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Research Article

Complete Asymptotic and Bifurcation Analysis for a Difference Equation with Piecewise Constant Control

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We consider a difference equation involving three parameters and a piecewise constant control function with an additional positive threshold λ. Treating the threshold as a bifurcation parameter that varies between 0 and ∞, we work out a complete asymptotic and bifurcation analysis. Among other things, we show that all solutions either tend to a limit 1-cycle or to a limit 2-cycle and, we find the exact regions of attraction for these cycles depending on the size of the threshold. In particular, we show that when the threshold is either small or large, there is only one corresponding limit 1-cycle which is globally attractive. It is hoped that the results obtained here will be useful in understanding interacting network models involving piecewise constant control functions.

1. Introduction

Let $N = \{0, 1, 2, \ldots\}$. In [1], Ge et al. obtained a complete asymptotic and bifurcation analysis of the following difference equation:

$$x_n = ax_{n-2} + bf_\lambda(x_{n-1}), \quad n \in N, \tag{1.1}$$

where $a \in (0,1)$, $b \in (0,\infty)$, and $f_\lambda : R \to R$ is a nonlinear signal filtering control function of the form

$$f_\lambda(x) = \begin{cases} 1, & x \in (0,\lambda], \\ 0, & x \in (-\infty,0] \cup (\lambda,\infty), \end{cases} \tag{1.2}$$

in which the *positive* number λ can be regarded as a threshold *bifurcation* parameter.

By adding a positive constant c to the right hand side of (1.1), we obtain the following equation:

$$x_n = ax_{n-2} + bf_\lambda(x_{n-1}) + c, \quad n \in N. \tag{1.3}$$

Since c can be an arbitrary small positive number, (1.1) may be regarded as a limiting case of (1.3). Therefore, it would appear that the qualitative behavior of (1.3) will "degenerate into" that of (1.1) when c tends to 0. However, it is our intention to derive a complete asymptotic and bifurcation analysis for our new equation and show that, among other things, our expectation is not quite true and perhaps such discrepancy is due to the nonlinear nature of our model at hand.

Indeed, we are dealing with a dynamical system with piecewise constant nonlinearities (see e.g., [2–6]), and the usual linear and continuity arguments cannot be applied to our (1.3). Fortunately, we are able to achieve our goal by means of completely elementary considerations.

To this end, we first recall a few concepts. Note that given $x_{-2}, x_{-1} \in R$, we may compute from (1.3) the numbers $x_0, x_1, x_2, \ldots$ in a unique manner. The corresponding sequence $\{x_n\}_{n=-2}^{\infty}$ is called the solution of (1.1) determined by or originated from the initial vector (x_{-2}, x_{-1}).

Recall also that a positive integer η is a period of the sequence $\{w_n\}_{n=\alpha}^{\infty}$ if $w_{\eta+n} = w_n$ for all $n \geq \alpha$ and that τ is the least or prime period of $\{w_n\}_{n=\alpha}^{\infty}$ if τ is the least among all periods of $\{w_n\}_{n=\alpha}^{\infty}$. The sequence $\{w_n\}_{n=\alpha}^{\infty}$ is said to be τ-periodic if τ is its least period. The sequence $w = \{w_n\}_{n=\alpha}^{\infty}$ is said to be asymptotically periodic if there exist real numbers $w^{(0)}, w^{(1)}, \ldots, w^{(\omega-1)}$, where ω is a positive integer, such that

$$\lim_{n\to\infty} w_{\omega n+i} = w^{(i)}, \quad i = 0, 1, \ldots, \omega - 1. \tag{1.4}$$

In case $\{w^{(0)}, w^{(1)}, \ldots, w^{(\omega-1)}, w^{(0)}, w^{(1)}, \ldots, w^{(\omega-1)}, \ldots\}$ is an ω-periodic sequence, we say that w is an asymptotically ω-periodic sequence tending to the limit ω-cycle (This term is introduced since the underlying concept is similar to that of the limit cycle in the theory of ordinary differential equations.) $\langle w^{(0)}, w^{(1)}, \ldots, w^{(\omega-1)} \rangle$. In particular, an asymptotically 1-periodic sequence is a convergent sequence and conversely.

Suppose that S is the set of all solutions of (1.1) that tend to the limit cycle Q. Then, the set

$$\left\{(x_{-2}, x_{-1}) \in R^2 \mid \{x_n\}_{n=-2}^{\infty} \in S\right\} \tag{1.5}$$

is called the the region of attraction of the limit cycle Q. In other words, Q attracts all solutions originated from its region of attraction.

Equation (1.3) is related to several linear recurrence and functional inequality relations of the form

$$x_{2k} = ax_{2k-2} + d, \quad k \in N, \tag{1.6}$$

$$x_{2k+1} = ax_{2k-1} + d, \quad k \in N, \tag{1.7}$$

$$x_{2k} \geq ax_{2k-2} + d, \quad k \in N, \tag{1.8}$$

$$x_{2k+1} \geq ax_{2k-1} + d, \quad k \in N, \tag{1.9}$$

where $a \in (0,1)$ and $d > 0$. Therefore, the following facts will be needed, which can easily be established by induction.

