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Research Article

Asymptotically Almost Periodic Solutions for Abstract Partial Neutral Integro-Differential Equation

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The existence of asymptotically almost periodic mild solutions for a class of abstract partial neutral integro-differential equations with unbounded delay is studied.

1. Introduction

In this paper, we study the existence of asymptotically almost periodic mild solutions for a class of abstract partial neutral integro-differential equations modelled in the form

$$\frac{d}{dt}\left(x(t)+f(t,x_t)\right)=Ax(t)+\int_0^t B(t-s)x(s)ds+g(t,x_t), \tag{1.1}$$

where $A : D(A) \subset X \to X$ and $B(t) : D(B(t)) \subset X \to X$, $t \geq 0$, are closed linear operators; $(X, \|\cdot\|)$ is a Banach space; the history $x_t : (-\infty, 0] \to X$, $x_t(\theta) = x(t+\theta)$, belongs to some abstract phase space $\mathcal{B}$ defined axiomatically $f, g : I \times \mathcal{B} \to X$ are appropriated functions.

The study of abstract neutral equations is motivated by different practical applications in different technical fields. The literature related to ordinary neutral functional differential equations is very extensive and we refer the reader to Chukwu [1], Hale and Lunel [2], Wu [3], and the references therein. As a practical application, we note that the equation

$$\frac{d}{dt}\left[u(t)-\lambda\int_{-\infty}^{t}C(t-s)u(s)ds\right]=Au(t)+\lambda\int_{-\infty}^{t}B(t-s)u(s)ds-p(t)+q(t) \tag{1.2}$$

arises in the study of the dynamics of income, employment, value of capital stock, and cumulative balance of payment; see [1] for details. In the above system, λ is a real number, the state $u(t)\in\mathbb{R}^n$, $C(\cdot)$, $B(\cdot)$ are $n\times n$ continuous functions matrices, A is a constant $n\times n$ matrix, $p(\cdot)$ represents the government intervention, and $q(\cdot)$ the private initiative. We note that by assuming the solution u is known on $(-\infty,0]$, we can transform this system into an abstract system with unbounded delay described as (1.1).

Abstract partial neutral differential equations also appear in the theory of heat conduction. In the classic theory of heat conduction, it is assumed that the internal energy and the heat flux depend linearly on the temperature u and on its gradient ∇u. Under these conditions, the classic heat equation describes sufficiently well the evolution of the temperature in different types of materials. However, this description is not satisfactory in materials with fading memory. In the theory developed in [4, 5], the internal energy and the heat flux are described as functionals of u and u_x. The next system, see for instance [6–9], has been frequently used to describe this phenomenon,

$$\begin{gathered}\frac{d}{dt}\left[u(t,x)+\int_{-\infty}^{t}k_1(t-s)u(s,x)ds\right]=c\Delta u(t,x)+\int_{-\infty}^{t}k_2(t-s)\Delta u(s,x)ds,\\ u(t,x)=0,\quad x\in\partial\Omega.\end{gathered} \tag{1.3}$$

In this system, $\Omega\subset\mathbb{R}^n$ is open, bounded, and with smooth boundary; $(t,x)\in[0,\infty)\times\Omega$; $u(t,x)$ represents the temperature in x at the time t; c is a physical constant $k_i:\mathbb{R}\to\mathbb{R}$, $i=1,2$, are the internal energy and the heat flux relaxation, respectively. By assuming that the solution u is known on $(-\infty,0]$ and $k_2\equiv 0$, we can transform this system into an abstract system with unbounded delay described in the form (1.1).

Recent contributions on the existence of solutions with some of the previously enumerated properties or another type of almost periodicity to neutral functional differential equations have been made in [10, 11], for the case of neutral ordinary differential equations, and in [12–15] for partial functional differential systems.

The purpose of this work is to study the existence of asymptotically almost periodic mild solutions for the neutral system (1.1). To this end, we study the existence and qualitative properties of an exponentially stable resolvent operator for the integro-differential system

$$\begin{gathered}\frac{dx(t)}{dt}=Ax(t)+\int_{0}^{t}B(t-s)x(s)ds,\quad t\geq 0,\\ x(0)=z\in X.\end{gathered} \tag{1.4}$$

There exists an extensive literature related to the existence and qualitative properties of resolvent operator for integro-differential equations. We refer the reader to the book by

Gripenberg et al. [16] which contains an overview of the theory for the case where the underlying space X has finite dimension. For abstract integro-differential equations described on infinite dimensional spaces, we cite the Prüss book [17] and the papers [18–20], Da Prato et al. [21, 22], and Lunardi [9, 23]. To finish this short description of the related literature, we cite the papers [24–26] where some of the above topics for the case of abstract neutral integro-differential equations with unbounded delay are treated.

To the best of our knowledge, the study of the existence of asymptotically almost periodic solutions of neutral integro-differential equations with unbounded delay described in the abstract form (1.1) is an untreated topic in the literature and this is the main motivation of this article.

To finish this section, we emphasize some notations used in this paper. Let $(Z, \|\cdot\|_Z)$ and $(W, \|\cdot\|_W)$ be Banach spaces. We denote by $\mathcal{L}(Z, W)$ the space of bounded linear operators from Z into W endowed with norm of operators, and we write simply $\mathcal{L}(Z)$ when $Z = W$. By $\mathbf{R}(Q)$, we denote the range of a map Q, and for a closed linear operator $P : D(P) \subseteq Z \to W$, the notation $[D(P)]$ represents the domain of P endowed with the graph norm, $\|z\|_1 = \|z\|_Z + \|Pz\|_W$, $z \in D(P)$. In the case $Z = W$, the notation $\rho(P)$ stands for the resolvent set of P, and $R(\lambda, P) = (\lambda I - P)^{-1}$ is the resolvent operator of P. Furthermore, for appropriate functions $K : [0, \infty) \to Z$ and $S : [0, \infty) \to \mathcal{L}(Z, W)$, the notation $\widehat{K}$ denotes the Laplace transform of K and $S * K$ the convolution between S and K, which is defined by $S * K(t) = \int_0^t S(t-s)K(s)ds$. The notation $B_r(x, Z)$ stands for the closed ball with center at x and radius $r > 0$ in Z. As usual, $C_0([0, \infty), Z)$ represents the subspace of $C_b([0, \infty), Z)$ formed by the functions which vanish at infinity.

2. Preliminaries

In this work, we will employ an axiomatic definition of the phase space $\mathcal{B}$ similar to that in [27]. More precisely, $\mathcal{B}$ will denote a vector space of functions defined from $(-\infty, 0]$ into X endowed with a seminorm denoted by $\|\cdot\|_{\mathcal{B}}$ and such that the following axioms hold.

(A) If $x : (-\infty, \sigma + b) \to X$, with $b > 0$, is continuous on $[\sigma, \sigma + b)$ and $x_\sigma \in \mathcal{B}$, then for each $t \in [\sigma, \sigma + b)$ the following conditions hold:

(i) x_t is in $\mathcal{B}$,

(ii) $\|x(t)\| \le H\|x_t\|_{\mathcal{B}}$,

(iii) $\|x_t\|_{\mathcal{B}} \le K(t - \sigma) \sup\{\|x(s)\| : \sigma \le s \le t\} + M(t - \sigma)\|x_\sigma\|_{\mathcal{B}}$, where $H > 0$ is a constant, and $K, M : [0, \infty) \mapsto [1, \infty)$ are functions such that $K(\cdot)$ and $M(\cdot)$ are respectively continuous and locally bounded, and H, K, M are independent of $x(\cdot)$.

(A1) If $x(\cdot)$ is a function as in (A), then x_t is a $\mathcal{B}$-valued continuous function on $[\sigma, \sigma + b)$.

(B) The space $\mathcal{B}$ is complete.

(C2) If $(\varphi^n)_{n\in\mathbb{N}}$ is a sequence in $C_b((-\infty, 0], X)$ formed by functions with compact support such that $\varphi^n \to \varphi$ uniformly on compact, then $\varphi \in \mathcal{B}$ and $\|\varphi^n - \varphi\|_{\mathcal{B}} \to 0$ as $n \to \infty$.

Remark 2.1. In the remainder of this paper, $\mathfrak{L} > 0$ is such that

$$\|\varphi\|_{\mathcal{B}} \leq \mathfrak{L} \sup_{\theta \leq 0} \|\varphi(\theta)\| \tag{2.1}$$

for every $\varphi : (-\infty, 0] \to X$ continuous and bounded; see [27, Proposition 7.1.1] for details.

Definition 2.2. Let $S(t) : \mathcal{B} \to \mathcal{B}$ be the C_0-semigroup defined by $S(t)\varphi(\theta) = \varphi(0)$ on $[-t, 0]$ and $S(t)\varphi(\theta) = \varphi(t+\theta)$ on $(-\infty, -t]$. The phase space $\mathcal{B}$ is called a fading memory if $\|S(t)\varphi\|_{\mathcal{B}} \to 0$ as $t \to \infty$ for each $\varphi \in \mathcal{B}$ with $\varphi(0) = 0$.

Remark 2.3. In this work, we suppose that there exists a positive $\mathfrak{K}$ such that

$$\max\{K(t), M(t)\} \leq \mathfrak{K} \tag{2.2}$$

for each $t \geq 0$. Observe that this condition is verified, for example, if $\mathcal{B}$ is a fading memory, see [27, Proposition 7.1.5].

