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Research Article

Existence of Positive Solutions for Semipositone Higher-Order BVPS on Time Scales

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We offer conditions on semipositone function $f(t, u_0, u_1, \ldots, u_{n-2})$ such that the boundary value problem, $u^{\Delta^n}(t) + f(t, u(\sigma^{n-1}(t)), u^{\Delta}(\sigma^{n-2}(t)), \ldots, u^{\Delta^{n-2}}(\sigma(t))) = 0$, $t \in (0,1) \cap \mathbb{T}$, $n \geq 2$, $u^{\Delta^i}(0) = 0$, $i = 0, 1, \ldots, n-3$, $\alpha u^{\Delta^{n-2}}(0) - \beta u^{\Delta^{n-1}}(0) = 0$, $\gamma u^{\Delta^{n-2}}(\sigma(1)) + \delta u^{\Delta^{n-1}}(\sigma(1)) = 0$, has at least one positive solution, where $\mathbb{T}$ is a time scale and $f(t, u_0, u_1, \ldots, u_{n-2}) \in C([0,1] \times \mathbb{R}[0,\infty)^{n-1}, \mathbb{R}(-\infty,\infty))$ is continuous with $f(t, u_0, u_1, \ldots, u_{n-2}) \geq -M$ for some positive constant M.

1. Introduction

Throughout this paper, let $\mathbb{T}$ be a time scale, for any $a, b \in \mathbb{R} = (-\infty, +\infty)(b > a)$, the interval $[a,b]$ defined as $[a,b] = \{t \in \mathbb{T} \mid a \leq t \leq b\}$. Analogous notations for open and half-open intervals will also be used in the paper. We also use the notation $\mathbb{R}[c,d]$ to denote the real interval $\{t \in \mathbb{R} \mid c \leq t \leq d\}$. To understand further knowledge about dynamic equations on time scales, the reader may refer to [1–3] for an introduction to the subject.

In this paper, we present results governing the existence of positive solutions to the differential equation on time scales of the form

$$u^{\Delta^n}(t) + f\Big(t, u\Big(\sigma^{n-1}(t)\Big), u^{\Delta}\Big(\sigma^{n-2}(t)\Big), \ldots, u^{\Delta^{n-2}}(\sigma(t))\Big) = 0, \quad t \in (0,1),\ n \geq 2 \tag{1.1}$$

subject to the two-point boundary conditions

$$\begin{gathered} u^{\Delta^i}(0) = 0, \quad i = 0, 1, \ldots, n-3, \\ \alpha u^{\Delta^{n-2}}(0) - \beta u^{\Delta^{n-1}}(0) = 0, \\ \gamma u^{\Delta^{n-2}}(\sigma(1)) + \delta u^{\Delta^{n-1}}(\sigma(1)) = 0, \end{gathered} \tag{1.2}$$

where $\alpha, \gamma, \beta, \delta \geq 0, d := \alpha\delta + \gamma\beta + \alpha\gamma\sigma(1) > 0$ and

$$\delta + \gamma\left(\sigma(1) - \sigma^2(1)\right) \geq 0. \tag{1.3}$$

Throughout, we assume that $f(t, u_0, u_1, \ldots, u_{n-2}) \in C([0,1] \times \mathbb{R}[0,\infty)^{n-1}, \mathbb{R}(-\infty,\infty))$ is continuous with $f(t, u_0, u_1, \ldots, u_{n-2}) \geq -M$ for some positive constant M.

Let $C^n_{rd}[0,1]$ denote the space of functions

$$C^n_{rd}[0,1] = \left\{u : u \in C[0,\sigma^n(1)], \ldots, u^{\Delta^{n-1}} \in C[0,\sigma(1)], u^{\Delta^n} \in C_{rd}[0,1]\right\}. \tag{1.4}$$

We say that $u(t)$ is a positive solution of BVP (1.1) and (1.2), if $u(t) \in C^n_{rd}[0,1]$ is a solution of BVP (1.1) and (1.2) and $u^{\Delta^i}(t) > 0, t \in (0, \sigma^{n-i}(1)), i = 0, 1, \ldots, n-2$.

Various cases of BVP (1.1) and (1.2) have attracted a lot of attention in the literature. When $n = 2$, BVP (1.1) and (1.2) has been studied by many specialists. For example, Agarwal et al. [4] have established the existence of positive solutions for continuous case of the semipositone Sturm-Liouville BVPs. Erbe and Peterson [5] andHao et al. [6] dealt with Sturm-Liouville BVPs on time scale of positone nonlinear term. In addition, Agarwal and O'Regan [7] obtained positive solution of second-order right focal BVPs on time scale by using nonlinear alternative of Leray-Schauder type. In 2005, Chyan and Wong [8] obtained triple solutions of the same BVPs with [7]. Recently,Sun and Li [9, 10] investigated semipositone Dirichlet BVPs on time scale. For higher-order BVPs, continuous case of BVP (1.1) and (1.2) have been investigated by Agarwal and Wong [11], Wong and Agarwal [12] and Wong [13]. The discrete positone case of BVP (1.1) and (1.2) has been tackled by using a fixed point theorem for mappings that are decreasing with respect to a cone in [14]. Especially, time-scale case of (1.1) with four-point boundary condition has been studied by Liu and Sang [15]. Besides, BVP (1.1) and (1.2) of nonlinear positone term $f(t, u(t), u'(t), \ldots, u^{(n-1)}(t))$ which satisfied Nagumo-type conditions have been dealt with in [16]. Motivated by the works mentioned above, the purpose of this paper is to tackle semipositone BVP (1.1) and (1.2). In fact, BVPs appeared in [7–14] can be looked at as special case of BVP (1.1) and (1.2) in this paper. For other related works, we also refer to [17–19].