(i) If $\{x_n\}_{n=-2}^{\infty}$ is a sequence which satisfies (1.6), then

$$x_{2k} = a^{k+1}x_{-2} + \frac{d(1-a^{k+1})}{1-a}, \quad k \in N. \tag{1.10}$$

(ii) If $\{x_n\}_{n=-2}^{\infty}$ is a sequence which satisfies (1.7), then

$$x_{2k+1} = a^{k+1}x_{-1} + \frac{d(1-a^{k+1})}{1-a}, \quad k \in N. \tag{1.11}$$

(iii) If $\{x_n\}_{n=-2}^{\infty}$ is a sequence which satisfies (1.8), then

$$x_{2k} \geq a^{k+1}x_{-2} + \frac{d(1-a^{k+1})}{1-a}, \quad k \in N. \tag{1.12}$$

(iv) If $\{x_n\}_{n=-2}^{\infty}$ is a sequence which satisfies (1.9), then

$$x_{2k+1} \geq a^{k+1}x_{-1} + \frac{d(1-a^{k+1})}{1-a}, \quad k \in N. \tag{1.13}$$

We will discuss solutions $\{x_n\}_{n=-2}^{\infty}$ of (1.3) originated from different x_{-2} and x_{-1} in R. For this reason, we let $B_0 = 0$ and

$$aB_{j+1} + b + c = B_j, \quad j \in N. \tag{1.14}$$

Then, for $j \in N$,

$$B_{j+1} - B_j = -\frac{b+c}{a^{j+1}} < 0. \tag{1.15}$$

Since

$$B_j = -\frac{b+c}{a^j(1-a)} + \frac{b+c}{1-a}, \tag{1.16}$$

we see that $\lim_{j\to\infty} B_j = -\infty$ and

$$(-\infty, 0] = \bigcup_{k=0}^{\infty} (B_{k+1}, B_k]. \tag{1.17}$$

Similarly, let $C_0 = 0$ and

$$aC_{j+1} + c = C_j, \quad j \in N. \tag{1.18}$$

Then,

$$\begin{aligned} C_{j+1} - C_j &= -\frac{c}{a^{j+1}} < 0, \quad j \in N, \\ C_j &= -\frac{c}{a^j(1-a)} + \frac{c}{1-a}, \quad j \in N, \end{aligned} \tag{1.19}$$

Since $\lim_{j\to\infty} C_j = -\infty$, we see further that

$$(-\infty, 0] = \bigcup_{k=0}^{\infty} (C_{k+1}, C_k]. \tag{1.20}$$

Note that (1.3) is equivalent to the following two dimensional dynamical system

$$(u_n, v_n) = \left(v_{n-1}, au_{n-1} + bf_\lambda(v_{n-1})\right) + (0, c), \quad n \in N, \tag{1.21}$$

by means of the identification $(x_{n-1}, x_n) = (u_n, v_n)$ for $n = -1, 0, 1, \ldots$. Therefore, our subsequent results can be interpreted in terms of the dynamics of plane vector sequences defined by (1.21).

In particular, the following result states that a solution $\{(u_n, v_n)\}_{n=-1}^{\infty}$ of (1.21) with $(u_{-1}, v_{-1}) \in (-\infty, 0]^2$ will have one of its terms in $(-\infty, 0] \times (0, c]$.

Lemma 1.1. *Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of* (1.3). *If $(x_{-2}, x_{-1}) \in (-\infty, 0]^2$, then there is $n_0 \in N$ such that $x_{-2}, x_{-1}, \ldots, x_{n_0-1} \le 0$ and $x_{n_0} \in (0, c]$.*

Proof. Suppose to the contrary that $x_p \le 0$ for all $p \ge -2$. Then, by (1.3),

$$x_n = ax_{n-2} + c, \quad n \in N. \tag{1.22}$$

This, in view of (1.10) and (1.11), leads us to

$$0 \ge \lim_{p\to\infty} x_{2p} = \lim_{p\to\infty} x_{2p+1} = \frac{c}{1-a} > 0, \tag{1.23}$$

which is a contradiction. Thus, there is $n_0 \in N$ such that $x_{-2}, x_{-1}, \dots, x_{n_0-1} \le 0$ and $x_{n_0} > 0$. Furthermore,

$$x_{n_0} = ax_{n_0-2} + bf(x_{n_0-1}) + c = ax_{n_0-2} + c \le c. \tag{1.24}$$

The proof is complete. □

In the following discussions, we will allow the bifurcation parameter λ to vary from 0^+ to $+\infty$. Indeed, we will consider five cases: (i) $0 < \lambda < c/(1-a)$, (ii) $\lambda = c/(1-a)$, (iii) $c/(1-a) < \lambda < (b+c)/(1-a)$, (iv) $\lambda = (b+c)/(1-a)$, and (v) $\lambda > (b+c)/(1-a)$ and show that each solution of (1.1) tend to the limit cycles

$$\left\langle \frac{c}{1-a} \right\rangle, \left\langle \frac{b+c}{1-a} \right\rangle \quad \text{or} \quad \left\langle \frac{c}{1-a}, \frac{b+c}{1-a} \right\rangle. \tag{1.25}$$

Furthermore, in each case, we find the exact regions of attraction of the limit cycles. Then we describe our results in terms of our phase plane model (1.21) and compare them with what we have obtained for the phase plane model of (1.1).

We remark that since we need to find the exact regions of attraction, we need to consider initial vectors (x_{-2}, x_{-1}) belonging to (up to 9) different parts of the plane. Therefore the following derivations will seem to be repetitive. Fortunately, the principles behind our derivations are quite similar, and therefore some of the repetitive arguments can be simplified.

For the sake of convenience, if no confusion is caused, the function f_λ is also denoted by f in the sequel.

2. The Case Where $\lambda > (b+c)/(1-a)$

In this section, we assume that $\lambda > (b+c)/(1-a)$.

