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Research Article

A Note on a Semilinear Fractional Differential Equation of Neutral Type with Infinite Delay

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We deal in this paper with the mild solution for the semilinear fractional differential equation of neutral type with infinite delay: $D^\alpha x(t) + Ax(t) = f(t, x_t)$, $t \in [0, T]$, $x(t) = \phi(t)$, $t \in]-\infty, 0]$, with $T > 0$ and $0 < \alpha < 1$. We prove the existence (and uniqueness) of solutions, assuming that $-A$ is a linear closed operator which generates an analytic semigroup $(T(t))_{t\geq 0}$ on a Banach space $\mathbb{X}$ by means of the Banach's fixed point theorem. This generalizes some recent results.

1. Introduction

We investigate in this paper the existence and uniqueness of the mild solution for the fractional differential equation with infinite delay

$$\begin{gathered} D^\alpha x(t) + Ax(t) = f(t, x_t), \quad t \in I = [0, T], \\ x(t) = \phi(t), \quad t \in]-\infty, 0], \end{gathered} \tag{1.1}$$

where $T > 0, 0 < \alpha < 1, -A$ is a generator of an analytic semigroup $(T(t))_{t\geq 0}$ on a Banach space $\mathbb{X}$ such that $\|T(t)\| \leq K$ for all $t \geq 0$ and $\|AT(t)x\| \leq K/t\|x\|$ for every $x \in X$ and $t > 0$. The function $f : I \times \mathcal{B} \to \mathbb{X}$ is continuous functions with additional assumptions.

The fractional derivative D^α is understood here in the Caputo sense, that is,

$$D^\alpha h(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha} h'(s)ds. \tag{1.2}$$

$\phi \in \mathcal{B}$ where $\mathcal{B}$ is called phase space to be defined in Section 2. For any function x defined on $]-\infty, T]$ and any $t \in I$, we denote by x_t the element of $\mathcal{B}$ defined by

$$x_t(\theta) = x(t+\theta), \quad \theta \in]-\infty, 0]. \tag{1.3}$$

The function x_t represents the history of the state from $-\infty$ up to the present time t.

The theory of functional differential equations has emerged as an important branch of nonlinear analysis. It is worthwhile mentioning that several important problems of the theory of ordinary and delay differential equations lead to investigations of functional differential equations of various types (see the books by Hale and Verduyn Lunel [1], Wu [2], Liang et al. [3], Liang and Xiao [4–9], and the references therein). On the other hand the theory of fractional differential equations is also intensively studied and finds numerous applications in describing real world problems (see e.g., the monographs of Lakshmikantham et al. [10], Lakshmikantham [11], Lakshmikantham and Vatsala [12, 13], Podlubny [14], and the papers of Agarwal et al. [15], Benchohra et al. [16], Anguraj et al. [17], Mophou and N'Guérékata [18], Mophou et al. [19], Mophou and N'Guérékata [20], and the references therein).

Recently we studied in our paper [20] the existence of solutions to the fractional semilinear differential equation with nonlocal condition and delay-free

$$\begin{gathered} D^\alpha\, x(t) = Ax(t) + t^n f(t, x(t), Bx(t)), \quad t \in [0,T],\ n \in \mathbb{Z}^+, \\ x(0) = x_0 + g(x), \end{gathered} \tag{1.4}$$

where T is a positive real, $0 < \alpha < 1$, A is the generator of a C_0-semigroup $(S(t))_{t\geq 0}$ on a Banach space $\mathbb{X}$, $Bx(t) := \int_0^t K(t,s)x(s)ds$, $K \in C(D, \mathbb{R}^+)$ with D defined as above and

$$B^* = \sup_{t\in[0,T]}\int_0^t K(t,s)ds < \infty, \tag{1.5}$$

$f : \mathbb{R} \times \mathbb{X} \times \mathbb{X} \to X$ is a nonlinear function, $g : \mathcal{C}([0,T], \mathbb{X}) \to D(A)$ is continuous, and $0 < q < 1$. The derivative D^α is understood here in the Riemann-Liouville sense.