Example 2.4 (The phase space $C_r \times L^p(\rho, X)$). Let $r \geq 0, 1 \leq p < \infty$, and let $\rho : (-\infty, -r] \to \mathbb{R}$ be a nonnegative measurable function which satisfies the conditions (g-5) and (g-6) in the terminology of [27]. Briefly, this means that ρ is locally integrable, and there exists a nonnegative, locally bounded function γ on $(-\infty, 0]$ such that $\rho(\xi + \theta) \leq \gamma(\xi)\rho(\theta)$, for all $\xi \leq 0$ and $\theta \in (-\infty, -r) \setminus N_\xi$, where $N_\xi \subseteq (-\infty, -r)$ is a set with Lebesgue measure zero. The space $C_r \times L^p(\rho, X)$ consists of all classes of functions $\varphi : (-\infty, 0] \to X$ such that φ is continuous on $[-r, 0]$, Lebesgue-measurable, and $\rho\|\varphi\|^p$ is Lebesgue integrable on $(-\infty, -r)$. The seminorm in $C_r \times L^p(\rho, X)$ is defined by

$$\|\varphi\|_{\mathcal{B}} : \sup\{\|\varphi(\theta)\| : -r \leq \theta \leq 0\} + \left(\int_{-\infty}^{-r} \rho(\theta)\|\varphi(\theta)\|^p d\theta\right)^{1/p}. \tag{2.3}$$

The space $\mathcal{B} = C_r \times L^p(\rho; X)$ satisfies axioms (A), (A-1), and (B). Moreover, when $r = 0$ and $p = 2$, we can take $H = 1$, $M(t) = \gamma(-t)^{1/2}$, and $K(t) = 1 + (\int_{-t}^{0} \rho(\theta) d\theta)^{1/2}$, for $t \geq 0$; see [27, Theorem 1.3.8] for details.

Now, we need to introduce some concepts, definitions, and technicalities on almost periodical functions.

Definition 2.5. A function $f \in C(\mathbb{R}, Z)$ is almost periodic (a.p.) if for every $\varepsilon > 0$, there exists a relatively dense subset of $\mathbb{R}$, denoted by $\mathscr{H}(\varepsilon, f, Z)$, such that

$$\|f(t+\xi) - f(t)\|_Z < \varepsilon, \quad t \in \mathbb{R},\ \xi \in \mathscr{H}(\varepsilon, f, Z). \tag{2.4}$$

Definition 2.6. A function $f \in C([0, \infty), Z)$ is asymptotically almost periodic (a.a.p.) if there exists an almost periodic function $g(\cdot)$ and $w \in C_0([0, \infty), Z)$ such that $f(\cdot) = g(\cdot) + w(\cdot)$.

The next lemmas are useful characterizations of a.p and a.a.p functions.

Lemma 2.7 (see [28, Theorem 5.5]). *A function $f \in C([0,\infty), Z)$ is asymptotically almost periodic if and only if for every $\varepsilon > 0$ there exist $L(\varepsilon, f, Z) > 0$ and a relatively dense subset of $[0,\infty)$, denoted by $\mathcal{T}(\varepsilon, f, Z)$, such that*

$$\|f(t+\xi) - f(t)\|_Z < \varepsilon, \quad t \geq L(\varepsilon, f, Z),\ \xi \in \mathcal{T}(\varepsilon, f, Z). \tag{2.5}$$

In this paper, $AP(Z)$ and $AAP(Z)$ are the spaces

$$\begin{gathered} AP(Z) = \{f \in C(\mathbb{R}, Z) : f \text{ is a.p.}\}, \\ AAP(Z) = \{f \in C([0,\infty), Z) : f \text{ is a.a.p.}\} \end{gathered} \tag{2.6}$$

endowed with the norms $|||u|||_Z = \sup_{s\in\mathbb{R}} \|u(s)\|_Z$ and $\|u\|_Z = \sup_{s\geq 0} \|u(s)\|_Z$, respectively. We know from the result in [28] that $AP(Z)$ and $AAP(Z)$ are Banach spaces.

Next, $(Z, \|\cdot\|_Z)$ and $(W, \|\cdot\|_W)$ are abstract Banach spaces.

Definition 2.8. Let Ω be an open subset of W.

(a) A continuous function $f \in C(\mathbb{R} \times \Omega, Z)$ (resp., $f \in C([0,\infty) \times \Omega; Z)$) is called pointwise almost periodic (p.a.p.), (resp., pointwise asymptotically almost periodic (p.a.a.p.) if $f(\cdot, x) \in AP(Z)$ (resp., $f(\cdot, x) \in AAP(Z)$) for every $x \in \Omega$.

(b) A continuous function $F \in C(\mathbb{R} \times \Omega, Z)$ is called uniformly almost periodic (u.a.p.), if for every $\varepsilon > 0$ and every compact $K \subset \Omega$ there exists a relatively dense subset of $\mathbb{R}$, denoted by $\mathscr{H}(\varepsilon, f, K, Z)$, such that

$$\|f(t+\xi, y) - f(t, y)\|_Z \leq \varepsilon, \quad (t, \xi, y) \in \mathbb{R} \times \mathscr{H}(\varepsilon, f, K, Z) \times K. \tag{2.7}$$

(c) A continuous function $f : C([0,\infty) \times \Omega, Z)$ is called uniformly asymptotically almost periodic (u.a.a.p.), if for every $\varepsilon > 0$ and every compact $K \subset \Omega$ there exists a relatively dense subset of $[0,\infty)$, denoted by $\mathcal{T}(\varepsilon, f, K, Z)$, and $L(\varepsilon, f, K, Z) > 0$ such that

$$\|f(t+\xi, y) - f(t, y)\|_Z \leq \varepsilon, \quad t \geq L(\varepsilon, f, K, Z),\ (\xi, y) \in \mathcal{T}(\varepsilon, f, K, Z) \times K. \tag{2.8}$$

The next lemma summarizes some properties which are fundamental to obtain our results.

Lemma 2.9 (see [29, Theorem 1.2.7]). *Let $\Omega \subset W$ be an open set. Then the following properties hold.*

(a) *If $f \in C(\mathbb{R} \times \Omega, Z)$ is p.a.p. and satisfies a local Lipschitz condition at $x \in \Omega$, uniformly at t, then f is u.a.p.*

(b) *If $f \in C([0,\infty) \times \Omega, Z)$ is p.a.a.p. and satisfies a local Lipschitz condition at $x \in \Omega$, uniformly at t, then f is u.a.a.p.*

(c) *If $x \in AP(X)$, then $t \mapsto x_t \in AP(\mathcal{B})$. Moreover, if $\mathcal{B}$ is a fading memory space and $z \in C(\mathbb{R}, X)$ is such that $z_0 \in \mathcal{B}$ and $z \in AAP(X)$, then $t \mapsto z_t \in AAP(\mathcal{B})$.*

(d) *If* $f \in C(\mathbb{R} \times \Omega, Z)$ *is u.a.p. and* $y \in AP(W)$ *is such that* $\overline{\{y(t) : t \in \mathbb{R}\}}^W \subset \Omega$, *then* $t \mapsto f(t, y(t)) \in AP(Z)$.

(e) *If* $f \in C([0,\infty) \times \Omega, Z)$ *is u.a.a.p and* $y \in AAP(W)$ *is such that* $\overline{\{y(t) : t \in \mathbb{R}\}}^W \subset \Omega$, *then* $t \mapsto f(t, y(t)) \in AAP(Z)$.

3. Resolvent Operators

In this section, we study the existence and qualitative properties of an exponentially resolvent operator for the integro-differential abstract Cauchy problem

$$\begin{gathered} \frac{dx(t)}{dt} = Ax(t) + \int_0^t B(t-s)x(s)ds, \\ x(0) = x \in X. \end{gathered} \tag{3.1}$$

The results obtained for the resolvent operator in this section are similar to those that can be found, for instance, in the papers [21, 23, 30]. In this paper, we prove the necessary estimates for the proof of an existence theorem of asymptotically almost periodic solutions for (1.1). For the better comprehension of the subject, we will introduce the following definitions, hypothesis, and results.

We introduce the following concept of resolvent operator for integro-differential problem (3.1).

Definition 3.1. A one-parameter family of bounded linear operators $(\mathcal{R}(t))_{t\geq 0}$ on X is called a resolvent operator of (3.1) if the following conditions are verified.

(a) Function $\mathcal{R}(\cdot) : [0,\infty) \to \mathcal{L}(X)$ is strongly continuous and $\mathcal{R}(0)x = x$ for all $x \in X$.

(b) For $x \in D(A)$, $\mathcal{R}(\cdot)x \in C([0,\infty), [D(A)]) \bigcap C^1([0,\infty), X)$, and

$$\frac{d\mathcal{R}(t)x}{dt} = A\mathcal{R}(t)x + \int_0^t B(t-s)\mathcal{R}(s)xds, \tag{3.2}$$

$$\frac{d\mathcal{R}(t)x}{dt} = \mathcal{R}(t)Ax + \int_0^t \mathcal{R}(t-s)B(s)xds, \tag{3.3}$$

for every $t \geq 0$,

(c) There exist constants $M > 0, \beta$ such that $\|R(t)\| \leq Me^{\beta t}$ for every $t \geq 0$.

Definition 3.2. A resolvent operator $(\mathcal{R}(t))_{t\geq 0}$ of (3.1) is called exponentially stable if there exist positive constants M, α such that $\|\mathcal{R}(t)\| \leq Me^{-\alpha t}$.

In this work, we always assume that the following conditions are verified.

(H1) The operator $A : D(A) \subseteq X \to X$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on X, and there are constants $M_0 > 0, \omega \in \mathbb{R}$, and $\vartheta \in (\pi/2, \pi)$ such that $\rho(A) \supseteq \Lambda_{\omega,\vartheta} = \{\lambda \in \mathbb{C} : \lambda \neq \omega, |\arg(\lambda-\omega)| < \vartheta\}$ and $\|R(\lambda, A)\| \leq M_0/|\lambda-\omega|$ for all $\lambda \in \Lambda_{\omega,\vartheta}$.

(H2) For all $t \geq 0, B(t) : D(B(t)) \subseteq X \rightarrow X$ is a closed linear operator, $D(A) \subseteq D(B(t))$, and $B(\cdot)x$ is strongly measurable on $(0,\infty)$ for each $x \in D(A)$. There exists $b(\cdot) \in L^1_{\text{loc}}(\mathbb{R}^+)$ such that $\widehat{b}(\lambda)$ exists for $\text{Re}(\lambda) > 0$ and $\|B(t)x\| \leq b(t)\|x\|_1$ for all $t > 0$ and $x \in D(A)$. Moreover, the operator valued function $\widehat{B} : \Lambda_{\omega,\pi/2} \rightarrow \mathcal{L}([D(A)], X)$ has an analytical extension (still denoted by $\widehat{B}$) to $\Lambda_{\omega,\vartheta}$ such that $\|\widehat{B}(\lambda)x\| \leq \|\widehat{B}(\lambda)\| \, \|x\|_1$ for all $x \in D(A)$, and $\|\widehat{B}(\lambda)\| = O(1/|\lambda|)$ as $|\lambda| \rightarrow \infty$.