Lemma 2.1. *Suppose that $\lambda > (b+c)/(1-a)$. Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of (1.3). Then, there is $m \in \{-2, -1, 0, \dots\}$ such that $0 < x_m, x_{m+1} \le \lambda$.*

Proof. If $x_k \notin (0, \lambda]$ for all $k \ge -2$, then by (1.3), $x_k = ax_{k-2} + c$ for $k \in N$. One sees from (1.10) and (1.11) that $\lim_{k\to\infty} x_k = c/(1-a) \in (0, \lambda)$ which is a contradiction. Hence, there must exist a m_0 such that $x_{m_0} \in (0, \lambda]$. If $x_{m_0+1} \in (0, \lambda]$, we are done. Otherwise, one sees that

$$x_{m_0+2} = ax_{m_0} + bf(x_{m_0+1}) + c = ax_{m_0} + c \in (0, \lambda]. \tag{2.1}$$

Repeating the argument we either find $m > m_0$ such that $x_m, x_{m+1} \in (0, \lambda]$, or one has that the subsequence x_{m_0+2k} lies in $(0, \lambda]$ whereas $x_{m_0+2k+1} \notin (0, \lambda]$. This would mean that the subsequence $\{x_{m_0+2k+1}\}$ satisfies (1.6) or (1.7) for $d = b + c$, and hence $\lim_{k\to\infty} x_{m_0+2k+1} = (b + c)/(1 - a) < \lambda$, a contradiction. The proof is complete. □

Theorem A

Suppose $\lambda > (b+c)/(1-a)$. Then every solution $\{x_n\}_{n=-2}^{\infty}$ of (1.3) converges to $(b+c)/(1-a)$.

Proof. In view of Lemma 2.1, we may suppose without loss of generality that $0 < x_{-2}, x_{-1} \le \lambda$. Since $a\lambda + b + c < \lambda$, we have

$$\begin{aligned} 0 &< x_0 = ax_{-2} + bf(x_{-1}) + c = ax_{-2} + b + c < \lambda, \\ 0 &< x_1 = ax_{-1} + bf(x_0) + c = ax_{-1} + b + c < \lambda, \end{aligned} \tag{2.2}$$

and by induction $0 < x_{2k}, x_{2k+1} < \lambda$ for all $k \in N$. Thus, by (1.3), $x_n = ax_n + b + c$ for $n \in N$. In view of (1.10) and (1.11), $\lim_{k\to\infty} x_{2k} = \lim_{k\to\infty} x_{2k+1} = (b + c)/(1 - a)$. The proof is complete. □

3. The Case Where $\lambda = (b + c)/(1 - a)$

In this section, we suppose that $\lambda = (b + c)/(1 - a)$. Then, $\lambda = a\lambda + b + c$. Let $D_0 = \lambda$ and

$$aD_{j+1} + c = D_j, \quad j \in N. \tag{3.1}$$

Then,

$$\begin{aligned} D_j &= \frac{\lambda(1-a) - c}{a^j(1-a)} + \frac{c}{1-a}, \quad j \in N, \\ D_{j+1} - D_j &= -\frac{1}{a^{j+1}}(c - \lambda + a\lambda) = \frac{b}{a^{j+1}} > 0, \quad j \in N, \\ \lim_{j\to\infty} D_j &= \lim_{j\to\infty} \left\{ \frac{\lambda(1-a) - c}{a^j(1-a)} + \frac{c}{1-a} \right\} = +\infty. \end{aligned} \tag{3.2}$$

For the sake of convenience, let us set

$$\Phi = (-\infty, \lambda]^2 \cup \left(\bigcup_{k=0}^{\infty} \{(-\infty, C_{k+1}] \times (D_k, D_{k+1}]\} \right) \cup \left(\bigcup_{k=0}^{\infty} \{(D_k, D_{k+1}] \times (-\infty, C_{k+1}]\} \right). \tag{3.3}$$

Lemma 3.1. *Suppose that $\lambda = (b+c)/(1-a)$. Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of* (1.3). *If $(x_{-2}, x_{-1}) \in \Phi$, then there is $m \in N$ such that $0 < x_m, x_{m+1} \le \lambda$.*

Proof. We break up $(-\infty, \lambda]^2$ into four different parts $\Omega_1 = (0, \lambda]^2$, $\Omega_2 = (-\infty, 0] \times (0, \lambda]$, $\Omega_3 = (0, \lambda] \times (-\infty, 0]$, and $\Omega_4 = (-\infty, 0]^2$. We also let $\Omega_5 = \bigcup_{k=0}^{\infty}\{(-\infty, C_{k+1}] \times (D_k, D_{k+1}]\}$ and $\Omega_6 = \bigcup_{k=0}^{\infty}\{(D_k, D_{k+1}] \times (-\infty, C_{k+1}]\}$.

Clearly, there is nothing to prove if $(x_{-2}, x_{-1}) \in \Omega_1$.

Next, suppose that $(x_{-2}, x_{-1}) \in \Omega_2$. Then $(x_{-2}, x_{-1}) \in (B_{k+1}, B_k] \times (0, \lambda]$ for some $k \in N$. If $x_{-2} \in (B_1, B_0] = (-(b+c)/a, 0]$, then by (1.3),

$$0 < x_0 = ax_{-2} + bf(x_{-1}) + c = ax_{-2} + b + c \le b + c < \lambda. \tag{3.4}$$

That is, $(x_{-1}, x_0) \in \Omega_1$. If $x_{-2} \in (B_{k+1}, B_k]$ for some $k > 0$, then

$$\begin{gathered} B_k = aB_{k+1} + b + c < x_0 = ax_{-2} + bf(x_{-1}) + c = ax_{-2} + b + c \le aB_k + b + c = B_{k-1}, \\ 0 < x_1 = ax_{-1} + bf(x_0) + c = ax_{-1} + c < \lambda. \end{gathered} \tag{3.5}$$

Hence, $(x_0, x_1) \in (B_k, B_{k-1}] \times (0, \lambda]$. By induction, we see that $(x_{2k-2}, x_{2k-1}) \in (B_1, B_0] \times (0, \lambda]$ and hence, $(x_{2k-1}, x_{2k}) \in \Omega_1$.