The paper is outlined as follows. In Section 2, we will present some notations and lemmas which will be used later. In Section 3, by using Krasnoselskii's fixed point theorem in a cone, we offer criteria for the existence of positive solution of BVP (1.1) and (1.2).

2. Preliminary

In this section, we offer some notations and lemmas, which will be used in main results. Throughout this paper, we always use the following notations:

(C_1) $K(t,s)$ is the Green's function of the differential equation $-u^{\Delta^n}(t) = 0$, $t \in (0,1)$ subject to the boundary conditions (1.2);

(C_2) $k(t,s)$ is the Green's function of the differential equation $-u^{\Delta\Delta}(t) = 0$, $t \in (0,1)$ subject to the boundary conditions

$$\alpha u(0) - \beta u^{\Delta}(0) = 0, \qquad \gamma u(\sigma(1)) + \delta u^{\Delta}(\sigma(1)) = 0; \tag{2.1}$$

(C_3) Define $T_i : [0,1] \to \mathbb{R}$, $i = 0,1,\ldots,n-2$ as

$$T_0(t) \equiv q(t), \quad T_i(t) = \int_0^t T_{i-1}(\tau)\Delta\tau, \quad i = 1,2,\ldots,n-2, \tag{2.2}$$

where

$$q(t) := \min_{t\in[0,\sigma^2(1)]} \left\{ \frac{\alpha t + \beta}{\alpha\sigma^2(1) + \beta}, \frac{\gamma(\sigma(1) - t) + \delta}{\gamma\sigma(1) + \delta} \right\}. \tag{2.3}$$

Lemma 2.1. *For the Green's function $k(t,s)$ the following hold:*

$$0 \le q(t)k(\sigma(s),s) \le k(t,s) \le k(\sigma(s),s), \quad (t,s) \in \left[0,\sigma^2(1)\right] \times [0,1]. \tag{2.4}$$

Proof. It is clear that

$$k(t,s) = K^{\Delta_t^{n-2}}(t,s) = \begin{cases} \dfrac{1}{d}\{\alpha t + \beta\}\{\gamma(\sigma(1) - \sigma(s)) + \delta\}, & t \le s, \\ \dfrac{1}{d}\{\alpha\sigma(s) + \beta\}\{\gamma(\sigma(1) - t) + \delta\}, & \sigma(s) \le t. \end{cases} \tag{2.5}$$

From the expression of $k(t,s)$, we can easily obtain

$$0 \le q(t)k(\sigma(s),s) \le k(t,s) \le k(\sigma(s),s), \quad (t,s) \in \left[0,\sigma^2(1)\right] \times [0,1]. \tag{2.6}$$

□

Lemma 2.2. *Let $w(t)$ be the solution of BVP*

$$\begin{gathered} -u^{\Delta^n}(t) = M, \quad t \in [0,1], \\ u^{\Delta^i}(0) = 0, \quad i = 0,1,\ldots,n-3, \\ \alpha u^{\Delta^{n-2}}(0) - \beta u^{\Delta^{n-1}}(0) = 0, \\ \gamma u^{\Delta^{n-2}}(\sigma(1)) + \delta u^{\Delta^{n-1}}(\sigma(1)) = 0. \end{gathered} \tag{2.7}$$

Then

$$0 \le w^{\Delta^i}(t) \le cMT_{n-2-i}(t), \quad t \in \left[0,\sigma^{n-i}(1)\right], \; i = 0,1,\ldots,n-2, \tag{2.8}$$

where

$$c := \frac{(\gamma\sigma(1) + \delta)(\alpha\sigma^2(1) + \beta)}{d}\sigma(1), \tag{2.9}$$

and $M \in \mathbb{R}(0,\infty)$ is a positive constant.

Proof. For $t \leq s$,

$$\begin{aligned} k(t,s) &= \frac{1}{d}\{\alpha t + \beta\}\{\gamma(\sigma(1) - \sigma(s)) + \delta\} \leq \frac{1}{d}\{\alpha t + \beta\}\{\gamma(\sigma(1) - t) + \delta\} \\ &= \frac{\alpha t + \beta}{\alpha\sigma^2(1) + \beta} \cdot \frac{\gamma(\sigma(1) - t) + \delta}{\gamma\sigma(1) + \delta} \cdot \frac{(\gamma\sigma(1) + \delta)(\alpha^2\sigma(1) + \beta)}{d} \\ &\leq \frac{cq(t)}{\sigma(1)}. \end{aligned} \tag{2.10}$$

For $\sigma(s) \leq t$,

$$\begin{aligned} k(t,s) &= \frac{1}{d}\{\alpha\sigma(s) + \beta\}\{\gamma(\sigma(1) - t) + \delta\} \leq \frac{1}{d}\{\alpha t + \beta\}\{\gamma(\sigma(1) - t) + \delta\} \\ &= \frac{\gamma(\sigma(1) - t) + \delta}{\gamma\sigma(1) + \delta} \cdot \frac{\alpha t + \beta}{\alpha\sigma^2(1) + \beta} \cdot \frac{(\gamma\sigma(1) + \delta)(\alpha\sigma^2(1) + \beta)}{d} \\ &\leq \frac{cq(t)}{\sigma(1)}. \end{aligned} \tag{2.11}$$