(H3) There exist a subspace $D \subseteq D(A)$ dense in $[D(A)]$ and positive constants C_i, $i = 1,2$, such that $A(D) \subseteq D(A)$, $\widehat{B}(\lambda)(D) \subseteq D(A)$, and $\|A\widehat{B}(\lambda)x\| \leq C_1\|x\|$ for every $x \in D$ and all $\lambda \in \Lambda_{\omega,\vartheta}$.

In the sequel, for $r > 0$, $\theta \in (\pi/2, \vartheta)$, and $w \in \mathbb{R}$, set

$$\Lambda_{r,\omega,\theta} = \left\{\lambda \in \mathbb{C} : \lambda \neq \omega, \ |\lambda| > r, \left|\arg(\lambda - \omega)\right| < \theta\right\}, \tag{3.4}$$

and for $\omega + \Gamma^i_{r,\theta}$, $i = 1,2,3$, the paths

$$\begin{aligned} \omega + \Gamma^1_{r,\theta} &= \left\{\omega + te^{i\theta} : t \geq r\right\}, \\ \omega + \Gamma^2_{r,\theta} &= \left\{\omega + re^{i\xi} : -\theta \leq \xi \leq \theta\right\}, \\ \omega + \Gamma^3_{r,\theta} &= \left\{\omega + te^{-i\theta} : t \geq r\right\}, \end{aligned} \tag{3.5}$$

with $\omega + \Gamma_{r,\theta} = \bigcup_{i=1}^3 \omega + \Gamma^i_{r,\theta}$ are oriented counterclockwise. In addition, $\Psi(G)$ is the set

$$\Psi(G) = \left\{\lambda \in \mathbb{C} : G(\lambda) := \left(\lambda I - A - \widehat{B}(\lambda)\right)^{-1} \in \mathcal{L}(X)\right\}. \tag{3.6}$$

We next study some preliminary properties needed to establish the existence of a resolvent operator for the problem (3.1).

Lemma 3.3. *There exists $r_1 > 0$ such that $\Lambda_{r_1,\omega,\vartheta} \subseteq \Psi(G)$ and the function $G : \Lambda_{r_1,\omega,\vartheta} \rightarrow \mathcal{L}(X)$ is analytic. Moreover,*

$$G(\lambda) = R(\lambda, A)\left[I - \widehat{B}(\lambda)R(\lambda, A)\right]^{-1}, \tag{3.7}$$

and there exist constants M_i for $i = 1,2,3$ such that

$$\|G(\lambda)\| \leq \frac{M_1}{|\lambda - \omega|}, \tag{3.8}$$

$$\|AG(\lambda)x\| \leq \frac{M_2}{|\lambda - \omega|}\|x\|_1, \quad x \in D(A), \tag{3.9}$$

$$\|AG(\lambda)\| \leq M_3, \tag{3.10}$$

for every $\lambda \in \Lambda_{r_1,\omega,\vartheta}$.

Proof. Since

$$\left\|\widehat{B}(\lambda)R(\lambda,A)\right\| \le \left\|\widehat{B}(\lambda)\right\| \|R(\lambda,A)\|_1 \le \left(\frac{M_0\left\|\widehat{B}(\lambda)\right\|}{|\lambda-\omega|} + \frac{M_0|\lambda|\left\|\widehat{B}(\lambda)\right\|}{|\lambda-\omega|} + \left\|\widehat{B}(\lambda)\right\|\right), \tag{3.11}$$

fixed $\varepsilon < 1$, there exists a positive number r_1 such that $\|\widehat{B}(\lambda)R(\lambda,A)\| \le \varepsilon$ for $\lambda \in \Lambda_{r_1,\omega,\vartheta}$. Consequently, the operator $I - \widehat{B}(\lambda)R(\lambda,A)$ has a continuous inverse with $\|(I - \widehat{B}(\lambda)R(\lambda,A))^{-1}\| \le 1/(1-\varepsilon)$. Moreover, for $x \in X$, we have

$$\left(\lambda I - \widehat{B}(\lambda) - A\right)R(\lambda,A)\left(I - \widehat{B}(\lambda)R(\lambda,A)\right)^{-1}x = x, \tag{3.12}$$

and for $x \in D(A)$,

$$R(\lambda,A)\left(I - \widehat{B}(\lambda)R(\lambda,A)\right)^{-1}\left(\lambda I - \widehat{B}(\lambda) - A\right)x = x, \tag{3.13}$$

which shows (3.7) and that $\Lambda_{r_1,\omega,\vartheta} \subseteq \Psi(G)$. Now, from (3.7) we obtain $\mathbf{R}(G(\lambda)) \subseteq D(A)$ and

$$AG(\lambda) = (\lambda R(\lambda,A) - I)\left(I - \widehat{B}(\lambda)R(\lambda,A)\right)^{-1}. \tag{3.14}$$

Consequently,

$$\|AG(\lambda)\| \le \frac{1}{1-\varepsilon}\|\lambda R(\lambda,A) - I\| \le \frac{1}{1-\varepsilon}\left(M_0 + \frac{M_0|\omega|}{|\lambda-\omega|} + 1\right) \le M_3, \tag{3.15}$$

the functions $G, AG : \Lambda_{r_1,\omega,\vartheta} \to \mathcal{L}(X)$ are analytic, and estimates (3.8), and (3.10) are valid. In addition, for $x \in D(A)$, we can write

$$\begin{aligned}
\|AG(\lambda)x\| &\le \left\|AR(\lambda,A)\left(I - \widehat{B}(\lambda)R(\lambda,A)\right)^{-1}x - AR(\lambda,A)x\right\| + \|R(\lambda,A)Ax\| \\
&= \left\|\left[AR(\lambda,A)\left(I - \left(I - \widehat{B}(\lambda)R(\lambda,A)\right)\right)\right]\left(I - \widehat{B}(\lambda)R(\lambda,A)\right)^{-1}x\right\| + \|R(\lambda,A)Ax\| \\
&= \left\|AG(\lambda)\widehat{B}(\lambda)R(\lambda,A)x\right\| + \|R(\lambda,A)Ax\|
\end{aligned}$$

$$\begin{aligned}&\leq M_3\|\widehat{B}(\lambda)\|\|R(\lambda,A)x\|_1+\|R(\lambda,A)Ax\|\\&\leq \frac{M_2}{|\lambda-\omega|}\|x\|_1,\end{aligned}\tag{3.16}$$

for $|\lambda|$ sufficiently large. This proves (3.9) and completes the proof. □

Observation 1. If $\mathcal{R}(\cdot)$ is a resolvent operator for (3.1), it follows from (3.3) that $\widehat{\mathcal{R}}(\lambda)(\lambda I - A - \widehat{B}(\lambda))x = x$ for all $x \in D(A)$. Applying Lemma 3.3 and the properties of the Laplace transform, we conclude that $\mathcal{R}(\cdot)$ is the unique resolvent operator for (3.1).

In the remainder of this section, r and θ are numbers such that $r > r_1$ and $\theta \in (\pi/2, \vartheta)$. Moreover, we denote by C a generic constant that represents any of the constants involved in the statements of Lemma 3.3 as well as any other constant that arises in the estimate that follows. We now define the operator family $(\mathcal{R}(t))_{t\geq 0}$ by

$$\mathcal{R}(t)=\begin{cases}\dfrac{1}{2\pi i}\displaystyle\int_{\omega+\Gamma_{r,\theta}} e^{\lambda t}G(\lambda)d\lambda, & t>0,\\ I, & t=0.\end{cases}\tag{3.17}$$

We will next establish that $(\mathcal{R}(t))_{t\geq 0}$ is a resolvent operator for (3.1).

Lemma 3.4. *The function $\mathcal{R}(\cdot)$ is exponentially bounded in $\mathcal{L}(X)$.*

Proof. If $t > 1$, from (3.17) and estimate (3.8), we get

$$\begin{aligned}\|\mathcal{R}(t)\| &\leq \frac{C}{\pi}\int_r^\infty e^{t(\omega+s\cos\theta)}\frac{ds}{s}+\frac{C}{2\pi}\int_{-\theta}^{\theta}e^{t(\omega+r\cos\xi)}d\xi\\&\leq\left(\frac{C}{\pi r|\cos\theta|}+\frac{C\theta}{\pi}e^{rt}\right)e^{\omega t}.\end{aligned}\tag{3.18}$$

On the other hand, using that $G(\cdot)$ is analytic on $\Lambda_{r,\omega,\theta}$, for $t \in (0,1)$, we obtain

$$\begin{aligned}\|\mathcal{R}(t)\| &= \left\|\frac{1}{2\pi i}\int_{\omega+\Gamma_{r/t,\theta}}e^{\lambda t}G(\lambda)d\lambda\right\|\\&\leq \frac{C}{\pi}\int_{r/t}^\infty e^{t(\omega+s\cos\theta)}\frac{ds}{s}+\frac{C}{2\pi}\int_{-\theta}^{\theta}e^{t\omega+r\cos\xi}d\xi\\&\leq\left(\frac{C}{\pi}\int_r^\infty e^{u\cos\theta}\frac{du}{u}+\frac{C}{2\pi}\int_{-\theta}^{\theta}e^{r\cos\xi}d\xi\right)e^{\omega t}\\&\leq\left(\frac{C}{\pi r|\cos\theta|}+\frac{C\theta}{\pi}e^{r}\right)e^{\omega t}.\end{aligned}\tag{3.19}$$

This complete the proof. □

Lemma 3.5. *The operator function $\mathcal{R}(\cdot)$ is exponentially bounded in $\mathcal{L}([D(A)])$.*