Suppose $(x_{-2}, x_{-1}) \in \Omega_3$. Then by (1.3), $0 < x_0 = ax_{-2} + bf(x_{-1}) + c = ax_{-2} + c \le a\lambda + c < \lambda$. Hence, $(x_{-1}, x_0) \in \Omega_2$

Suppose that $(x_{-2}, x_{-1}) \in \Omega_4$. Then by Lemma 1.1, there is $n_0 \in N$ such that $(x_{n_0-1}, x_{n_0}) \in (-\infty, 0] \times (0, c] \subset \Omega_2$.

Suppose that $(x_{-2}, x_{-1}) \in \Omega_5$. Then $(x_{-2}, x_{-1}) \in (-\infty, C_{k+1}] \times (D_k, D_{k+1}]$ for some $k \in N$. If $(x_{-2}, x_{-1}) \in (-\infty, C_1] \times (D_0, D_1] = (-\infty, -c/a] \times (\lambda, (\lambda - c)/a]$, then in view of (1.3),

$$\begin{gathered} x_0 = ax_{-2} + bf(x_{-1}) + c = ax_{-2} + c \le 0, \\ 0 < x_1 = ax_{-1} + bf(x_0) + c = ax_{-1} + c \le \lambda. \end{gathered} \tag{3.6}$$

Hence, $(x_0, x_1) \in \Omega_2$. If $(x_{-2}, x_{-1}) \in (-\infty, C_{k+1}] \times (D_k, D_{k+1}]$ for some $k > 0$, then by (1.3),

$$\begin{gathered} x_0 = ax_{-2} + bf(x_{-1}) + c = ax_{-2} + c \le aC_{k+1} + c = C_k, \\ D_{k-1} = aD_k + c < x_1 = ax_{-1} + bf(x_0) + c = ax_{-1} + c \le aD_{k+1} + c = \mathrm{D}_k. \end{gathered} \tag{3.7}$$

Hence, $(x_0, x_1) \in (-\infty, C_k] \times (D_{k-1}, D_k]$, and by induction, $(x_{2k-2}, x_{2k-1}) \in (-\infty, C_1] \times (D_0, D_1]$. Thus $(x_{2k}, x_{2k+1}) \in (-\infty, 0] \times (0, \lambda] \subset \Omega_2$.

Suppose $(x_{-2}, x_{-1}) \in \Omega_6$. Then $(x_{-2}, x_{-1}) \in (D_k, D_{k+1}] \times (-\infty, C_{k+1}]$ for some $k \in N$. As in the previous case, we may show by similar arguments that $(x_{2k}, x_{2k+1}) \in \Omega_3$.

Therefore, in the last four cases, we may apply the first two cases to conclude our proof. The proof is complete. □

Theorem B

Suppose that $\lambda = (b+c)/(1-a)$. Then, every solution of (1.3) with $(x_{-2}, x_{-1}) \in \Phi$ tends to $(b+c)/(1-a)$.

Proof. Indeed, in view of Lemma 3.1, we may assume without loss of generality that $0 < x_{-2}, x_{-1} \le \lambda$. Then, the same arguments in the proof of Theorem A holds so that $\lim_{n\to\infty} x_n = (b+c)/(1-a)$. □

Lemma 3.2. *Suppose that $\lambda = (b+c)/(1-a)$. Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of* (1.3). *If $(x_{-2}, x_{-1}) \in R^2 \setminus \Phi$, then there is $m \in N$ such that $0 < x_m \leq \lambda$ and $x_{m+1} > \lambda$.*

Proof. We break up $R^2 \setminus \Phi$ into five different parts $\Gamma_1 = (0, \lambda] \times (\lambda, +\infty)$, $\Gamma_2 = (\lambda, +\infty) \times (0, \lambda]$, $\Gamma_3 = (\lambda, +\infty) \times (\lambda, +\infty)$, $\Gamma_4 = \bigcup_{k=0}^{\infty}\{(C_{k+1}, C_k] \times (D_k, +\infty)\}$, and $\Gamma_5 = \bigcup_{k=0}^{\infty}\{(D_k, +\infty) \times (C_{k+1}, C_k]\}$.

Clearly, there is nothing to prove if $(x_{-2}, x_{-1}) \in \Gamma_1$.

Next, suppose that $(x_{-2}, x_{-1}) \in \Gamma_2$. Then, by (1.3), $x_0 = ax_{-2} + bf(x_{-1}) + c = ax_{-2} + b + c > a\lambda + b + c = \lambda$. Hence, $(x_{-1}, x_0) \in \Gamma_1$.

Next, suppose that $(x_{-2}, x_{-1}) \in \Gamma_3$. If $x_k > \lambda$ for all $k \geq -2$, then, by (1.3), $x_n = ax_n + c$ for $n \in N$. In view of (1.10) and (1.11),

$$\lambda \leq \lim_{k \to \infty} x_{2k} = \lim_{k \to \infty} x_{2k+1} = \frac{c}{1-a} < \frac{b+c}{1-a} = \lambda, \tag{3.8}$$

which is a contradiction. Thus there is $\mu \in N$ such that $x_{-2}, \ldots, x_{\mu-1} \in (\lambda, +\infty)$ and $x_\mu \in (0, \lambda]$. Hence, $(x_\mu, x_{\mu+1}) \in \Gamma_1$.