So

$$0 \leq k(t,s) \leq \frac{cq(t)}{\sigma(1)}, \quad (t,s) \in \left[0, \sigma^2(1)\right] \times [0,1]. \tag{2.12}$$

By defining $w(t)$ as $w(t) = \int_0^{\sigma(1)} K(t,s)M\,ds, t \in [0, \sigma^n(1)]$, it is clear that

$$w^{\Delta^{n-2}}(t) = \int_0^{\sigma(1)} k(t,s)M\,ds, \quad t \in \left[0, \sigma^2(1)\right]. \tag{2.13}$$

Then

$$0 \leq w^{\Delta^{n-2}}(t) \leq \frac{cq(t)}{\sigma(1)} \int_0^{\sigma(1)} M\Delta s = cMq(t), \quad t \in \left[0, \sigma^2(1)\right]. \tag{2.14}$$

Further, since $w^{\Delta^i}(t) = 0, \ i = 0, 1, \ldots, n-3$, we get

$$0 \leq w^{\Delta^i}(t) \leq cMT_{n-2-i}(t), \quad t \in \left[0, \sigma^{n-i}(1)\right], \ i = 0, 1, \ldots, n-2. \tag{2.15}$$

□

Lemma 2.3 (see [20]). *Let E be a Banach space, and let $C \subset E$ be a cone in E. Assume that Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let $T : C \cap (\overline{\Omega}_2 \setminus \Omega_1) \to C$ be a completely continuous operator such that either*

(i) $\|Tu\| \le \|u\|, u \in C \cap \partial\Omega_1$; $\|Tu\| \ge \|u\|, u \in C \cap \partial\Omega_2$ *or*

(ii) $\|Tu\| \ge \|u\|, u \in C \cap \partial\Omega_1$; $\|Tu\| \le \|u\|, u \in C \cap \partial\Omega_2$.

Then, T has a fixed point in $C \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Main Results

In this section, by using Lemma 2.3, we offer criteria for the existence of positive solution of BVP (1.1) and (1.2).

Let E denote the space of functions

$$E = \left\{u : u \in C[0, \sigma^n(1)], \dots, u^{\Delta^{n-3}} \in C\left[0, \sigma^3(1)\right], u^{\Delta^{n-2}} \in C\left[0, \sigma^2(1)\right]\right\}. \tag{3.1}$$

Let $\mathcal{B} = \{u \in E : u^{\Delta^i}(0) = 0, i = 0, 1, \dots, n-3\}$ be a Banach space with the norm $\|u\| = \sup_{t\in[0,\sigma^2(1)]} |u^{\Delta^{n-2}}(t)|$, and let

$$C = \left\{u \in \mathcal{B} : u^{\Delta^{n-2}}(t) \ge q(t)\|u\|, t \in \left[0, \sigma^2(1)\right]\right\}. \tag{3.2}$$

It is obvious that C is a cone in $\mathcal{B}$. From $u^{\Delta^i}(0) = 0, i = 0, 1, \dots, n-3$, it follows that for all $u \in C$,

$$T_{n-2-i}(t)\|u\| \le u^{\Delta^i}(t) \le \sigma_0\|u\|, \quad t \in \left[0, \sigma^{n-i}(1)\right], \ i = 0, 1, \dots, n-2, \tag{3.3}$$

where

$$\sigma_0 := [\sigma^n(1)]^{n-2}. \tag{3.4}$$

Throughout the rest of the section, we assume that the set $[0, \sigma(1)]$ is such that

$$\xi = \min\left\{\tau \in \mathbb{T} : \tau \ge \frac{\sigma(1)}{4}\right\}, \qquad \zeta = \min\left\{\tau \in \mathbb{T} : \tau \le \frac{3\sigma(1)}{4}\right\} \tag{3.5}$$

exist and satisfy

$$\frac{\sigma(1)}{4} \le \xi < \zeta \le \frac{3\sigma(1)}{4}. \tag{3.6}$$

In addition, we denote that

$$\eta_i = \min_{t\in\left[\xi,\ \sigma^{n-i-1}(\zeta)\right]} T_{n-2-i}(t), \quad i = 0, 1, \dots, n-2. \tag{3.7}$$

In order to obtain positive solutions of BVP (1.1) and (1.2), we need to consider the following boundary value problem:

$$\begin{gathered} u^{\Delta^n}(t) + f^*\left(t, v\left(\sigma^{n-1}(t)\right), v^{\Delta}\left(\sigma^{n-2}(t)\right), \dots, v^{\Delta^{n-2}}(\sigma(t))\right) = 0, \quad t \in (0,1), \\ u^{\Delta^i}(0) = 0, \quad i = 0, 1, \dots, n-3, \\ \alpha u^{\Delta^{n-2}}(0) - \beta u^{\Delta^{n-1}}(0) = 0, \qquad \gamma u^{\Delta^{n-2}}(\sigma(1)) + \delta u^{\Delta^{n-1}}(\sigma(1)) = 0, \end{gathered} \tag{3.8}$$

where

$$\begin{gathered} v(t) = u(t) - w(t), \quad (w(t) \text{ is as in Lemma 2.2}), \\ f^*(t, u_0, u_1, \dots, u_{n-2}) = f\left(t, \rho_0, \rho_1, \dots, \rho_{n-2}\right) + M \end{gathered} \tag{3.9}$$

and for all $i = 0, 1, \dots, n-2$,

$$\rho_i = \begin{cases} u_i, & u_i \geq 0, \\ 0, & u_i < 0. \end{cases} \tag{3.10}$$