In the present paper we deal with an infinite time delay. Note that in this case, the phase space $\mathcal{B}$ plays a crucial role in the study of both qualitative and quantitative aspects of theory of functional equations. Its choice is determinant as can be seen in the important paper by Hale and Kato [21].

Similar works to the present paper include the paper by Benchohra et al. [16], where the authors studied an existence result related to the nonlinear functional differential equation

$$D^{\alpha}\ x(t) = f(t, x_t), \quad t \in I = [0, T],\ 0 < \alpha < 1,$$
$$x(t) = \phi(t), \quad t \in\ ]-\infty, 0], \tag{1.6}$$

where D^{α} is the standard Riemann-Liouville fractional derivative, ϕ in the phase space $\mathcal{B}$, with $\phi(0) = 0$.

2. Preliminaries

From now on, we set $I = [0, T]$. We denote by $\mathbb{X}$ a Banach space with norm $\|\cdot\|$, $C(I; \mathbb{X})$ the space of all $\mathbb{X}$-valued continuous functions on I, and $L(\mathbb{X})$ the Banach space of all linear and bounded operators on $\mathbb{X}$.

We assume that the phase space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a seminormed linear space of functions mapping $]-\infty, 0]$ into $\mathbb{X}$, and satisfying the following fundamental axioms due to Hale and Kato (see e.g., in [21]).

(A_0) If $x :\ ]-\infty, T] \to \mathbb{X}$, is continuous on I and $x_0 \in \mathcal{B}$, then for every $t \in I$ the following conditions hold:

(i) x_t is in $\mathcal{B}$,

(ii) $\|x(t)\| \le H\|x_t\|_{\mathcal{B}}$,

(iii) $\|x_t\|_{\mathcal{B}} \le C_1(t)\sup_{0\le s\le t}\|x(s)\| + C_2(t)\|x_0\|_{\mathcal{B}}$,

where $H \ge 0$ is a constant, $C_1 :\ [0, +\infty[\to [0, +\infty[$ is continuous, $C_2 :\ [0, +\infty[\to [0, +\infty[$ is locally bounded, and H, C_1, C_2 are independent of $x(\cdot)$.

(A_1) For the function $x(\cdot)$ in (A_0), x_t is a $\mathcal{B}$-valued continuous function on I.

(A_2) The space $\mathcal{B}$ is complete.

Remark 2.1. Condition (ii) in (A_0) is equivalent to $\|\phi(0)\| \le H\|\phi\|_{\mathcal{B}}$, for all $\phi \in \mathcal{B}$.

Let us recall some examples of phase spaces.

Example 2.2. (E1) $BUC(]-\infty, 0]); \mathbb{X})$ the Banach space of all bounded and uniformly continuous functions $\phi :] -\infty, 0] \to \mathbb{X}$ endowed with the supnorm.

(E2) $C^0(]-\infty, 0]) : \mathbb{X})$ the Banach space of all bounded and continuous functions $\phi :] -\infty, 0] \to \mathbb{X}$ such that $\lim_{\theta\to-\infty}\phi(\theta) = 0$ endowed with the norm

$$|\phi| := \sup_{\theta\le 0}|\phi(\theta)|. \tag{2.1}$$

(E3) $C_{\gamma} := \{\phi \in C(]-\infty, 0] : \mathbb{X}) : \lim_{\theta\to-\infty} e^{\gamma\theta}\phi(\theta)$exists in $\mathbb{X}\}$ endowed with the norm

$$|\phi| = \sup_{-\infty<\theta\le 0} e^{\gamma\theta}|\phi(\theta)|. \tag{2.2}$$

Note that the space C_{γ} is a uniform fading memory for $\gamma > 0$.