Next suppose that $(x_{-2}, x_{-1}) \in \Gamma_4$. Then, $(x_{-2}, x_{-1}) \in (C_{k+1}, C_k] \times (D_k, +\infty)$ for some $k \in N$. If $(x_{-2}, x_{-1}) \in (C_1, C_0] \times (D_0, +\infty) = (-c/a, 0] \times (\lambda, +\infty)$, then by (1.3),

$$\begin{aligned} 0 < x_0 &= ax_{-2} + bf(x_{-1}) + c = ax_{-2} + c \leq c < \lambda, \\ x_1 &= ax_{-1} + bf(x_0) + c = ax_{-1} + b + c > a\lambda + b + c = \lambda. \end{aligned} \tag{3.9}$$

Hence, $(x_0, x_1) \in \Gamma_1$. If $(x_{-2}, x_{-1}) \in (C_{k+1}, C_k] \times (D_k, +\infty)$ for some $k > 0$, then

$$\begin{aligned} C_k = aC_{k+1} + c < x_0 &= ax_{-2} + bf(x_{-1}) + c = ax_{-2} + c \leq aC_k + c = C_{k-1}, \\ x_1 &= ax_{-1} + b\mathrm{f}(x_0) + c = ax_{-1} + c \geq aD_k + c = D_{k-1}, \end{aligned} \tag{3.10}$$

we see that $(x_0, x_1) \in (C_k, C_{k-1}] \times (D_{k-1}, +\infty)$. By induction, we may further see that $(x_{2k-2}, x_{2k-1}) \in (C_1, C_0] \times (D_0, +\infty)$. Hence, $(x_{2k}, x_{2k+1}) \in \Gamma_1$.

Next, suppose that $(x_{-2}, x_{-1}) \in \Gamma_5$. Then $(x_{-2}, x_{-1}) \in (D_k, +\infty) \times (C_{k+1}, C_k]$ for some $k \in N$. By arguments similar to the previous case, we may then, show that $(x_{2k+1}, x_{2k+2}) \in \Gamma_1$. The proof is complete. □

Theorem C

Suppose that $\lambda = (b+c)/(1-a)$. Then, any solution $\{x_n\}_{n=-2}^{\infty}$ with $(x_{-2}, x_{-1}) \in R^2 \setminus \Phi$ tends to the limit 2-cycle $\langle c/(1-a), (b+c)/(1-a) \rangle$.

Proof. In view of Lemma 3.2, we may assume without loss of generality that $0 < x_{-2} \leq \lambda$ and $x_{-1} > \lambda$. Then, by (1.3),

$$\begin{aligned} 0 < x_0 &= ax_{-2} + bf(x_{-1}) + c = ax_{-2} + c \leq a\lambda + c < \lambda, \\ x_1 &= ax_{-1} + bf(x_0) + c = ax_{-1} + b + c > a\lambda + b + c = \lambda, \end{aligned} \tag{3.11}$$

and by induction $x_{2k} \in (0, \lambda)$ and $x_{2k+1} \in (\lambda, +\infty)$ for all $k \geq 0$. Hence, by (1.3), $x_{2k} = ax_{2k-2} + c$ and $x_{2k+1} = ax_{2k-1} + b + c$ for $k \in N$. In view of (1.10) and (1.11), $\lim_{k\to\infty} x_{2k} = c/(1-a)$ and $\lim_{k\to\infty} x_{2k+1} = (b+c)/(1-a)$. The proof is complete. □

4. The Case Where $c/(1-a) < \lambda < (b+c)/(1-a)$

In this section, we suppose $c/(1-a) < \lambda < (b+c)/(1-a)$. Then, $a\lambda + c < \lambda < a\lambda + b + c$.

Lemma 4.1. *Suppose that $c/(1-a) < \lambda < (b+c)/(1-a)$. Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of* (1.3). *Then, there is $m \in \{-2, -1, 0, \ldots\}$ such that $0 < x_m \leq \lambda$ and $x_{m+1} > \lambda$.*

Proof. We break up the plane into seven different parts: $\Gamma_1 = (0, \lambda] \times (\lambda, +\infty)$, $\Gamma_2 = (\lambda, +\infty) \times (0, \lambda]$, $\Gamma_3 = (0, \lambda]^2$, $\Gamma_4 = (\lambda, +\infty)^2$, $\Gamma_5 = (-\infty, 0] \times (0, +\infty)$, $\Gamma_6 = (0, +\infty) \times (-\infty, 0]$, and $\Gamma_7 = (-\infty, 0]^2$.

Clearly there is nothing to prove if $(x_{-2}, x_{-1}) \in \Gamma_1$.

Next, suppose that $(x_{-2}, x_{-1}) \in \Gamma_2$. Then, by (1.3), $x_0 = ax_{-2} + bf(x_{-1}) + c = ax_{-2} + b + c > a\lambda + b + c > \lambda$, and hence $(x_{-1}, x_0) \in \Gamma_1$.

Next, suppose that $(x_{-2}, x_{-1}) \in \Gamma_3$. If $x_k \in (0, \lambda]$ for all $k \geq -2$, then by (1.3), $x_n = ax_{n-2} + b + c$ for $n \in N$, which leads us to $\lambda \geq \lim_{k\to\infty} x_{2k} = \lim_{k\to\infty} x_{2k+1} = (b+c)/(1-a) > \lambda$, which is a contradiction. Hence, there is $\mu \in N$ such that $x_{-2}, x_{-1}, \ldots, x_{\mu-1} \in (0, \lambda]$ and $x_\mu \in (\lambda, +\infty)$. Therefore, $(x_{\mu-1}, x_\mu) \in \Gamma_1$.

Next, suppose that $(x_{-2}, x_{-1}) \in \Gamma_4$. If $x_k \in (\lambda, +\infty)$ for all $k \geq -2$, then, by (1.3), $x_n = ax_{n-2} + c$ for $n \in N$, which leads us to the contradiction $\lambda \leq \lim_{k\to\infty} x_{2k} = \lim_{k\to\infty} x_{2k+1} = c/(1-a) < \lambda$. Thus there is $\mu \in N$ such that $x_{-2}, x_{-1}, \ldots, x_{\mu-1} \in (\lambda, +\infty)$ and $x_\mu \in (0, \lambda]$. Then $(x_{\mu-1}, x_\mu) \in \Gamma_2$ and hence, $(x_\mu, x_{\mu+1}) \in \Gamma_1$.