Let the operator $S : C \to \mathcal{B}$ be defined by

$$\begin{gathered} (Su)(t) = \int_0^{\sigma(1)} K(t,s) f^*\left(s, v\left(\sigma^{n-1}(s)\right), v^{\Delta}\left(\sigma^{n-2}(s)\right), \dots, v^{\Delta^{n-2}}(\sigma(s))\right) \Delta s, \quad t \in [0, \sigma^n(1)], \\ (Su)^{\Delta^{n-2}}(t) = \int_0^{\sigma(1)} k(t,s) f^*\left(s, v\left(\sigma^{n-1}(s)\right), v^{\Delta}\left(\sigma^{n-2}(s)\right), \dots, v^{\Delta^{n-2}}(\sigma(s))\right) \Delta s, \quad t \in \left[0, \sigma^2(1)\right]. \end{gathered} \tag{3.11}$$

Lemma 3.1. *The operator S maps C into C.*

Proof. From Lemma 2.1, we know that for $t \in [0, \sigma^2(1)]$,

$$\begin{aligned} (Su)^{\Delta^{n-2}}(t) &= \int_0^{\sigma(1)} k(t,s) f^*\left(s, v\left(\sigma^{n-1}(s)\right), v^{\Delta}\left(\sigma^{n-2}(s)\right), \dots, v^{\Delta^{n-2}}(\sigma(s))\right) \Delta s \\ &\leq \int_0^{\sigma(1)} k(\sigma(s), s) f^*\left(s, v\left(\sigma^{n-1}(s)\right), v^{\Delta}\left(\sigma^{n-2}(s)\right), \dots, v^{\Delta^{n-2}}(\sigma(s))\right) \Delta s. \end{aligned} \tag{3.12}$$

So for $t \in [0, \sigma^2(1)]$,

$$\|Su\| \leq \int_0^{\sigma(1)} k(\sigma(s), s) f^*\left(s, v\left(\sigma^{n-1}(s)\right), v^{\Delta}\left(\sigma^{n-2}(s)\right), \dots, v^{\Delta^{n-2}}(\sigma(s))\right) \Delta s. \tag{3.13}$$

From Lemma 2.1 again, it follows that for $t \in [0, \sigma^2(1)]$,

$$
\begin{aligned}
(Su)^{\Delta^{n-2}}(t) &= \int_0^{\sigma(1)} k(t,s) f^*\left(s, v\left(\sigma^{n-1}(s)\right), v^{\Delta}\left(\sigma^{n-2}(s)\right), \ldots, v^{\Delta^{n-2}}(\sigma(s))\right) \Delta s \\
&\geq \int_0^{\sigma(1)} q(t) k(\sigma(s), s) f^*\left(s, v\left(\sigma^{n-1}(s)\right), v^{\Delta}\left(\sigma^{n-2}(s)\right), \ldots, v^{\Delta^{n-2}}(\sigma(s))\right) \Delta s \\
&\geq q(t) \|Su\|.
\end{aligned} \tag{3.14}
$$

Hence, S maps C into C. □

Lemma 3.2. *The operator $S : C \to C$ is completely continuous.*

Proof. First we shall prove that the operator S is continuous. Let u_m, $u \in C$ be such that $\lim_{m\to\infty} \|u_m - u\| = 0$. From $u^{\Delta^i}(0) = 0$, $i = 0, 1, \ldots, n-3$, we have

$$
\sup_{t\in[0,\sigma^{n-i}(1)]} \left| u_m^{\Delta^i}(t) - u^{\Delta^i}(t) \right| \longrightarrow 0, \quad i = 0, 1, \ldots, n-2. \tag{3.15}
$$

Then, it is easy to see that as $m \to \infty$

$$
\begin{aligned}
\rho_m := \sup_{s\in[0,1]} \Big| & f^*\left(s, u_m\left(\sigma^{n-1}(s)\right) - w\left(\sigma^{n-1}(s)\right), \ldots, u_m^{\Delta^{n-2}}(\sigma(s)) - w^{\Delta^{n-2}}(\sigma(s))\right) \\
& - f^*\left(s, u\left(\sigma^{n-1}(s)\right) - w\left(\sigma^{n-1}(s)\right), \ldots, u^{\Delta^{n-2}}(\sigma(s)) - w^{\Delta^{n-2}}(\sigma(s))\right) \Big| \longrightarrow 0.
\end{aligned} \tag{3.16}
$$