Proof. It follows from (3.9) that the integral in

$$S(t) = \frac{1}{2\pi i}\int_{\omega+\Gamma_{r,\theta}} e^{\lambda t} AG(\lambda)d\lambda, \quad t > 0, \tag{3.20}$$

is absolutely convergent in $\mathcal{L}([D(A)], X)$ and defines a linear operator $S(t) \in \mathcal{L}([D(A)], X)$. Using that A is closed, we can affirm that $S(t) = A\mathcal{R}(t)$. From Lemma 3.3, $G : \Lambda_{r,\omega,\vartheta} \to \mathcal{L}([D(A)])$ is analytic and $\|G(\lambda)\|_1 \leq C|\lambda - \omega|^{-1}$. If $t > 1$ and $x \in D(A)$, we have

$$\begin{aligned} \|A\mathcal{R}(t)x\| &\leq \left(\frac{C}{\pi}\int_r^\infty e^{t(\omega+s\cos\theta)}\frac{ds}{s} + \frac{C}{2\pi}\int_{-\theta}^{\theta} e^{t(\omega+r\cos\xi)}d\xi\right)\|x\|_1 \\ &\leq \left(\frac{C}{\pi r|\cos\theta|} + \frac{C\theta}{\pi}e^{rt}\right)e^{\omega t}\|x\|_1. \end{aligned} \tag{3.21}$$

For $t \in (0, 1)$ and $x \in D(A)$, we get

$$\begin{aligned} \|A\mathcal{R}(t)x\| &= \left\|\frac{1}{2\pi i}\int_{\omega+\Gamma_{r/t,\theta}} e^{\lambda t}AG(\lambda)xd\lambda\right\| \\ &\leq \left(\frac{C}{\pi}\int_{r/t}^\infty e^{t(\omega+s\cos\theta)}\frac{ds}{s} + \frac{C}{2\pi}\int_{-\theta}^{\theta} e^{t\omega+r\cos\xi}d\xi\right)\|x\|_1 \\ &\leq \left(\frac{C}{\pi r|\cos\theta|} + \frac{C\theta}{\pi}e^{r}\right)e^{\omega t}\|x\|_1. \end{aligned} \tag{3.22}$$

From before and Lemma 3.4, we infer that $\mathcal{R}(\cdot)$ is exponentially bounded in $\mathcal{L}([D(A)])$. The proof is finished. □

Lemma 3.6. *The function $\mathcal{R} : [0, \infty) \to \mathcal{L}(X)$ is strongly continuous.*

Proof. It is clear from (3.17) that $\mathcal{R}(\cdot)x$ is continuous at $t > 0$ for every $x \in X$. We next establish the continuity at $t = 0$. Let $\omega \geq 0$ and N be sufficiently large, using that

$$\frac{1}{2\pi i}\int_{\omega+\Gamma_{r,\theta}} \lambda^{-1}e^{\lambda t}d\lambda = \lim_{N\to\infty}\frac{1}{2\pi i}\int_{\{\omega+\Gamma_{r,\theta}:\ |r|\leq N\}\cup\ \omega+C_{N,\theta}} \lambda^{-1}e^{\lambda t}d\lambda = 1, \tag{3.23}$$

where $\omega + C_{N,\theta}$ represent the curve $\omega + Ne^{i\xi}$ for $\theta \leq \xi \leq 2\pi - \theta$.

For $x \in D(A)$ and $0 < t \leq 1$, we get

$$\begin{aligned} \mathcal{R}(t)x - x &= \frac{1}{2\pi i}\int_{\omega+\Gamma_{r,\theta}} \left(e^{\lambda t}G(\lambda)x - \lambda^{-1}e^{\lambda t}x\right)d\lambda \\ &= \frac{1}{2\pi i}\int_{\omega+\Gamma_{r,\theta}} e^{\lambda t}\lambda^{-1}G(\lambda)\left(A + \widehat{B}(\lambda)\right)xd\lambda. \end{aligned} \tag{3.24}$$

Furthermore, it follows from (3.8), and assumption (H2) that

$$\left\| e^{\lambda t}\lambda^{-1}G(\lambda)\left(A+\widehat{B}(\lambda)\right)x\right\| \le \frac{e^{\omega+r}C}{|\lambda||\lambda-\omega|} = \Phi(\lambda), \tag{3.25}$$

where $\Phi(\cdot)$ is integrable for $\lambda \in \omega+\Gamma_{r,\theta}$. From the Lebesgue dominated convergence theorem, we infer that

$$\lim_{t\to 0^+}(\mathcal{R}(t)x-x)\frac{1}{2\pi i}\int_{\omega+\Gamma_{r,\theta}} \lambda^{-1}G(\lambda)\left(A+\widehat{B}(\lambda)\right)x\,d\lambda. \tag{3.26}$$

Let now $\omega+C_{L,\theta}$ be the curve $\omega+Le^{i\xi}$ for $\theta \le \xi \le 2\pi-\theta$. Turning to apply Cauchy's theorem combining with the estimate

$$\left\|\int_{\omega+C_{L,\theta}} \lambda^{-1}G(\lambda)\left(A+\widehat{B}(\lambda)\right)x\,d\lambda\right\| \le \frac{C\theta L}{(L-\omega)^2}, \tag{3.27}$$

we obtain

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\omega+\Gamma_{r,\theta}} \lambda^{-1}G(\lambda)\left(A+\widehat{B}(\lambda)\right)x\,d\lambda \\ &\quad= \lim_{L\to\infty}\frac{1}{2\pi i}\int_{\{\omega+\Gamma_{r,\theta}:\ |r|\le L\}\cup\, \omega+C_{L,\theta}} \lambda^{-1}G(\lambda)\left(A+\widehat{B}(\lambda)\right)x\,d\lambda = 0,\end{aligned} \tag{3.28}$$

we can affirm that $\lim_{t\to 0^+}(\mathcal{R}(t)x-x)=0$ for all $x\in D(A)$, which completes and the proof since $D(A)$ is dense in X and $\mathcal{R}(\cdot)$ is bounded on $[0,1]$.

Notice that $\omega<0$, the sectors $\Lambda_{r,0,\vartheta} \subseteq \Lambda_{r,\omega,\vartheta}$, from Lemma 3.3, $G:\Lambda_{r,\omega,\vartheta}\to \mathcal{L}(X)$ is analytic. Consider the contours

$$\begin{aligned}\gamma_1 &= \{\lambda = s-iN\sin(\theta) : \cos(\theta)\le s\le \omega+\cos(\theta)\},\\ \gamma_2 &= \{\lambda = s+iN\sin(\theta) : \omega+\cos(\theta)\le s\le \cos(\theta)\},\\ \omega+\Gamma^N_{r,\theta} &= \left\{\omega+te^{i\theta} : r\le t\le N\right\}\cup\left\{\omega+re^{i\xi} : -\theta\le\xi\le\theta\right\}\cup\left\{\omega+te^{-i\theta} : r\le t\le N\right\},\\ 0+\Gamma^N_{r,\theta} &= \left\{te^{i\theta} : r\le t\le N\right\}\cup\left\{re^{i\xi} : -\theta\le\xi\le\theta\right\}\cup\left\{te^{-i\theta} : r\le t\le N\right\},\end{aligned} \tag{3.29}$$

and $R_N = \omega+\Gamma^N_{r,\theta}\cup\gamma_2\cup 0+\Gamma^N_{r,\theta}\cup\gamma_1$, oriented counterclockwise. By Cauchy theorem for $0<t\le 1$, we obtain

$$\int_{R_N} e^{\lambda t}G(\lambda)\,d\lambda = 0. \tag{3.30}$$

The following estimate:

$$\begin{aligned}\left\|\int_{\gamma_1} e^{\lambda t}G(\lambda)d\lambda\right\| &\le \int_{\cos(\theta)}^{\omega+\cos(\theta)} e^{\mathrm{Re}(s-iN\sin(\theta))t}\frac{C}{|s-iN\sin(\theta)-\omega|}ds \\ &\le \int_{\cos(\theta)}^{\omega+\cos(\theta)} e^{st}\frac{C}{N|\sin(\theta)|}ds \le \frac{C}{N|\sin(\theta)|}\left(\frac{e^{\omega t}-1}{t}\right)e^{\cos(\theta)t} \\ &\le \frac{C}{N|\sin(\theta)|}\,e\end{aligned} \tag{3.31}$$

is the one responsible for the fact that the integral $\int_{\gamma_1} e^{\lambda t}G(\lambda)d\lambda$ tends to 0 as N tend to $+\infty$, in a similar way the integral $\int_{\gamma_2} e^{\lambda t}G(\lambda)d\lambda$, tend to 0 as N tend to $+\infty$, so that

$$\frac{1}{2\pi i}\int_{\omega+\Gamma_{r,\theta}} e^{\lambda t}G(\lambda)d\lambda \frac{1}{2\pi i}\int_{0+\Gamma_{r,\theta}} e^{\lambda t}G(\lambda)d\lambda. \tag{3.32}$$

For $x \in D(A)$, we obtain

$$\begin{aligned}\mathcal{R}(t)x - x &= \frac{1}{2\pi i}\int_{0+\Gamma_{r,\theta}} \left(e^{\lambda t}G(\lambda)x - \lambda^{-1}e^{\lambda t}x\right)d\lambda \\ &= \frac{1}{2\pi i}\int_{0+\Gamma_{r,\theta}} e^{\lambda t}\lambda^{-1}G(\lambda)\left(A + \widehat{B}(\lambda)\right)x d\lambda,\end{aligned} \tag{3.33}$$

and proceeding as before, we obtain $\lim_{t\to 0^+}(\mathcal{R}(t)x - x) = 0$ for all $x \in X$, which ends the proof. □

The following result can be proved with an argument similar to that used in the proof of the preceding lemma with changing $[D(A)]$ by D.

Lemma 3.7. *The function $\mathcal{R} : [0,\infty) \to \mathcal{L}([D(A)])$ is strongly continuous.*

We next set $\delta = \min\{\vartheta - \pi/2, \pi - \vartheta\}$.