Next, suppose that $(x_{-2}, x_{-1}) \in \Gamma_5$. Then, by (1.3) and induction, it is easily seen that $x_{2k-1} > 0$ for all $k \geq 0$. If $x_{2k} \leq 0$ for all $k \geq 0$, then by (1.3),

$$x_{2k} = ax_{2k-2} + bf(x_{2k-1}) + c \geq ax_{2k-2} + c, \quad k \in N. \tag{4.1}$$

In view of (1.12), $0 \geq \lim_{k\to\infty} x_{2k} \geq c/(1-a) > 0$, which is a contradiction. Hence, there is $n_0 \in N$ such that $(x_{2n_0-1}, x_{2n_0}) \in (0, +\infty)^2 = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$.

Next, suppose that $(x_{-2}, x_{-1}) \in \Gamma_6$. Then, $x_0 = ax_{-2} + bf(x_{-1}) + c = ax_{-2} + c > 0$. Hence, $(x_{-1}, x_0) \in (-\infty, 0] \times (0, +\infty) = \Gamma_5$.

Finally, suppose that $(x_{-2}, x_{-1}) \in \Gamma_7$. Then, by Lemma 1.1, there is $n_0 \in N$ such that $(x_{n_0-1}, x_{n_0}) \in (-\infty, 0] \times (0, c] \subset \Gamma_5$.

Therefore, in the last three cases, we may apply the conclusions in the first four cases to conclude our proof. The proof is complete. □

Theorem D

Suppose that $c/(1-a) < \lambda < (b+c)/(1-a)$. Then any solution $\{x_n\}_{n=-2}^{\infty}$ of (1.3) tends to the limit 2-cycle $\langle c/(1-a), (b+c)/(1-a) \rangle$.

Proof. Indeed, in view of Lemma 4.1, we may assume without loss of generality that $0 < x_{-2} \leq \lambda$ and $x_{-1} > \lambda$. Then the same arguments in the proof of Theorem C then shows that $\lim_{k\to\infty} x_{2k} = c/(1-a)$ and $\lim_{k\to\infty} x_{2k+1} = (b+c)/(1-a)$. □

5. The Case Where $\lambda = c/(1-a)$

In this section, we assume that $\lambda = c/(1-a)$. Then $\lambda = a\lambda + c$.

Lemma 5.1. *Suppose that $\lambda = c/(1-a)$. Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of* (1.3). *If $(x_{-2}, x_{-1}) \in R^2 \setminus (\lambda, +\infty)^2$, Then, there is $m \in \{-2,-1,0,\dots\}$ such that $0 < x_m \le \lambda$ and $x_{m+1} > \lambda$.*

Proof. We break up the set $R^2 \setminus (\lambda,+\infty)^2$ into eight different parts: $\Omega_1 = (0,\lambda] \times (\lambda,+\infty)$, $\Omega\_2 = (0,\lambda]^2$, $\Omega_3 = (\lambda,+\infty) \times (0,\lambda]$, $\Omega_4 = (-\infty,0] \times (0,\lambda]$, $\Omega_5 = (-\infty,0] \times (\lambda,+\infty)$, $\Omega_6 = (0,\lambda] \times (-\infty,0]$, $\Omega_7 = (\lambda,+\infty) \times (-\infty,0]$, and $\Omega_8 = (-\infty,0]^2$.

Clearly, there is nothing to prove if $(x_{-2}, x_{-1}) \in \Omega_1$.

Next, suppose that $(x_{-2}, x_{-1}) \in \Omega_2$. If $x_k \in (0,\lambda]$ for all $k \ge -2$, then by (1.3), $x_n = ax_{n-2} + b + c$ for $n \in N$. In view of (1.10) and (1.11), we obtain the contradiction.

$$\lambda \ge \lim_{k\to\infty} x_{2k} = \lim_{k\to\infty} x_{2k+1} = \frac{b+c}{1-a} > \lambda. \tag{5.1}$$

Hence, there is n_0 such that $0 < x_{-2}, x_{-1}, \dots, x_{n_0-1} \le \lambda$ and $x_{n_0} \in (\lambda,+\infty)$. Thus $(x_{n_0-1}, x_{n_0}) \in \Omega_1$.

Next, suppose that $(x_{-2}, x_{-1}) \in \Omega_3$. Then, by (1.3), $x_0 = ax_{-2} + bf(x_{-1}) + c = ax_{-2} + b + c > a\lambda + b + c > \lambda$. Hence, $(x_{-1}, x_0) \in \Omega_1$.

Next, suppose that $(x_{-2}, x_{-1}) \in \Omega_4$. Then, $(x_{-2}, x_{-1}) \in \bigcup_{k=0}^{\infty} (B_{k+1}, B_k] \times (0,\lambda]$ for some $k \in N$. If $x_{-2} \in (B_1, B_0] = (-(b+c)/a, 0]$, then

$$x_0 = ax_{-2} + bf(x_{-1}) + c = ax_{-2} + b + c > 0. \tag{5.2}$$

Hence, $(x_{-1}, x_0) \in (0,\lambda] \times (0,+\infty) = \Omega_1 \cup \Omega_2$. If $x_{-2} \in (B_{k+1}, B_k]$ for some $k > 0$, then

$$\begin{gathered} B_k = aB_{k+1} + b + c < x_0 = ax_{-2} + bf(x_{-1}) + c = ax_{-2} + b + c \le aB_k + b + c = B_{k-1}, \\ 0 < x_1 = ax_{-1} + bf(x_0) + c = ax_{-1} + c \le a\lambda + c < \lambda, \end{gathered} \tag{5.3}$$

we see that $(x_0, x_1) \in (B_k, B_{k-1}] \times (0,\lambda]$. By induction, we see that $(x_{2k-2}, x_{2k-1}) \in (B_1, B_0] \times (0,\lambda]$. Hence, $(x_{2k-1}, x_{2k}) \in (0,\lambda] \times (0,+\infty) = \Omega_1 \cup \Omega_2$.