Hence, we get from Lemma 2.1 that for $t \in [0, \sigma^2(1)]$,

$$
\begin{aligned}
&\left| (Su_m)^{\Delta^{n-2}}(t) - (Su)^{\Delta^{n-2}}(t) \right| \\
&\quad = \Big| \int_0^{\sigma(1)} k(t,s) \Big[f^*\left(s, u_m\left(\sigma^{n-1}(s)\right) - w\left(\sigma^{n-1}(s)\right), \ldots, u_m^{\Delta^{n-2}}(\sigma(s)) - w^{\Delta^{n-2}}(\sigma(s))\right) \\
&\qquad - f^*\left(s, u\left(\sigma^{n-1}(s)\right) - w\left(\sigma^{n-1}(s)\right), \ldots, u^{\Delta^{n-2}}(\sigma(s)) - w^{\Delta^{n-2}}(\sigma(s))\right) \Big] \Delta s \Big| \\
&\quad \leq \rho_m \int_0^{\sigma(1)} k(t,s) \Delta s \leq \rho_m \int_0^{\sigma(1)} k(\sigma(s), s) \Delta s \longrightarrow 0, \quad \text{as } m \longrightarrow \infty.
\end{aligned} \tag{3.17}
$$

This shows that $S : C \to C$ is continuous.

Next, to show complete continuity, we will apply Arzela-Ascoli theorem. Let Ω be a bounded subset of C. Then there exists $L > 0$ such that for all $u \in \Omega$,

$$
\sup \left| u^{\Delta^{n-2}} \right| \leq L, \quad \sup \left| u^{\Delta^i} \right| \leq \sigma_0 L, \quad i = 0, 1, \ldots, n-3, \tag{3.18}
$$

where σ_0 is given in (3.4). Let

$$\widetilde{M} = \sup_{(s,\rho_0,\rho_1,\ldots,\rho_{n-2})\in[0,1]\times\mathbb{R}[0,\sigma_0 L]^{n-2}\times\mathbb{R}[0,L]} f\left(s,\rho_0,\rho_1,\ldots,\rho_{n-2}\right) + M. \tag{3.19}$$

Clearly, we have for $t \in [0,\sigma^2(1)]$,

$$\left|(Su)^{\Delta^{n-2}}(t)\right| \leq \widetilde{M}\int_0^{\sigma(1)} k(t,s)\Delta s \leq \widetilde{M}\int_0^{\sigma(1)} k(\sigma(s),s)\Delta s \tag{3.20}$$

and for $t,t' \in [0,\sigma^2(1)]$,

$$\left|(Su)^{\Delta^{n-2}}(t) - (Su)^{\Delta^{n-2}}(t')\right| \leq \widetilde{M}\int_0^{\sigma(1)} \left|k(t,s) - k(t',s)\right|\Delta s. \tag{3.21}$$

The Arzela-Ascoli theorem guarantees that $S\Omega$ is relatively compact, so $S : C \to C$ is completely continuous. □

Theorem 3.3. *Assume that the following conditions hold:*

(i) *there exist* $r \in \mathbb{R}(cM,\infty)$ *such that for any* $(u_0,u_1,\ldots,u_{n-2}) \in \Gamma_r := \mathbb{R}[0,\sigma_0 r]^{n-2} \times \mathbb{R}[0,r]$,

$$A(u_0,u_1,\ldots,u_{n-2}) := \int_0^{\sigma(1)} k(\sigma(s),s)\left[f(s,u_0,u_1,\ldots,u_{n-2}) + M\right]\Delta s \leq r, \tag{3.22}$$

(ii) *there exist* $R \in \mathbb{R}(cM,\infty)$ *with* $R \neq r$ *such that for any* $(u_0,u_1,\ldots,u_{n-2}) \in \Gamma_R := \mathbb{R}[\epsilon\eta_0 R,\sigma_0 R] \times \mathbb{R}[\epsilon\eta_1 R,\sigma_0 R] \times \cdots \times \mathbb{R}[\epsilon\eta_{n-2}R,R]$,

$$B(u_0,u_1,\ldots,u_{n-2}) := \eta_{n-2}\int_\xi^\zeta k(\sigma(s),s)\left[f(s,u_0,u_1,\ldots,u_{n-2}) + M\right]\Delta s \geq R, \tag{3.23}$$

where σ_0 *is given in* (3.4), ξ,ζ *are given in* (3.5), η_i, $i = 0,1,\ldots,n-2$ *are given in* (3.7), *and*

$$\epsilon = 1 - \frac{cM}{R}. \tag{3.24}$$

Then BVP (1.1) *and* (1.2) *has a positive solution.*

Proof. Without loss of generality, we assume that $r < R$. Now we seek positive solutions of BVP (3.8). Let

$$\Omega_1 = \{u \in B : \|u\| \leq r\}. \tag{3.25}$$

For $u \in \partial\Omega_1 \cap C$, it follows from (3.3) that

$$0 \le u^{\Delta^i}(t) \le \sigma_0 r, \quad t \in \left[0, \sigma^{n-i}(1)\right],\ i = 0, 1, \ldots, n-3. \tag{3.26}$$

From (i), we obtain that for $u \in \partial\Omega_1 \cap C$,

$$\begin{aligned}(Su)^{\Delta^{n-2}}(t) &= \int_0^{\sigma(1)} k(t,s) f^*\left(s, v\left(\sigma^{n-1}(s)\right), v^{\Delta}\left(\sigma^{n-2}(s)\right), \ldots, v^{\Delta^{n-2}}(\sigma(s))\right)\Delta s \\ &\le \int_0^{\sigma(1)} k(\sigma(s), s) f^*\left(s, v\left(\sigma^{n-1}(s)\right), v^{\Delta}\left(\sigma^{n-2}(s)\right), \ldots, v^{\Delta^{n-2}}(\sigma(s))\right)\Delta s \\ &\le r.\end{aligned} \tag{3.27}$$