Lemma 3.8. *The function $\mathcal{R} : (0,\infty) \to \mathcal{L}(X)$ has an analytic extension to $\Lambda_{\delta,0}$, and*

$$\frac{d\mathcal{R}(z)}{dz} = \frac{1}{2\pi i}\int_{\omega+\Gamma_{r,\theta}} \lambda e^{\lambda z}G(\lambda)d\lambda, \quad z \in \Lambda_{\delta,0}. \tag{3.34}$$

Proof. For $\lambda \in \omega + \Gamma_{r,\theta}$ and $z \in \Lambda_{\delta,0}$, we can write $\lambda z = \omega|z|e^{i\arg(z)} + s|z|e^{i(\arg(z)+\xi)}$, where $\pi/2 < \arg(z) + \xi < \pi$, $-\theta \le \xi \le \theta$ and $s \ge r$. If $|z| > 1$, from (3.8) and (3.17), we obtain

$$\begin{aligned}
\|\mathcal{R}(z)\| &\le \frac{1}{2\pi i}\int_{\omega+\Gamma_{r,\theta}} e^{\mathrm{Re}(\lambda z)}\frac{C}{|\lambda-\omega|}\,|d\lambda| \\
&\le \frac{C}{\pi}\int_r^{\infty} e^{\omega|z|\cos(\arg(z))+s|z|\cos(\arg(z)+\theta)}\frac{ds}{s} \\
&\quad+\frac{C}{2\pi}\int_{-\theta}^{\theta} e^{\omega|z|\cos(\arg(z))+r|z|\cos(\arg(z)+\xi)}d\xi \\
&\le \left(\frac{C}{\pi r\left|\cos(\arg(z)+\theta)\right|}+\frac{C\theta}{\pi}e^{r|z|}\right)e^{\omega|z|\cos(\arg(z))}.
\end{aligned} \tag{3.35}$$

Using that $G(\cdot)$ is analytic on $\Lambda_{r,\omega,\theta}$, for $z \in \Lambda_{\delta,0}$, $0 < |z| < 1$, we get

$$\begin{aligned}
\|\mathcal{R}(z)\| &= \left\|\frac{1}{2\pi i}\int_{\omega+\Gamma_{r/|z|,\theta}} e^{\lambda z}G(\lambda)d\lambda\right\| \\
&\le \frac{C}{\pi}\int_{r/|z|}^{\infty} e^{\omega|z|\cos(\arg(z))+s|z|\cos(\arg(z)+\theta)}\frac{ds}{s} \\
&\quad+\frac{C}{2\pi}\int_{-\theta}^{\theta} e^{\omega|z|\cos(\arg(z))+r\cos(\arg(z)+\xi)}d\xi \\
&\le \left(\frac{C}{\pi}\int_r^{\infty} e^{u\cos(\arg(z)+\theta)}\frac{du}{u}+\frac{C}{2\pi}\int_{-\theta}^{\theta} e^{r\cos(\arg(z)+\xi)}d\xi\right)e^{\omega|z|\cos(\arg(z))} \\
&\le \left(\frac{C}{\pi r\left|\cos(\arg(z)+\theta)\right|}+\frac{C\theta}{\pi}e^{r}\right)e^{\omega|z|\cos(\arg(z))}.
\end{aligned} \tag{3.36}$$

This property allows us to define the extension $\mathcal{R}(z)$ by this integral.

Similarly, the integral on the right hand side of (3.34) is also absolutely convergent in $\mathcal{L}(X)$ and strong, continuous on X for $z \in \Lambda_{\delta,0}$. For $\lambda \in \omega + \Gamma_{r,\theta}$,

$$\begin{gathered}
\left\|\frac{e^{\lambda(z+h)}-e^{\lambda z}}{h}\,G(\lambda)-\lambda e^{\lambda z}G(\lambda)\right\| \le \left|\frac{e^{\lambda(z+h)}-e^{\lambda z}}{h}-\lambda e^{\lambda z}\right|\frac{C}{r} \longrightarrow 0 \quad \text{as } |h| \longrightarrow 0, \\
\left\|\frac{e^{\lambda(z+h)}-e^{\lambda z}}{h}\,G(\lambda)-\lambda e^{\lambda z}G(\lambda)\right\| \le e^{\mathrm{Re}(\lambda z)}\frac{C}{|\lambda-\omega|} = \Sigma(\lambda),
\end{gathered} \tag{3.37}$$

where $\Sigma(\cdot)$ is integrable for $\lambda \in \omega + \Gamma_{r,\theta}$. From the Lebesgue dominated convergence theorem, we obtain that $\mathcal{R}'(z)$ verifies (3.34). The proof is ended. □

Lemma 3.9. *For every* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}(\lambda) > \max\{0, \omega + r\}$, $\widehat{\mathcal{R}}(\lambda) = G(\lambda)$.

Proof. Using that $G(\cdot)$ is analytic on $\Lambda_{r,\omega,\theta}$ and that the integrals involved in the calculus are absolutely convergent, we have

$$\begin{aligned}
\widehat{\mathcal{R}}(\lambda) = \int_0^\infty e^{-\lambda t}\mathcal{R}(t)dt &= \int_0^\infty \frac{1}{2\pi i}\int_{\omega+\Gamma_{r,\theta}} e^{-(\lambda-\gamma)t}G(\gamma)d\gamma dt \\
&= \frac{1}{2\pi i}\int_{\omega+\Gamma_{r,\theta}} (\lambda-\gamma)^{-1}G(\gamma)d\gamma \\
&= \lim_{L\to\infty}\left(\frac{1}{2\pi i}\int_{\{\omega+\Gamma_{r,\theta}:\ |r|\le L\}\cup\ \omega+C_{L,\theta}} (\lambda-\gamma)^{-1}G(\gamma)d\gamma\right) \\
&= G(\lambda).
\end{aligned} \tag{3.38}$$

□

Theorem 3.10. *The function* $\mathcal{R}(\cdot)$ *is a resolvent operator for the system* (3.1).

Proof. Let $x \in D(A)$. From Lemma 3.9, for $\operatorname{Re}(\lambda) > \max\{0, \omega + r\}$,

$$\widehat{\mathcal{R}}(\lambda)\left[\lambda I - A - \widehat{B}(\lambda)\right]x = x, \tag{3.39}$$

which implies

$$\widehat{\mathcal{R}}(\lambda)x = \frac{1}{\lambda}x + \frac{1}{\lambda}\widehat{\mathcal{R}}(\lambda)A + \frac{1}{\lambda}\widehat{\mathcal{R}}(\lambda)\widehat{B}(\lambda)x. \tag{3.40}$$

Applying [31, Proposition 1.6.4, Corollary 1.6.5], we get

$$\mathcal{R}(t)x = x + \int_0^t \mathcal{R}(s)Axds + \int_0^t\int_0^s \mathcal{R}(s-\xi)B(\xi)xd\xi ds \tag{3.41}$$

which in turn implies that

$$\frac{d\mathcal{R}(t)}{dt}x = \mathcal{R}(t)Ax + \int_0^t \mathcal{R}(t-s)B(s)xds. \tag{3.42}$$

Arguing as above but using the equality $[\lambda I - A - \widehat{B}(\lambda)]\widehat{\mathcal{R}}(\lambda)x = x$, we obtain that (3.2) holds.

On the other hand, by Lemma 3.8 we infer that $\mathcal{R}(\cdot)x \in C^1((0,\infty), X)$. Next, we analyze the differentiability on $t = 0$. Let $a > 0$ and $x \in D(A)$, for all $\varepsilon > 0$; we can choose $\delta \in [0, a]$ such that

$$\sup_{t\in(0,\delta]} \|\mathcal{R}(t)Ax + \int_0^t \mathcal{R}(t-s)B(s)ds - Ax\| < \varepsilon. \tag{3.43}$$

For $\zeta \in X'$ and $t \in (0, \delta)$, there exists $c_{\zeta,t} \in (0, t)$ such that

$$\frac{\zeta \circ \mathcal{R}(t)x - \zeta \circ \mathcal{R}(0)x}{t} = \zeta\left(\mathcal{R}(c_{\zeta,t})Ax + \int_0^{c_{\zeta,t}} \mathcal{R}(c_{\varphi,t} - s)B(s)x\,ds\right). \tag{3.44}$$

Consequently, for $t \in (0, \delta)$ we have that

$$\begin{aligned} \left\| \frac{\mathcal{R}(t)x - \mathcal{R}(0)x}{t} - Ax \right\| &= \sup_{\|\zeta\| \le 1} \left| \frac{\zeta \circ \mathcal{R}(t)x - \zeta \circ \mathcal{R}(0)x}{t} - \zeta(Ax) \right| \\ &\le \sup_{s \in (0,\delta]} \left\| \mathcal{R}(s)Ax + \int_0^s \mathcal{R}(s-\tau)B(\tau)x\,d\tau - Ax \right\|, \end{aligned} \tag{3.45}$$

which proves the existence of the right derivative of $\mathcal{R}(\cdot)$ at zero and that $(d/dt)\mathcal{R}(t)|_{t=0} = Ax$. This proves that resolvent equation (3.3) is valid for every $t \ge 0$ and $\mathcal{R}(\cdot)x \in C^1([0,\infty); X)$ for every $x \in D(A)$. This completes the proof. □

Corollary 3.11. *If $\omega + r < 0$, then the function $\mathcal{R}(\cdot)$ is an exponentially stable resolvent operator for the system* (3.1).

In the next result, we denote by $(-A)^{\vartheta}$ the fractional power of the operator $(-A)$ (see [32] for details).

Theorem 3.12. *Suppose that the conditions $(H1)$–$(H3)$ are satisfied. Then there exists a positive number C such that*

$$\left\| (-A)^{\vartheta} \mathcal{R}(t) \right\| \le \begin{cases} Ce^{(r+\omega)t}, & t \ge 1, \\ Ce^{(r+\omega)t} t^{-\vartheta}, & t \in (0,1), \end{cases} \tag{3.46}$$

for all $\vartheta \in (0,1)$.