Next, suppose that $(x_{-2}, x_{-1}) \in \Omega_5$. Then $(x_{-2}, x_{-1}) \in \bigcup_{k=0}^{\infty} (C_{k+1}, C_k] \times (\lambda,+\infty)$ for some $k \in N$. If $x_{-2} \in (C_1, C_0] = (-c/a, 0]$, then

$$\begin{gathered} 0 < x_0 = ax_{-2} + bf(x_{-1}) + c = ax_{-2} + c \le c < \lambda, \\ x_1 = ax_{-1} + bf(x_0) + c = ax_{-1} + b + c > a\lambda + b + c > \lambda. \end{gathered} \tag{5.4}$$

Hence, $(x_0, x_1) \in \Omega_1$. If $x_{-2} \in (C_{k+1}, C_k]$ for some $k > 0$, then

$$\begin{gathered} C_k = aC_{k+1} + c < x_0 = ax_{-2} + bf(x_{-1}) + c = ax_{-2} + c \le aC_k + c = C_{k-1}, \\ x_1 = ax_{-1} + bf(x_0) + c = ax_{-1} + c > a\lambda + c = \lambda, \end{gathered} \tag{5.5}$$

we see that $(x_0, x_1) \in (C_k, C_{k-1}] \times (\lambda,+\infty)$. By induction, $(x_{2k-2}, x_{2k-1}) \in (C_1, C_0] \times (\lambda,+\infty)$. Hence $(x_{2k-1}, x_{2k}) \in \Omega_1$.

Next, suppose that $(x_{-2}, x_{-1}) \in \Omega_6$. Then, by (1.3), $0 < x_0 = ax_{-2}+bf(x_{-1})+c = ax_{-2}+c \leq \lambda$. Hence, $(x_{-1}, x_0) \in (-\infty, 0] \times (0, \lambda] = \Omega_4$.

Next, suppose that $(x_{-2}, x_{-1}) \in \Omega_7$. Then, $x_0 = ax_{-2}+bf(x_{-1})+c = ax_{-2}+c > a\lambda+c = \lambda$, and hence, $(x_{-1}, x_0) \in (-\infty, 0] \times (\lambda, +\infty) = \Omega_5$.

Finally suppose $(x_{-2}, x_{-1}) \in \Omega_8$. Then, by Lemma 1.1, there is $n_0 \in N$ such that $(x_{n_0-1}, x_{n_0}) \in (-\infty, 0] \times (0, c] \subset \Omega_4$.

Therefore, in the fourth, sixth, seventh, and the eigth cases, we may use the conclusions in the other cases to conclude our proof. The proof is complete. □

Theorem E

Suppose that $\lambda = c/(1-a)$. Then, any solution $\{x_n\}_{n=-2}^{\infty}$ with $(x_{-2}, x_{-1}) \in R^2 \setminus (\lambda, +\infty)^2$ tends to the limit 2-cycle $\langle c/(1-a), (b+c)/(1-a)\rangle$.

Proof. Indeed, in view of Lemma 5.1, we may assume without loss of generality that $(x_{-2}, x_{-1}) \in (0, \lambda] \times (\lambda, +\infty)$. Then the same arguments in the proof of Theorem C shows that $\lim_{k\to\infty} x_{2k} = c/(1-a)$ and $\lim_{k\to\infty} x_{2k+1} = (b+c)/(1-a)$. □

Theorem F

Suppose that $\lambda = c/(1-a)$. Then, any solution $\{x_n\}_{n=-2}^{\infty}$ of (1.3) with $(x_{-2}, x_{-1}) \in (\lambda, +\infty)^2$ tends to $c/(1-a)$.

Proof. By (1.3),

$$\begin{aligned} x_0 &= ax_{-2} + bf(x_{-1}) + c = ax_{-2} + c > \lambda, \\ x_1 &= ax_{-1} + bf(x_0) + c = ax_{-1} + c > \lambda, \end{aligned} \tag{5.6}$$

and by induction, $x_k > \lambda$ for all $k \geq -2$. Hence, $x_n = ax_{n-2} + c$ for $n \in N$, which leads us to $\lim_{n\to\infty} x_n = c/(1-a)$. The proof is complete. □

6. The Case Where $0 < \lambda < c/(1-a)$

In this section, we suppose that $0 < \lambda < c/(1-a)$. Then $0 < \lambda < a\lambda + c$.

Lemma 6.1. *Suppose that $0 < \lambda < c/(1-a)$. Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of* (1.3)*. Then there is $m \in \{-2, -1, 0, \ldots\}$ such that $x_m, x_{m+1} > \lambda$.*

Proof. If $x_k \leq \lambda$ for all $k \geq -2$, then by (1.3), $x_k \geq ax_{k-2} + c$ for all $k \in N$. In view of (1.12) and (1.13), this is impossible, and hence, there must exist a m_0 such that $x_{m_0} \in (\lambda, +\infty)$. Then $x_{m_0+2} \geq ax_{m_0} + c > a\lambda + c > \lambda$. By induction, we see that the subsequence $\{x_{m_0+2k}\}$ lies in $(\lambda, +\infty)$, and hence $x_{m_0+2k+1} = ax_{m_0+2k-1} + c$ for all $k \in N$. In view of (1.11), $\lim_{k\to+\infty} x_{m_0+2k+1} = c/(1-a) > \lambda$. The proof is complete. □

Theorem G

Suppose that $0 < \lambda < c/(1-a)$. Then, every solution $\{x_n\}_{n=-2}^{\infty}$ of (1.3) converges to $c/(1-a)$.