So

$$\|Su\| \le \|u\|, \quad u \in \partial\Omega_1 \cap C. \tag{3.28}$$

Let

$$\Omega_2 = \{u \in B : \|u\| \le R\}. \tag{3.29}$$

For $u \in \partial\Omega_2 \cap C$, it follows from Lemma 2.2 and (3.3) that for $s \in [0, \sigma(1)]$,

$$\begin{aligned}v^{\Delta^i}\left(\sigma^{n-i-1}(s)\right) &= u^{\Delta^i}\left(\sigma^{n-i-1}(s)\right) - w^{\Delta^i}\left(\sigma^{n-i-1}(s)\right) \\ &\ge u^{\Delta^i}\left(\sigma^{n-i-1}(s)\right) - cMT_{n-2-i}\left(\sigma^{n-i-1}(s)\right) \\ &= u^{\Delta^i}\left(\sigma^{n-i-1}(s)\right) - \frac{cMT_{n-2-i}(\sigma^{n-i-1}(s))\|u\|}{R} \\ &\ge u^{\Delta^i}\left(\sigma^{n-i-1}(s)\right) - \frac{cMu^{\Delta^i}(\sigma^{n-i-1}(s))}{R} \\ &= \epsilon u^{\Delta^i}\left(\sigma^{n-i-1}(s)\right) \ge \epsilon R T_{n-2-i}\left(\sigma^{n-i-1}(s)\right), \quad i = 0, 1, \ldots, n-2.\end{aligned} \tag{3.30}$$

So

$$v^{\Delta^i}\left(\sigma^{n-i-1}(s)\right) \ge \epsilon \eta_i R, \quad s \in [\xi, \zeta],\ i = 0, 1, \ldots, n-2, \tag{3.31}$$

where η_i is given in (3.7) and ϵ is given in (3.24).

Combining Lemma 2.1, (3.3), and (ii) with (3.31), we obtain that for $u \in \partial\Omega_2 \cap C$,

$$\begin{aligned}
(Su)^{\Delta^{n-2}}(t) &= \int_0^{\sigma(1)} k(t,s) f^*\Big(s, v\Big(\sigma^{n-1}(s)\Big), v^{\Delta}\Big(\sigma^{n-2}(s)\Big), \dots, v^{\Delta^{n-2}}(\sigma(s))\Big)\Delta s \\
&\geq \int_{\xi}^{\zeta} k(t,s) f^*\Big(s, v\Big(\sigma^{n-1}(s)\Big), v^{\Delta}\Big(\sigma^{n-2}(s)\Big), \dots, v^{\Delta^{n-2}}(\sigma(s))\Big)\Delta s \\
&\geq \eta_{n-2} \int_{\xi}^{\zeta} k(\sigma(s), s) f^*\Big(s, v\Big(\sigma^{n-1}(s)\Big), v^{\Delta}\Big(\sigma^{n-2}(s)\Big), \dots, v^{\Delta^{n-2}}(\sigma(s))\Big)\Delta s \\
&\geq R.
\end{aligned} \tag{3.32}$$

So

$$\|Su\| \geq \|u\|, \quad u \in \partial\Omega_2 \cap C. \tag{3.33}$$

Therefore, it follows from Lemma 2.3 that BVP (3.8) has a solution $u_1 \in C$ such that $r \leq \|u_1\| \leq R$.

Finally, we will prove that $u_1(t) - w(t)$ is a positive solution of BVP (1.1) and (1.2). Let $u(t) = u_1(t) - w(t)$, then we have from Lemma 2.2 and (3.3) that for $i = 0, 1, \dots, n-2$,

$$\begin{aligned}
u^{\Delta^i}(t) &= u_1^{\Delta^i}(t) - w^{\Delta^i}(t) \geq u_1^{\Delta^i}(t) - cMT_{n-2-i}(t) \\
&\geq u_1^{\Delta^i}(t) - \frac{cMT_{n-2-i}(t)\|u_1\|}{r} \\
&\geq u_1^{\Delta^i}(t) - \frac{cMu_1^{\Delta^i}(t)}{r} \\
&= \left(1 - \frac{cM}{r}\right) u_1^{\Delta^i}(t) \geq (r - cM) T_{n-2-i}(t) > 0, \quad t \in \Big(0, \sigma^{n-i}(1)\Big).
\end{aligned} \tag{3.34}$$

In addition,

$$\begin{aligned}
u^{\Delta^n}(t) &= u_1^{\Delta^n}(t) + M \\
&= -f^*\Big(t, u_1\Big(\sigma^{n-1}(t)\Big) - w\Big(\sigma^{n-1}(t)\Big), \dots, u_1^{\Delta^{n-2}}(\sigma(t)) - w^{\Delta^{n-2}}(\sigma(t))\Big) + M \\
&= -f\Big(t, u_1\Big(\sigma^{n-1}(t)\Big) - w\Big(\sigma^{n-1}(t)\Big), \dots, u_1^{\Delta^{n-2}}(\sigma(t)) - w^{\Delta^{n-2}}(\sigma(t))\Big) \\
&= -f\Big(t, u\Big(\sigma^{n-1}(t)\Big), \dots, u^{\Delta^{n-2}}(\sigma(t))\Big).
\end{aligned} \tag{3.35}$$

