an eventually positive is similar to that of the proof of Theorem 2.3 or Theorem 2.5, and hence it is omitted.

Next, suppose that $x^{\Delta}(t) < 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. Then $x(t)$ is decreasing and $\lim_{t\to\infty} x(t) = b \geq 0$. We assert that $b = 0$. If not, then $x(\tau(t)) \geq x(t) \geq x(\sigma(t)) \geq b > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. Since $f(x(\tau(t))) \geq L(x(\tau(t)))^{\gamma}$, there exists a number $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $f(x(\tau(t))) \geq Lb^{\gamma}$ for $t \geq t_2$.

Defining the function $u(t) = \eta(t) r(t)(x^{\Delta}(t))^{\gamma}$, we obtain from (1.1)

$$\begin{aligned} u^{\Delta}(t) &= \eta^{\Delta}(t) r(t)\left(x^{\Delta}(t)\right)^{\gamma} + \eta^{\sigma}(t)\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} \\ &\leq -\eta^{\sigma}(t) p(t) f(x(\tau(t))) \leq -L\eta^{\sigma}(t) p(t)(x(\tau(t)))^{\gamma} \leq -Lb^{\gamma}\eta^{\sigma}(t) p(t), \quad t \in [t_2, \infty)_{\mathbb{T}}. \end{aligned} \tag{2.35}$$

Hence, for $t \in [t_2, \infty)_{\mathbb{T}}$, we have

$$u(t) \leq u(t_2) - Lb^{\gamma}\int_{t_2}^{t} \eta^{\sigma}(s) p(s)\Delta s \leq -Lb^{\gamma}\int_{t_2}^{t} \eta^{\sigma}(s) p(s)\Delta s, \tag{2.36}$$

because of $u(t_2) = \eta(t_2) r(t_2)(x^{\Delta}(t_2))^{\gamma} < 0$. So, we have

$$x(t) - x(t_2) = \int_{t_2}^{t} x^{\Delta}(s)\Delta s \leq -L^{1/\gamma} b \int_{t_2}^{t}\left[\frac{1}{\eta(s) r(s)}\int_{t_2}^{s} \eta^{\sigma}(\tau) p(\tau)\Delta\tau\right]^{1/\gamma}\Delta s. \tag{2.37}$$

By condition (2.34), we get $x(t) \to -\infty$ as $t \to \infty$, and this is a contradiction to the fact that $x(t) > 0$ for $t \geq t_1$. Thus $b = 0$ and then $x(t) \to 0$ as $t \to \infty$. The proof is completed. □

3. Application and Example

Hassan [7] considered the second-order half-linear dynamic equations on time scales

$$\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + p(t) x^{\gamma}(t) = 0, \quad t \in \mathbb{T}, \tag{3.1}$$

where $\gamma > 0$ is a quotient of odd positive integers, and r and p are positive, real-valued *rd*-continuous functions defined on $\mathbb{T}$, and he established some new oscillation criteria of (3.1). For example

Theorem 3.1 (Hassan [7, Theorem 2.1]). *Assume that* (1.9) *holds. Furthermore, assume that there exists a positive function* $\delta \in C_{rd}^1([t_0,\infty),\mathbb{R})$ *such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left(p(s)\alpha^{\gamma}(s)\delta^{\sigma}(s)-\frac{r(s)\left(\left(\delta^{\Delta}(s)\right)_+\right)^{\gamma+1}}{(\gamma+1)^{\gamma+1}\left(\delta^{\sigma}(s)\beta(s)\right)^{\gamma}}\right)\Delta s=\infty. \tag{3.2}$$

Then (3.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

We note that (1.1) becomes (3.1) when $f(x)\equiv x^{\gamma}, \tau(t)\equiv t$, and Theorem 2.3 becomes Theorem 3.1, and so Theorem 2.3 in this paper essentially includes results of Hassan [7, Theorem 2.1].

Similarly, Theorem 2.5 includes results of Hassan [7, Theorem 2.2], and Theorem 2.6 includes results of Hassan [7, Theorem 2.4]. One can easily see that nonlinear delay dynamic equations (1.8) considered by Erbe et al. [10] are the special cases of (1.1), and the results obtained in [10] cannot be applied in (1.1), and so our results are new.

Example 3.2. Consider the second-order delay dynamic equations

$$\left(\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}+\frac{(\sigma(\tau(t)))^{\gamma}}{t\tau^{\gamma}(t)}x^{\gamma}(\tau(t))=0,\quad t\in[t_0,\infty)_{\mathbb{T}}, \tag{3.3}$$

where $\gamma > 0$ is a quotient of odd positive integers, $\tau(t)\le t,$and $\tau(\mathbb{T})\subset\mathbb{T}$ is a time scale, $\lim_{t\to\infty}\tau(t)=\infty, \tau^{\Delta}(t)>0, \tau(\sigma(t))=\sigma(\tau(t))$.

Let $r(t)=1$, $p(t)=(\sigma(\tau(t)))^{\gamma}/t\tau^{\gamma}(t)$, $f(x)=x^{\gamma}$, $L=1$, $t\in[t_0,\infty)_{\mathbb{T}}$. Then condition (1.9) holds, $R(t)=t-t_1$, and we can find $0<k<1$ such that $\alpha(t)=R(t)/R(t)+\mu(t)=t-t_1/\sigma(t)-t_1\ge kt/\sigma(t), \alpha(\tau(t))\ge k\tau(t)/\sigma(\tau(t))$ for $t\ge t_k>t_1\ge t_0$. Take $\delta(t)=1$. By Theorem 2.3, we obtain

$$\begin{aligned}
&\limsup_{t\to\infty}\int_{t_1}^{t}\left(Lp(s)\alpha^{\gamma}(\tau(s))\delta^{\sigma}(s)-\frac{r(\tau(s))\left(\left(\delta^{\Delta}(s)\right)_+\right)^{\gamma+1}}{(\gamma+1)^{\gamma+1}\left(\delta^{\sigma}(s)\beta(\tau(s))\tau^{\Delta}(s)\right)^{\gamma}}\right)\Delta s\\
&\quad=\limsup_{t\to\infty}\int_{t_1}^{t}p(s)\alpha^{\gamma}(\tau(s))\Delta s\\
&\quad\ge k^{\gamma}\limsup_{t\to\infty}\int_{t_k}^{t}\frac{(\sigma(\tau(s)))^{\gamma}}{s\tau^{\gamma}(s)}\frac{\tau^{\gamma}(s)}{(\sigma(\tau(s)))^{\gamma}}\Delta s\\
&\quad=k^{\gamma}\limsup_{t\to\infty}\int_{t_k}^{t}\frac{\Delta s}{s}=\infty.
\end{aligned} \tag{3.4}$$

We conclude that (3.3) is oscillatory.

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