Proof. Indeed, in view of Lemma 6.1, we may assume without loss of generality that $x_{-2}, x_{-1} > \lambda$. Then the same arguments in the proof of Theorem F shows that $\lim_{k\to\infty} x_{2k} = \lim_{k\to\infty} x_{2k+1} = c/(1-a)$. □

7. Phase Plane Interpretation and Comparison Remarks

We first recall that (1.1) and (1.3) are equivalent to

$$(u_n, v_n) = \left(v_{n-1}, au_{n-1} + bf_\lambda(v_{n-1})\right), \quad n \in \mathrm{N}, \tag{7.1}$$

$$(u_n, v_n) = \left(v_{n-1}, au_{n-1} + bf_\lambda(v_{n-1})\right) + (0, c), \quad n \in \mathrm{N}, \tag{7.2}$$

respectively, by means of the identification $(x_{n-1}, x_n) = (u_n, v_n)$ for $n = -1, 0, 1, \ldots$

Then, Theorem G states that when $0 < \lambda < c/(1-a)$, all solutions $\{(u_n, v_n)\}_{n=-1}^{\infty}$ of (7.2) tends to the point $(c/(1-a), c/(1-a))$, or equivalently, all solutions of (7.2) are "attracted" to the limit 1-cycle $\langle (c/(1-a), c/(1-a)) \rangle$, or equivalently, the limit 1-cycle $\langle (c/(1-a), c/(1-a)) \rangle$ is a global attractor. For the sake of convenience, let us set

$$p = \frac{c}{1-a}, \qquad q = \frac{b+c}{1-a}. \tag{7.3}$$

Then the above statements can be restated as follows.

(i) If $0 < \lambda < p$, then the limit 1-cycle $\langle (p, p) \rangle$ attracts all solutions of (7.2).

Similarly, we may restate the other Theorems A–G obtained previously as follows.

(ii) If $\lambda = p$, then the limit 1-cycle $\langle (p, p) \rangle$ attracts all solutions of (7.2) originated from $(p, +\infty)^2$, and the limit 2-cycle $\langle (p, q), (q, p) \rangle$ attracts all other solutions of (7.2).

(iii) If $p < \lambda < q$, then the limit 2-cycle $\langle (p, q), (q, p) \rangle$ attracts all solutions of (7.2).

(iv) If $\lambda = q$, then the limit 1-cycle $\langle (q, q) \rangle$ attracts all solutions of (7.2) originated from Φ (see (3.3)), and the limit 2-cycle $\langle (p, q), (q, p) \rangle$ attracts all other solutions.

(v) If $\lambda > q$, then the limit 1-cycle $\langle (q, q) \rangle$ attracts all solutions of (7.2).

For comparison purposes, let us now recall the asymptotic results in [1]. Let us set

$$r = \frac{b}{1-a}. \tag{7.4}$$

(vi) If $0 < \lambda < r$, then the limit 1-cycle $\langle (0, 0) \rangle$ attracts all solutions of (7.1) originated from $(-\infty, 0)^2$, and the limit 2-cycle $\langle (r, 0), (0, r) \rangle$ attracts all other solutions.

(vii) If $\lambda = r$, then the limit 1-cycle $\langle (0, 0) \rangle$ attracts all solutions of (7.1) originated from $(-\infty, 0)^2$; the limit 2-cycle $\langle (r, 0), (0, r) \rangle$ attracts all solutions of (7.1) originated from $((r, +\infty) \times (0, +\infty)) \cup ((0, +\infty) \times (r, +\infty))$, and the limit 1-cycle $\langle (r, r) \rangle$ attracts all other solutions.

(viii) If $\lambda > r$, then the limit 1-cycle $\langle(0,0)\rangle$ attracts all solutions of (7.1) originated from $(-\infty,0)^2$; and the limit 1-cycle attracts all other solutions.

In view of these statements, we see that for a small positive λ, all solutions of (7.2) tend to a unique "lower" state vector, and for large λ, to another unique "higher" state vector. On the other hand, for a small positive λ, there are always solutions of (7.1) which tend to a limit 2-cycle, and solutions which tend to the limit 1-cycle $\langle(0,0)\rangle$, and for a large λ, there are solutions of (7.1) which tend to the limit 1-cycle $\langle(0,0)\rangle$ and solutions to the limit 1-cycle $\langle(r,r)\rangle$. These observations show that it is probably not appropriate to call (1.1) the limiting case of (1.3)!

Finally, we mention that network models such as the following

$$\begin{aligned} x_n &= ax_{n-\alpha} + bf_\lambda(y_{n-1}) + c, \\ y_n &= ry_{n-\beta} + sf_\tau(x_{n-1}) + t \end{aligned} \tag{7.5}$$

can be used to describe competing dynamics and it is hoped that our techniques, and results here will be useful in these studies.

References

[1] Q. Ge, C. M. Hou, and S. S. Cheng, "Complete asymptotic analysis of a nonlinear recurrence relation with threshold control," *Advances in Difference Equations*, vol. 2010, Article ID 143849, 19 pages, 2010.

[2] H. Zhu and L. Huang, "Asymptotic behavior of solutions for a class of delay difference equation," *Annals of Differential Equations*, vol. 21, no. 1, pp. 99–105, 2005.

[3] Z. Yuan, L. Huang, and Y. Chen, "Convergence and periodicity of solutions for a discrete-time network model of two neurons," *Mathematical and Computer Modelling*, vol. 35, no. 9-10, pp. 941–950, 2002.

[4] H. Sedaghat, *Nonlinear Difference Equations*, vol. 15 of *Mathematical Modelling: Theory and Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2003.

[5] M. di Bernardo, C. J. Budd, A. R. Champneys, and P. Kowalczyk, *Piecewise Smooth Dynamical Systems*, Springer, New York, NY, USA, 2008.

[6] C. Hou and S. S. Cheng, "Eventually periodic solutions for difference equations with periodic coefficients and nonlinear control functions," *Discrete Dynamics in Nature and Society*, vol. 2008, Article ID 179589, 21 pages, 2008.