Throughout this work f will be a continuous function $I\times\mathcal{B}\times \longrightarrow \mathbb{X}$. Let Ω be set defined by:

$$\Omega = \left\{x : \ ]-\infty,T] \longrightarrow \mathbb{X} \text{ such that } x|_{]-\infty,0]} \in \mathcal{B},\ x|_I \in \mathcal{C}(I;\mathbb{X})\right\}. \tag{2.3}$$

Remark 2.3. We recall that the Cauchy Problem

$$\begin{aligned} D^\alpha x(t) + Ax(t) = 0, \quad t \in [0,T], \\ x(0) = x_0 \in D, \quad t \in I, \end{aligned} \tag{2.4}$$

where $-A$ is a closed linear operator defined on a dense subset, $D \subset X$ is wellposed, and the unique solution is given by

$$x(t) = \int_0^\infty \zeta_\alpha(\sigma) T(t^\alpha \sigma) x_0 d\sigma, \tag{2.5}$$

where ζ_α is a probability density function defined on $(0,\infty)$ such that its Laplace transform is given by

$$\int_0^\infty e^{-\sigma x}\zeta_\alpha(\sigma)d\sigma = \sum_{i=0}^{\infty}\frac{(-x^i)}{\Gamma(1+\alpha i)}, \quad x > 0 \tag{2.6}$$

([22, cf. Theorem 2.1]).

Following [22, 23] we will introduce now the definition of mild solution to (1.1).

Definition 2.4. A function $x \in \Omega$ is said to be a mild solution of (1.1) if x satisfies

$$x(t) = \begin{cases} \phi(t), & t \in \ ]-\infty,0], \\ Q(t)\phi(0) + \displaystyle\int_0^t R(t-s) f(s,x_s) ds, & t \in I, \end{cases} \tag{2.7}$$

where

$$Q(t) = \int_0^\infty \zeta_\alpha(\sigma) T(t^\alpha\sigma) d\sigma, \qquad R(t) = \alpha\int_0^\infty \sigma t^{\alpha-1}\zeta_\alpha(\sigma) T(t^\alpha\sigma) d\sigma. \tag{2.8}$$

Remark 2.5. Note that

$$\|R(t)\|_{B(\mathbb{X})} \le \alpha K t^{\alpha-1}, \quad t \ge 0, \tag{2.9}$$

since $\int_0^\infty \sigma\zeta_\alpha(\sigma)d\sigma = 1$ (cf. [23]).

3. Main Results

We present now our result.

Theorem 3.1. *Assume the following.*

(H_1) *There exist* $\mu > 0$ *such that for all* $t \in I$, $(\psi, \varphi) \in \mathcal{B}^2$

$$\|f(t,\varphi) - f(t,\psi)\| \leq \mu\|\varphi - \psi\|_{\mathcal{B}} \tag{3.1}$$

(H_2) *There exists* δ, *with* $0 < \delta < 1$ *such that the function* $\Lambda : I \to]0, +\infty]$ *defined by:*

$$\Lambda(t) = \mu K C_1^* t^{\alpha} \tag{3.2}$$

satisfies $\Lambda(t) \leq \delta$ *for all* $t \in I$. *Here*

$$C_1^* = \sup_{t\in I} C_1(t). \tag{3.3}$$

Then (1.1) *has a unique mild solution on* $]-\infty, T]$.

Proof. Consider the operator $N : \Omega \to \Omega$ defined by

$$N(x)(t) = \begin{cases} \phi(t), & t \in]-\infty, 0], \\ Q(t)\phi(0) + \displaystyle\int_0^t R(t-s) f(s, x_s) ds, & t \in I. \end{cases} \tag{3.4}$$

Let $y(\cdot) :]-\infty, T] \to \mathbb{X}$ be the function defined by

$$y(t) = \begin{cases} \phi(t), & \text{if } t \in]-\infty, 0], \\ Q(t)\phi(0), & \text{if } t \in I. \end{cases} \tag{3.5}$$

Then $y_0 = \phi$. For each $z \in C(I, \mathbb{X})$ with $z(0) = 0$, we denote by $\overline{z}$ the function defined by

$$\overline{z}(t) = \begin{cases} 0, & t \in]-\infty, 0], \\ z(t), & t \in I. \end{cases} \tag{3.6}$$

If $x(\cdot)$ verifies (2.7) then writing $x(t) = y(t) + \overline{z}(t)$ for $t \in I$, we have $x_t = y_t + \overline{z}_t$ for $t \in I$ and

$$z(t) = \int_0^t R(t-s) f(s, y_s + \overline{z}_s) ds. \tag{3.7}$$

Moreover $z_0 = 0$.