Proof. Let $\vartheta \in (0,1)$. From [32, Theorem 6.10], there exists $C_{\vartheta} > 0$ such that

$$\left\| (-A)^{\vartheta} x \right\| \le C_{\vartheta} \|Ax\|^{\vartheta} \|x\|^{1-\vartheta}, \quad x \in D(A). \tag{3.47}$$

Since $G(\cdot)$ is a $D(A)$ valued function, for all $x \in X$

$$\begin{aligned} \left\| (-A)^{\vartheta} G(\lambda)x \right\| &\le C_{\vartheta} \|AG(\lambda)x\|^{\vartheta} \|G(\lambda)x\|^{1-\vartheta} \\ &\le C_{\vartheta} M_3^{\vartheta} \|x\|^{\vartheta} \frac{M_1^{1-\vartheta}}{|\lambda - \omega|^{1-\vartheta}} \|x\|^{1-\vartheta} \\ &\le \frac{C}{|\lambda - \omega|^{1-\vartheta}} \|x\|, \end{aligned} \tag{3.48}$$

where C is independent of λ. From (3.48), we get for $t \geq 1$

$$\begin{aligned}\left\|(-A)^{\vartheta}\mathcal{R}(t)\right\| &\leq \left\|\frac{1}{2\pi i}\int_{\omega+\Gamma_{r,\theta}} e^{\lambda t}(-A)^{\vartheta}G(\lambda)d\lambda\right\| \\ &\leq \frac{C}{\pi}\int_{r}^{\infty} e^{t(\omega+s\cos\theta)}\frac{ds}{s^{1-\vartheta}} + \frac{C}{2\pi}\int_{-\theta}^{\theta} e^{t(\omega+r\cos\xi)}\frac{rd\xi}{r^{1-\vartheta}} \\ &\leq \left(\frac{C}{\pi r^{1-\vartheta}|\cos\theta|} + \frac{C\theta r^{\vartheta}}{\pi}e^{rt}\right)e^{\omega t} \leq Ce^{(r+\omega)t}.\end{aligned} \tag{3.49}$$

On the other hand, using that $G(\cdot)$ is analytic on $\Lambda_{r,\omega,\theta}$, for $t \in (0,1)$, we get

$$\begin{aligned}\left\|(-A)^{\vartheta}\mathcal{R}(t)\right\| &= \left\|\frac{1}{2\pi i}\int_{\omega+\Gamma_{r/t,\theta}} e^{\lambda t}(-A)^{\vartheta}G(\lambda)d\lambda\right\| \\ &\leq \frac{C}{\pi}\int_{r/t}^{\infty} e^{t(\omega+s\cos\theta)}\frac{ds}{s^{1-\vartheta}} + \frac{C}{2\pi}\int_{-\theta}^{\theta} e^{t\omega+r\cos\xi}\frac{rt^{-1}d\xi}{r^{1-\vartheta}t^{\vartheta-1}} \\ &\leq \left(\frac{C}{\pi}\int_{r}^{\infty} e^{u\cos\theta}\frac{t^{-1}du}{u^{1-\vartheta}t^{\vartheta-1}} + \frac{C}{2\pi r^{-\vartheta}}\int_{-\theta}^{\theta} e^{r\cos\xi}\frac{rt^{-1}d\xi}{r^{1-\vartheta}t^{\vartheta-1}}\right)e^{\omega t} \\ &\leq \left(\frac{C}{\pi r^{1-\vartheta}|\cos\theta|} + \frac{C\theta r^{\vartheta}}{\pi}e^{r}\right)\frac{e^{\omega t}}{t^{\vartheta}}.\end{aligned} \tag{3.50}$$

From the previous facts, we conclude that

$$\left\|(-A)^{\vartheta}\mathcal{R}(t)\right\| \leq Ce^{(r+\omega)t}t^{-\vartheta}, \quad t \in (0,1), \tag{3.51}$$

which ends the proof. □

Corollary 3.13. *If $\omega + r < 0$ and $\vartheta \in (0,1)$, then there exists $\phi \in L^1([0,\infty))$ such that*

$$\left\|(-A)^{\vartheta}\mathcal{R}(t)\right\| \leq \phi(t). \tag{3.52}$$

In the remainder of this section, we discuss the existence and regularity of solutions of

$$\frac{dx(t)}{dt} = Ax(t) + \int_0^t B(t-s)x(s)ds + f(t), \quad t \in [0,a], \tag{3.53}$$

$$x(0) = z \in X, \tag{3.54}$$

where $f \in L^1([0,a],X)$. In the sequel, $\mathcal{R}(\cdot)$ is the operator function defined by (3.17). We begin by introducing the following concept of classical solution.

Definition 3.14. A function $x : [0,b] \to X$, $0 < b \leq a$, is called a classical solution of (3.53)-(3.54) on $[0,b]$ if $x \in C([0,b],[D(A)]) \cap C^1((0,b],X)$, the condition (3.54) holds and (3.53) is verified on $[0,a]$.

The next result has been established in [30].

Theorem 3.15 ([30, Theorem 2]). *Let $z \in X$. Assume that $f \in C([0,a],X)$ and $x(\cdot)$ is a classical solution of* (3.53)-(3.54) *on* $[0,a]$. *Then*

$$x(t) = \mathcal{R}(t)z + \int_0^t \mathcal{R}(t-s)f(s)ds, \quad t \in [0,a]. \tag{3.55}$$

An immediate consequence of the above theorem is the uniqueness of classical solutions.

Corollary 3.16. *If u, v are classical solutions of* (3.53)-(3.54) *on* $[0,b]$, *then* $u = v$ *on* $[0,b]$.

Motivated by (3.55), we introduce the following concept.

Definition 3.17. A function $u \in C([0,a],X)$ is called a mild solution of (3.53)-(3.54) if

$$u(t) = \mathcal{R}(t)z + \int_0^t \mathcal{R}(t-s)f(s)ds, \quad t \in [0,a]. \tag{3.56}$$

4. Existence Result of Asymptotically Almost Periodic Solutions

In this section, we study the existence of asymptotically almost periodic mild solutions for the abstract integro-differential system (1.1). To establish our existence result, motivated by the previous section we introduce the following assumptions.

(P_1) There exists a Banach space $(Y, \|\cdot\|_Y)$ continuously included in X such that the following conditions are verified.

(a) For every $t \in (0,\infty)$, $\mathcal{R}(t) \in \mathcal{L}(X) \cap \mathcal{L}(Y,[D(A)])$ and $B(t) \in \mathcal{L}(Y,X)$. In addition, $A\mathcal{R}(\cdot)x, B(\cdot)x \in C((0,\infty),X)$ for every $x \in Y$.

(b) There are positive constants M, β such that

$$\max\{\|\mathcal{R}(s)\|, \|B(s)\|_{\mathcal{L}(Y,X)}\} \leq Me^{-\beta t}, \quad s \geq 0. \tag{4.1}$$

(c) There exists $\phi \in L^1([0,\infty))$ such that $\|A\mathcal{R}(t)\|_{\mathcal{L}(Y,X)} \leq \phi(t), t \geq 0$.

(P_2) The continuous function $f : \mathbb{R} \times \mathcal{B} \to Y$ is p.a.a.p, and there exists a continuous function $L_f : [0,\infty) \to [0,\infty)$, such that

$$\|f(t,\psi_1) - f(t,\psi_2)\|_Y \leq L_f(r)\|\psi_1 - \psi_2\|_{\mathcal{B}}, \quad (t,\psi_j) \in \mathbb{R} \times B_r(0,\mathcal{B}). \tag{4.2}$$

(P_3) The continuous function $g : \mathbb{R} \times \mathcal{B} \to X$ is p.a.a.p, and there exists a continuous function $L_g : [0,\infty) \to [0,\infty)$ such that

$$\|g(t,\psi_1) - g(t,\psi_2)\| \leq L_g(r)\|\psi_1 - \psi_2\|_{\mathcal{B}}, \quad (t,\psi_j) \in \mathbb{R} \times B_r(0,\mathcal{B}). \tag{4.3}$$

Motivated by the theory of resolvent operator, we introduce the following concept of mild solution for (1.1).

Definition 4.1. A function $u : (-\infty, b] \to X$, $0 < b \leq a$, is called a mild solution of (1.1) on $[0,b]$, if $u_0 = \varphi \in \mathcal{B}; u|_{[0,b]} \in C([0,b] : X)$; the functions $\tau \mapsto A\mathcal{R}(t-\tau)f(\tau,u_\tau)$ and $\tau \mapsto \int_0^\tau B(\tau-\xi)f(\xi,u_\xi)d\xi$ are integrable on $[0,t)$ for every $t \in (0,b]$ and

$$\begin{aligned} u(t) = {} & \mathcal{R}(t)(\varphi(0) + f(0,\varphi)) - f(t,u_t) - \int_0^t A\mathcal{R}(t-s)f(s,u_s)ds \\ & - \int_0^t \mathcal{R}(t-s)\int_0^s B(s-\xi)f(\xi,u_\xi)d\xi ds + \int_0^t \mathcal{R}(t-s)g(s,u_s)ds, \quad t \in [0,b]. \end{aligned} \tag{4.4}$$

Lemma 4.2. *Let condition* (P_1)*—*(c) *hold and let v be a function in $AAP(Y)$. If $u : [0,\infty) \to X$ is the function defined by $u(t) = \int_0^t A\mathcal{R}(t-s)v(s)ds$, then $u(\cdot) \in AAP(X)$.*

Proof. Let $\eta = \max\{2\|v\|_Y, \int_0^\infty \phi(s)ds\}$. Let $\mathcal{T}((\varepsilon/3)\eta^{-1}, v, Y)$, $T = T((\varepsilon/3)\eta^{-1}, v, Y)$ be as in Lemma 2.7 and $T_1 > 1$ such that $\int_{T_1}^\infty \phi(s)ds \leq (\varepsilon/3)\eta^{-1}$. For $t \geq T + T_1$ and $\xi \in \mathcal{T}((\varepsilon/3)\eta^{-1}, v, Y)$, we get

$$\begin{aligned} \|u(t+\xi) - u(t)\| &\leq \int_{-\xi}^0 \|A\mathcal{R}(t-s)\|_{\mathcal{L}(Y,X)}\|v(s+\xi)\|_Y ds \\ &\quad + \int_0^t \|A\mathcal{R}(t-s)\|_{\mathcal{L}(Y,X)}\|v(s+\xi) - v(s)\|_Y ds \\ &\leq \int_{-\xi}^0 \phi(t-s)\|v(s+\xi)\|_Y ds \\ &\quad + \int_0^T \phi(t-s)\|v(s+\xi) - v(s)\|_Y ds \\ &\quad + \int_T^t \phi(t-s)\|v(s+\xi) - v(s)\|_Y ds \\ &\leq \|v\|_Y \int_t^{t+\xi} \phi(s)ds + 2\|v\|_Y \int_{t-T}^t \phi(s)ds + \varepsilon \int_0^{t-T} \phi(s)ds \\ &\leq \|v\|_Y \int_{T_1}^\infty \phi(s)ds + 2\|v\|_Y \int_{T_1}^\infty \phi(s)ds + \frac{\varepsilon}{3}\eta^{-1} \int_0^\infty \phi(s)ds \end{aligned} \tag{4.5}$$

which implies that

$$\|u(t+\xi)-u(t)\| \le \varepsilon, \quad t \ge T\left(\frac{\varepsilon}{3}\eta^{-1}, v, Y\right) + T_1,\ \xi \in \mathcal{T}\left(\frac{\varepsilon}{3}\eta^{-1}, v, Y\right). \tag{4.6}$$

Now, from inequality (4.6) and Lemma 2.7, we conclude that $u(\cdot)$ is a.a.p. The proof is complete. □

Lemma 4.3. *Assume that the condition* (P_1) *is fulfilled. Let* $v \in AAP(Y)$ *and let* $w(\cdot) : [0,\infty) \to X$ *be the function defined by*

$$w(t) = \int_0^t \mathcal{R}(t-s)\int_0^s B(s-\xi)v(\xi)d\xi ds, \quad t \ge 0. \tag{4.7}$$

Then $w(\cdot) \in AAP(X)$.