So, $u(t) = u_1(t) - w(t)$ is a positive solution of BVP (1.1) and (1.2). This completes the proof. □

Corollary 3.4. *Assume that*

(a) *for any* $(t, u_0, u_1, \ldots, u_{n-2}) \in [0,1] \times \mathbb{R}[0,\infty)^{n-1}$,

$$f(t, u_0, u_1, \ldots, u_{n-2}) + M \le \mu(t) g(u_0, u_1, \ldots, u_{n-2}), \tag{3.36}$$

where $g : \mathbb{R}[0,\infty)^{n-1} \to \mathbb{R}[0,\infty)$ *is a continuous function which is nondecreasing in* u_j *for each fixed* $(u_0, \ldots, u_{j-1}, u_{j+1}, \ldots, u_{n-2})$ *and* $\mu(t)$ *is a continuous nonnegative function on* $[0,1]$,

(b) *for any* $(t, u_0, u_1, \ldots, u_{n-2}) \in [\xi, \zeta] \times \mathbb{R}[0,\infty)^{n-1}$,

$$f(t, u_0, u_1, \ldots, u_{n-2}) + M \ge \nu(t) h(u_0, u_1, \ldots, u_{n-2}), \tag{3.37}$$

where $h : \mathbb{R}[0,\infty)^{n-1} \to \mathbb{R}[0,\infty)$ *is a continuous function which is nondecreasing in* u_j *for each fixed* $(u_0, \ldots, u_{j-1}, u_{j+1}, \ldots, u_{n-2})$ *and* $\nu(t)$ *is a continuous nonnegative function on* $[0,1]$,

(c) *there exists* $r \in \mathbb{R}(cM, \infty)$ *such that*

$$g(\sigma_0 r, \ldots, \sigma_0 r, r) \int_0^{\sigma(1)} k(\sigma(s), s) \mu(s) \Delta s \le r, \tag{3.38}$$

(d) *there exists* $R \in \mathbb{R}(cM, \infty)$ *with* $R \ne r$ *such that*

$$h(\epsilon\eta_0 R, \epsilon\eta_1 R, \ldots, \epsilon\eta_{n-2} R) \eta_{n-2} \int_\xi^\zeta k(\sigma(s), s) \nu(s) \Delta s \ge R. \tag{3.39}$$

Then BVP (1.1) *and* (1.2) *has a positive solution.*

Proof. From (a) and (c), we obtain that for $(u_0, u_1, \ldots, u_{n-2}) \in \Gamma_r$,

$$\begin{aligned} A(u_0, u_1, \ldots, u_{n-2}) &= \int_0^{\sigma(1)} k(\sigma(s), s) \left[f(s, u_0, u_1, \ldots, u_{n-2}) + M \right] \Delta s \\ &\le \int_0^{\sigma(1)} k(\sigma(s), s) \mu(s) g(u_0, u_1, \ldots, u_{n-2}) \Delta s \\ &\le g(\sigma_0 r, \ldots, \sigma_0 r, r) \int_0^{\sigma(1)} k(\sigma(s), s) \mu(s) \Delta s \le r. \end{aligned} \tag{3.40}$$

So, condition (i) of Theorem 3.3 is satisfied. From (b) and (d), we obtain that for $(u_0, u_1, \dots, u_{n-2}) \in \Gamma_R$,

$$\begin{aligned} B(u_0, u_1, \dots, u_{n-2}) &= \eta_{n-2} \int_{\xi}^{\zeta} k(\sigma(s), s)\left[f(s, u_0, u_1, \dots, u_{n-2}) + M\right]\Delta s \\ &\geq \eta_{n-2} \int_{\xi}^{\zeta} k(\sigma(s), s)\nu(s) h(u_0, u_1, \dots, u_{n-2})\Delta s \\ &\geq h(\epsilon\eta_0 R, \epsilon\eta_1 R, \dots, \epsilon\eta_{n-2} R)\eta_{n-2} \int_{\xi}^{\zeta} k(\sigma(s), s)\nu(s)\Delta s \geq R. \end{aligned} \tag{3.41}$$

So, condition (ii) of Theorem 3.3 is satisfied.

Therefore, from Theorem 3.3, BVP(1.1) and (1.2) has a positive solution. □

Corollary 3.5. *Assume that conditions* (*a*) *and* (*c*) *of Corollary 3.4 and the following condition hold:*

$$\lim_{u_0+u_1+\dots+u_{n-2}\to\infty} \min_{t\in[\xi,\zeta]} \frac{f(t, u_0, u_1, \dots, u_{n-2}) + M}{u_0 + u_1 + \cdots + u_{n-2}} \in \mathbb{R}\left(\frac{D_1}{\epsilon\sum_{i=0}^{n-2}\eta_i}, \infty\right), \tag{3.42}$$

where $D_1 = [\eta_{n-2}\int_{\xi}^{\zeta} k(\sigma(s), s)\Delta s]^{-1}$. *Then BVP* (1.1) *and* (1.2) *has one positive solution.*