Proof. Let $\mathcal{T}((\varepsilon/3)\eta^{-1}, v, Y)$, $T = T((\varepsilon/3)\eta^{-1}, v, Y)$ be as in Lemma 2.7 and $T_1 > 1$ such that

$$\int_{T_1}^{\infty} e^{-\beta s} ds \le \frac{\varepsilon}{3}\eta^{-1}, \qquad te^{-\beta(t-T)} \le \frac{\varepsilon}{3}\eta^{-1} \tag{4.8}$$

for $t \ge T_1$, where $\eta = \max\{3\|v\|_Y(M^2/\beta), \sup_{t\ge T_1}(M^2/\beta)te^{-\beta(t-T)}\}$. For $t \ge T + T_1$ and $\xi \in \mathcal{T}((\varepsilon/3)\eta^{-1}, v, Y)$, we get

$$\begin{aligned} w(t+\xi) - w(t) &= \int_0^{t+\xi} \mathcal{R}(t+\xi-s)\int_0^s B(s-u)v(u)du\, ds - \int_0^t \mathcal{R}(t-s)\int_0^s B(s-u)v(u)du\, ds \\ &= \int_0^t \mathcal{R}(t-s)\int_0^s B(s-u)(v(u+\xi)-v(u))du\, ds \\ &\quad + \int_0^{\xi} \mathcal{R}(t+\xi-s)\int_0^s B(s-u)v(u)du\, ds \\ &\quad + \int_0^t \mathcal{R}(t-s)\int_0^{\xi} B(s+\xi-u)v(u)du\, ds. \end{aligned} \tag{4.9}$$

We obtain

$$
\begin{aligned}
&\|w(t+\xi)-w(t)\| \\
&\quad\leq \int_0^t \|\mathcal{R}(t-s)\| \int_0^T \|B(s-u)\|_{\mathcal{L}(Y,X)} \|v(u+\xi)-v(u)\|_Y du\, ds \\
&\qquad + \int_0^t \|\mathcal{R}(t-s)\| \int_T^s \|B(s-u)\|_{\mathcal{L}(Y,X)} \|v(u+\xi)-v(u)\|_Y du\, ds \\
&\qquad + \int_0^\xi \|\mathcal{R}(t+\xi-s)\| \int_0^s \|B(s-u)\|_{\mathcal{L}(Y,X)} \|v(u)\|_Y du\, ds \\
&\qquad + \int_0^t \|\mathcal{R}(t-s)\| \int_0^\xi \|B(s+\xi-u)\|_{\mathcal{L}(Y,X)} \|v(u)\|_Y du\, ds. \\
&\quad\leq \int_0^t Me^{-\beta(t-s)} \int_0^T Me^{-\beta(s-u)} \|v(u+\xi)-v(u)\|_Y du\, ds \\
&\qquad + \int_0^t Me^{-\beta(t-s)} \int_T^t Me^{-\beta(s-u)} \|v(u+\xi)-v(u)\|_Y du\, ds \\
&\qquad + \int_0^\xi Me^{-\beta(t+\xi-s)} \int_0^s Me^{-\beta(s-u)} \|v(u)\|_Y du\, ds \\
&\qquad + \int_0^t Me^{-\beta(t-s)} \int_0^\xi Me^{-\beta(s+\xi-u)} \|v(u)\|_Y du\, ds. \\
&\quad\leq 2\|v\|_Y \int_0^t Me^{-\beta t} \int_0^T Me^{\beta u} du\, ds + \varepsilon \int_0^t Me^{-\beta t} \int_T^t Me^{\beta u} du\, ds \\
&\qquad + \|v\|_Y \int_0^\xi Me^{-\beta(t+\xi-s)} \int_0^s Me^{-\beta(s-u)} du\, ds \\
&\qquad + \|v\|_Y \int_0^t Me^{-\beta(\mathrm{t}-s)} \int_0^\xi Me^{\beta(s+\xi-u)} du\, ds \\
&\quad\leq 2\|v\|_Y \frac{M^2}{\beta} te^{-\beta(t-T)} + \varepsilon \frac{M^2}{\beta} te^{-\beta(t-T)} \\
&\qquad + \|v\|_Y \frac{M^2}{\beta} \int_t^{t+\xi} e^{-\beta s} ds + \|v\|_Y \frac{M^2}{\beta} te^{-\beta(t-T)} \\
&\quad\leq 3\|v\|_Y \frac{M^2}{\beta} te^{-\beta(t-T)} + \varepsilon \frac{M^2}{\beta} te^{-\beta(t-T)} + \|v\|_Y \frac{M^2}{\beta} \int_{T_1}^\infty e^{-\beta s} ds,
\end{aligned}
\tag{4.10}
$$

which implies that

$$\|w(t+\xi)-w(t)\| \le \varepsilon, \quad t \ge T\left(\frac{\varepsilon}{3}\eta^{-1}, v, Y\right) + T_1,\ \xi \in \mathcal{T}\left(\frac{\varepsilon}{3}\eta^{-1}, v, Y\right). \tag{4.11}$$

From inequality (4.11) and Lemma 2.7, we conclude that $w(\cdot)$ is a.a.p., which ends the proof. □

Now, we can establish our existence result.

Theorem 4.4. *Assume that $\mathcal{B}$ is a fading memory space and (P_1), (P_2), and (P_3) are held. If $L_f(0) = L_g(0) = 0$ and $f(t,0) = g(t,0) = 0$ for every $t \in \mathbb{R}$, then there exists $\varepsilon > 0$ such that for each $\varphi \in B_\varepsilon(0,\mathcal{B})$, there exists a mild solution, $u(\cdot,\varphi)$, of (1.1) on $[0,\infty)$ such that $u(\cdot,\varphi) \in AAP(X)$ and $u_0(\cdot,\varphi) = \varphi$.*

Proof. Let $r > 0$ and $0 < \lambda < 1$ be such that

$$\begin{aligned}\Theta = {} & MH\lambda + ML_f(\lambda r)\lambda + L_f((\lambda+1)\mathfrak{K}r)(\lambda+1)\mathfrak{K}\left(\|i_c\|_{\mathcal{L}(Y,X)} + \|\phi\|_{L^1} + \frac{M^2}{\beta^2}\right) \\ & + L_g((\lambda+1)\mathfrak{K}r)(\lambda+1)\mathfrak{K}\frac{M}{\beta} < 1,\end{aligned} \tag{4.12}$$

where $\mathfrak{K}$ is the constant introduced in Remark 2.3. We affirm that the assertion holds for $\varepsilon = \lambda r$. Let $\varphi \in B_\varepsilon(0,\mathcal{B})$. On the space

$$\mathfrak{D} = \left\{x \in AAP(X) : x(0) = \varphi(0),\ \|x(t)\| \le r, t \ge 0\right\} \tag{4.13}$$

endowed with the metric $d(u,v) = \|u - v\|$, we define the operator $\Gamma : \mathfrak{D} \to C([0,\infty);X)$ by

$$\begin{aligned}\Gamma u(t) = {} & \mathcal{R}(t)(\varphi(0) + f(0,\varphi)) - f(t,\tilde{u}_t) - \int_0^t A\mathcal{R}(t-s)f(s,\tilde{u}_s)ds \\ & - \int_0^t \mathcal{R}(t-s)\int_0^s B(s-\xi)f(\xi,\tilde{u}_\xi)d\xi ds + \int_0^t \mathcal{R}(t-s)g(s,\tilde{u}_s)ds, \quad t \ge 0,\end{aligned} \tag{4.14}$$

where $\tilde{u} : \mathbb{R} \to X$ is the function defined by the relation $\tilde{u}_0 = \varphi$ and $\tilde{u} = u$ on $[0,\infty)$. From the hypothesis (P_1), (P_2), and (P_3), we obtain that Γu is well defined and that $\Gamma u \in C([0,\infty);X)$. Moreover, from Lemmas 4.2 and 4.3 it follows that $\Gamma u \in AAP(X)$.

Next, we prove that $\Gamma(\cdot)$ is a contraction from $\mathfrak{D}$ into $\mathfrak{D}$. If $u \in \mathfrak{D}$ and $t \geq 0$, we get

$$\begin{aligned}
\|\Gamma u(t)\| \leq{}& MH\lambda r + ML_f(\lambda r)\lambda r + \|i_c\|_{\mathcal{L}(Y,X)} L_f((\lambda+1)\mathfrak{K}r)(\lambda+1)\mathfrak{K}r \\
&+ \int_0^t \phi(t-s) L_f((\lambda+1)\mathfrak{K}r)(\lambda+1)\mathfrak{K}r\, ds \\
&+ \int_0^t Me^{-\beta(t-s)} \int_0^s Me^{-\beta(s-\xi)} L_f((\lambda+1)\mathfrak{K}r)(\lambda+1)\mathfrak{K}r\, d\xi ds \\
&+ \int_0^t Me^{-\beta(t-s)} L_g((\lambda+1)\mathfrak{K}r)(\lambda+1)\mathfrak{K}r\, ds \\
\leq{}& MH\lambda r + ML_f(\lambda r)\lambda r + L_f((\lambda+1)\mathfrak{K}r)(\lambda+1)\mathfrak{K}r \\
&+ \left(\int_0^\infty \phi(s)ds\right) L_f((\lambda+1)\mathfrak{K}r)(\lambda+1)\mathfrak{K}r \\
&+ \left(\int_0^\infty Me^{-\beta s}ds\right)\left(\int_0^\infty Me^{-\beta\xi}d\xi\right) L_f((\lambda+1)\mathfrak{K}r)(\lambda+1)\mathfrak{K}r \\
&+ \left(\int_0^\infty e^{-\beta s}ds\right) L_g((\lambda+1)\mathfrak{K})(\lambda+1)\mathfrak{K}r \\
\leq{}& \Theta r,
\end{aligned} \tag{4.15}$$

where the inequality $\|\tilde{u}_t\| \leq (\lambda+1)\mathfrak{K}r$ has been used and $i_c : Y \to X$ represent the continuous inclusion of Y on X. Thus, $\Gamma(\mathfrak{D}) \subset \mathfrak{D}$. On the other hand, for $u, v \in \mathfrak{D}$ we see that

$$\begin{aligned}
&\|\Gamma u(t) - \Gamma v(t)\| \\
&\quad\leq \|i_c\|_{\mathcal{L}(Y,X)} \left\| f(t,\tilde{u}_t) - f(t,\tilde{v}_t)\right\| + \int_0^t \|A\mathcal{R}(t-s)\|_{\mathcal{L}(Y,X)} \left\| f(s,\tilde{u}_s) - f(s,\tilde{v}_s)\right\|_Y ds \\
&\qquad+ \int_0^t \|\mathcal{R}(t-s)\| \int_0^s \|B(s-\xi)\|_{\mathcal{L}(Y,X)} \left\| f(\xi,\tilde{u}_\xi) - f(\xi,\tilde{v}_\xi)\right\|_Y d\xi ds \\
&\qquad+ \int_0^t \|\mathcal{R}(t-s)\| \left\| g(s,\tilde{u}_s) - g(s,\tilde{v}_s)\right\| ds \\
&\quad\leq L_f((\lambda+1)\mathfrak{K}r)\mathfrak{K}\left(\|i_c\|_{\mathcal{L}(Y,X)} + \|\phi\|_{L^1} + \frac{M^2}{\beta^2}\right)\|u-v\| \\
&\qquad+ L_g((\lambda+1)\mathfrak{K})\mathfrak{K}\frac{M}{\beta}\|u-v\| \leq \Theta\|u-v\|,
\end{aligned} \tag{4.16}$$

which shows that $\Gamma(\cdot)$ is a contraction from $\mathfrak{D}$ into $\mathfrak{D}$. The assertion is now a consequence of the contraction mapping principle. The proof is complete. □

5. Applications

In this section, we study the existence of asymptotically almost periodic solutions of the partial neutral integro-differential system

$$\begin{aligned} &\frac{\partial}{\partial t}\left[u(t,\xi)+\int_{-\infty}^{t}\int_{0}^{\pi}b(s-t,\eta,\xi)u(s,\eta)d\eta ds\right] \\ &\quad=\left(\frac{\partial^2}{\partial\xi^2}+\mu\right)\left[u(t,\xi)+\int_0^t e^{-\gamma(t-s)}u(s,\xi)ds\right]+\int_{-\infty}^{t}a_0(s-t)u(s,\xi)ds, \\ &u(t,0)=u(t,\pi)=0, \\ &u(\theta,\xi)=\varphi(\theta,\xi), \end{aligned} \tag{5.1}$$

for $(t,\xi)\in[0,a]\times[0,\pi]$, $\theta\leq 0,\mu<0$, and $\gamma>0$. Moreover, we have identified $\varphi(\theta)(\xi)=\varphi(\theta,\xi)$.

To represent this system in the abstract form (1.1), we choose the spaces $X=L^2([0,\pi])$ and $\mathcal{B}=C_0\times L^2(\rho,X)$; see Example 2.4 for details. We also consider the operators $A,B(t):D(A)\subseteq X\to X$, $t\geq 0$, given by $Ax=x''+\mu x$, $B(t)x=e^{-\gamma t}Ax$ for $x\in D(A)=\{x\in X: x''\in X, x(0)=x(\pi)=0\}$. Moreover, A has discrete spectrum, the eigenvalues are $-n^2+\mu$, $n\in\mathbb{N}$, with corresponding eigenvectors $z_n(\xi)=(2/\pi)^{1/2}\sin(n\xi)$, and the set of functions $\{z_n:n\in\mathbb{N}\}$ is an orthonormal basis of X and $T(t)x=\sum_{n=1}^{\infty}e^{-(n^2-\mu)t}<x,z_n>z_n$ for $x\in X$. For $\alpha\in(0,1)$, from [32] we can define the fractional power $(-A)^{\alpha}:D((-A)^{\alpha})\subset X\to X$ of A is given by $(-A)^{\alpha}x=\sum_{n=1}^{\infty}(n^2-\mu)^{\alpha}\langle x,z_n\rangle z_n$, where $D((-A)^{\alpha})=\{x\in X:(-A)^{\alpha}x\in X\}$. In the next theorem, we consider $Y=D((-A)^{1/2})$. We observe that $\rho(A)\supset\{\lambda\in\mathbb{C}:\mathrm{Re}(\lambda)\geq\mu\}$ and $\|\lambda R(\lambda,A)\|\leq M_1$ for $\mathrm{Re}(\lambda)\geq\mu$; from [33, Proposition 2.2.11], we obtain that A is a sectorial operator satisfying $\|R(\lambda,A)\|\leq M/|\lambda-\mu|, M>0$. Moreover, it is easy to see that conditions (H2)-(H3) in Section 3 are satisfied with $b(t)=e^{-\gamma t}$, and $D=C_0^{\infty}([0,\pi])$ is the space of infinitely differentiable functions that vanish at $\xi=0$ and $\xi=\pi$. Under the above conditions, we can represent the system

$$\begin{gathered} \frac{\partial u(t,\xi)}{\partial t}=\left(\frac{\partial^2}{\partial\xi^2}+\mu\right)\left[u(t,\xi)+\int_0^t e^{-\gamma(t-s)}u(s,\xi)ds\right], \\ u(t,\pi)=u(t,0)=0, \end{gathered} \tag{5.2}$$

in the abstract form

$$\begin{gathered} \frac{dx(t)}{dt}=Ax(t)+\int_0^t B(t-s)x(s)ds, \\ x(0)=z\in X. \end{gathered} \tag{5.3}$$

We define the functions $f, g : \mathcal{B} \to X$ by

$$f(\psi)(\xi) = \int_{-\infty}^{0} \int_{0}^{\pi} b(s,\eta,\xi)\psi(s,\eta)\,d\eta\,ds,$$
$$g(\psi)(\xi) = \int_{-\infty}^{0} a_0(s)\psi(s,\xi)\,ds, \tag{5.4}$$

where

(i) the functions $a_0 : \mathbb{R} \to \mathbb{R}$ are continuous and $L_g := (\int_{-\infty}^{0}((a_0(s))^2/\rho(s))ds)^{1/2} < \infty$.;

(ii) the functions $b(\cdot)$, $\partial b(s,\eta,\xi)/\partial\xi$ are measurable, $b(s,\eta,\pi) = b(s,\eta,0) = 0$ for all (s,η) and

$$L_f := \max\left\{ \left(\int_0^{\pi} \int_{-\infty}^{0} \int_0^{\pi} \rho^{-1}(\theta) \left(\frac{\partial^i}{\partial\xi^i} b(\theta,\eta,\xi) \right)^2 d\eta\, d\theta\, d\xi \right)^{1/2} : \ i = 0,1 \right\} < \infty. \tag{5.5}$$

Moreover, f, g are bounded linear operators, $\|f\|_{\mathcal{L}(\mathcal{B},X)} \le L_f$, $\|g\|_{\mathcal{L}(\mathcal{B},X)} \le L_g$, and a straightforward estimation using (ii) shows that $f(I \times \mathcal{B}) \subset D((-A)^{1/2})$ and

$$\left\| (-A)^{1/2} f(t,\cdot) \right\|_{\mathcal{L}(\mathcal{B},X)} \le L_f \tag{5.6}$$

for all $t \in I$. This allows us to rewrite the system (5.1) in the abstract form (1.1) with $u_0 = \varphi \in \mathcal{B}$.

Theorem 5.1. *Assume that the previous conditions are verified. Let* $2 < K < \gamma$ *and* $\mu < 0$ *such that* $|\mu| > \max\{M(K+1+\gamma), \gamma\}$, *then there exists a mild solution* $u(\cdot) \in AAP(X)$ *of* (5.1) *with* $u_0 = \varphi$.

Proof. For $\lambda = \mu + se^{i\theta}$, from $|\lambda + \gamma| \ge s - |\gamma + \mu|$, we obtain

$$\begin{aligned}
\left\| \widehat{B}(\lambda) R(\lambda, A) \right\| &\le \frac{1}{|\lambda+\gamma|} \left(1 + \frac{M}{|\lambda-\mu|} + \frac{M|\lambda|}{|\lambda-\mu|} \right) \\
&\le \frac{1}{|\lambda+\gamma|} + \left(\frac{1}{|\lambda+\gamma|} + \frac{|\lambda|}{|\lambda+\gamma|} \right) \frac{M}{|\lambda-\mu|} \\
&\le \frac{1}{|\lambda+\gamma|} + \left(\frac{1}{|\lambda+\gamma|} + 1 + \frac{|\gamma|}{|\lambda+\gamma|} \right) \frac{M}{|\lambda-\mu|}
\end{aligned}$$

$$\begin{aligned}&\leq \frac{1}{s-|\gamma+\mu|}+\left(\frac{1}{s-|\gamma+\mu|}+1+\frac{|\gamma|}{s-|\gamma+\mu|}\right)\frac{M}{|\lambda-\mu|}\\&\leq \frac{1}{K}+\left(\frac{1+K+\gamma}{K}\right)\frac{M}{|\lambda-\mu|}\\&\leq \frac{1}{K}+\frac{1}{K},\end{aligned} \tag{5.7}$$

since $s \geq r = \max\{M(K+1+\gamma), K+|\gamma+\mu|\}$. By using a similar procedure as in the proofs of Lemma 3.3 and Theorem 3.10, we obtain the existence of resolvent operator for (5.2). From the hypothesis, we obtain $\mu + r < 0$; by the Lemma 3.3, Corollaries 3.11 and 3.13, the assumption (P_1) is satisfied. From Theorem 4.4, the proof is complete. □

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