Proof. We only need to prove that (3.42) implies condition (ii) of Theorem 3.3. From (3.42), we know that there exists R (R may be chosen arbitrary large) such that for $(u_0, u_1, \dots, u_{n-2}) \in \mathbb{R}[\epsilon\eta_0 R, \infty) \times \cdots \times \mathbb{R}[\epsilon\eta_{n-2} R, \infty)$,

$$\min_{t\in[\xi,\zeta]} \frac{f(t, u_0, u_1, \dots, u_{n-2}) + M}{u_0 + u_1 + \cdots + u_{n-2}} \geq \frac{D_1}{\epsilon\sum_{i=0}^{n-2}\eta_i}. \tag{3.43}$$

Hence, for $(t, u_0, u_1, \dots, u_{n-2}) \in [\xi, \zeta] \times \Gamma_R$,

$$f(t, u_0, u_1, \dots, u_{n-2}) + M \geq \frac{D_1}{\epsilon\sum_{i=0}^{n-2}\eta_i}\sum_{i=0}^{n-2} u_i \geq \frac{D_1}{\epsilon\sum_{i=0}^{n-2}\eta_i} \cdot \epsilon\sum_{i=0}^{n-2}\eta_i R = D_1 R. \tag{3.44}$$

Thus, it follows that

$$\begin{aligned} B(u_0, u_1, \dots, u_{n-2}) &= \eta_{n-2} \int_{\xi}^{\zeta} k(\sigma(s), s)\left[f(s, u_0, u_1, \dots, u_{n-2}) + M\right]\Delta s \\ &\geq \eta_{n-2} \int_{\xi}^{\zeta} k(\sigma(s), s) D_1 R \Delta s = R. \end{aligned} \tag{3.45}$$

So, condition (ii) of Theorem 3.3 is satisfied. □

Finally we present an example to illustrate our result.

Example 3.6. Consider the following boundary value problem:

$$u^{\Delta^3}(t) + \sin\left[u^{\Delta}(\sigma(t))\right] + \frac{\exp\left(\left(u(\sigma^2(t)) + u^{\Delta}(\sigma(t))\right)/5\right)}{5(5+u(\sigma^2(t)))} = 0, \quad t \in (0,1) \cap \mathbb{T},$$
$$u(0) = 0, \qquad u^{\Delta}(0) - u^{\Delta^2}(0) = 0, \qquad u^{\Delta^2}(\sigma(1)) = 0, \tag{3.46}$$

where $\mathbb{T} = 0 \cup \{t/8 : t \in \mathbb{N}\}$, $f(t,u_0,u_1) = \sin u_1 + \exp((u_0+u_1)/5)/5(5+u_0)$, $M = 1$, $\alpha = \beta = \delta = 1$, and $\gamma = 0$. Obviously,

$$d = \alpha\delta + \gamma\beta + \alpha\gamma\sigma(1) = 1, \qquad \delta + \gamma\left(\sigma(1) - \sigma^2(1)\right) = 1 \geq 0, \qquad \xi = \frac{3}{8}, \qquad \zeta = \frac{6}{8}. \tag{3.47}$$

Let $\mu(t) = \nu(t) = 1, g(u_0,u_1) = 2 + \exp((u_0+u_1)/5)/5(5+u_0)$, and $h(u_0,u_1) = \exp((u_0+u_1)/5)/5(5+u_0)$. So conditions (a) and (b) in Corollary 3.4 are satisfied. By direct calculation, we obtain that $c = 81/32, \sigma_0 = 11/8, T_0(t) = q(t) = (4/9)(t+1), t \in [0, 10/8]$, and $T_1(t) = \int_0^t q(\tau)\Delta\tau = \sum_{\tau\in[0,t)}[\sigma(\tau)-\tau]q(\tau)$. Since $T_i(t),\ i = 0,1$ are nondecreasing, $\eta_0 = \min_{t\in[3/8,1]} T_1(t) = T_1(3/8) = 3/16$, $\eta_1 = \min_{t\in[3/8,7/8]} q(t) = q(3/8) = 11/18$. In addition, $\int_0^{\sigma(1)} k(\sigma(s),s)\Delta s = 117/64$, $\int_\xi^\zeta k(\sigma(s),s)\Delta s = 39/64$. Take $r = 5, R = 63$. So

$$\begin{aligned} &g(\sigma_0 r, r)\int_0^{\sigma(1)} k(\sigma(s),s)\mu(s)\Delta s \\ &= \left(2 + \frac{\exp([\sigma_0+1]r/5)}{5(5+\sigma_0 r)}\right)\int_0^{\sigma(1)} k(\sigma(s),s)\Delta s \approx 3.99 < 5 = r, \\ &h(\epsilon\eta_0 R, \epsilon\eta_1 R)\eta_1 \int_\xi^\zeta k(\sigma(s),s)\nu(s)\Delta s \\ &= \frac{\exp\left((R-cM)(\eta_0+\eta_1)/5\right)}{5\left[5+(R-cM)\eta_0\right]}\eta_1 \int_\xi^\zeta k(\sigma(s),s)\Delta s \approx 71.25 > 60 = R. \end{aligned} \tag{3.48}$$

Hence, conditions (c) and (d) in Corollary 3.4 are satisfied. Therefore from Corollary 3.4, (3.46) has at least one positive solution.

Remark 3.7. In Example 3.6, because nonlinear term f may attain negative value, the result in [15] is not applicable.

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