Let

$$Z_0 = \{z \in \Omega \text{ such that } z_0 = 0\}. \tag{3.8}$$

For any $z \in Z_0$, we have

$$\|z\|_{Z_0} = \sup_{t\in I}\|z(t)\| + \|z_0\|_{\mathcal{B}} = \sup_{t\in I}\|z(t)\|. \tag{3.9}$$

Thus $(Z_0, \|\cdot\|_{Z_0})$ is a Banach space. We define the operator $\Pi : Z_0 \to Z_0$ by

$$\Pi(z)(t) = \int_0^t R(t-s)f(s, y_s + \overline{z}_s)ds. \tag{3.10}$$

It is clear that the operator N has a unique fixed point if and only if Π has a unique fixed point. So let us prove that Π has a unique fixed point. Observe first that Π is obviously well defined. Now, consider $z, z^* \in Z_0$. For any $t \in I$, we have

$$\begin{aligned}
\|\Pi(z)(t) - \Pi(z^*)(t)\| &= \left|\int_0^t R(t-s)f(s, y_s + \overline{z}_s)ds - \int_0^t R(t-s)f\left(s, y_s + \overline{z^*}_s\right)ds\right| \\
&\le \int_0^t \|R(t-s)\| \left\|f(s, y_s + \overline{z}_s) - f\left(s, y_s + \overline{z^*}_s\right)\right\|ds \\
&\le \mu\int_0^t \|R(t-s)\| \left\|\overline{z}_s - \overline{z^*}_s\right\|_{\mathcal{B}}ds.
\end{aligned} \tag{3.11}$$

So using (A_0)-(iii), (2.9) and (3.3), we obtain for all $t \in I$

$$\begin{aligned}
\|\Pi(z)(t) - \Pi(z^*)(t)\| &\le \mu\alpha K\int_0^t (t-s)^{\alpha-1}C_1(s)\|z - z^*\|_{Z_0} \\
&\le \mu K C_1^* t^{\alpha}\|z - z^*\|_{Z_0}
\end{aligned} \tag{3.12}$$

which according to (H_2) gives

$$\begin{aligned}
\|\Pi(z)(t) - \Pi(z^*)(t)\| &\le \Lambda(t)\|z - z^*\|_{Z_0} \\
&\le \delta\|z - z^*\|_{Z_0}.
\end{aligned} \tag{3.13}$$

Therefore

$$\|\Pi z - \Pi z^*\|_{Z_0} \le \delta\|z - z^*\|_{Z_0}. \tag{3.14}$$

And since $0 \le \delta < 1$, we conclude by way of the Banach's contraction mapping principle that Π has a unique fixed point $z \in Z_0$. This means that N has a unique fixed point $x \in \Omega$ which is obviously a mild solution of (1.1) on $]-\infty, T]$. □

4. Application

To illustrate our result, we consider the following Lotka-Volterra model with diffusion:

$$D_t^{\alpha} u(t,\xi) = \frac{\partial^2}{\partial \xi^2} u(t,\xi) + \int_{-\infty}^{0} \eta(\sigma) u(t+\sigma,\xi) d\sigma, \quad 0 \leq \xi \leq \pi,$$
$$u(t,0) = u(t,\pi) = 0 \quad \text{for } t \in \mathbb{R}, \tag{4.1}$$

where $0 < \alpha < 1$ and η is a positive function on $(-\infty, 0]$ with $\int_{-\infty}^{0} \eta(\sigma) d\sigma < \infty$.

Now let $X = L^2(0,\pi)$ and consider the operator $A : D(A) \subset X \to X$ defined by

$$D(A) = H^2(0,\pi) \cap H_0^1(0,\pi) = \left\{ H^2(0,\pi) : z(0) = z(\pi) = 0 \right\},$$
$$Az = z''. \tag{4.2}$$

Clearly $D(A)$ is dense in $L^2(0,\pi)$.

Define

$$f(\phi)(\xi) := \int_{0}^{\infty} \eta(\sigma) \phi(\sigma)(\xi) d\sigma, \quad \xi \in [0,\pi], \ \phi \in \mathcal{B}. \tag{4.3}$$

We choose $\mathcal{B}$ as in Example (E3) above. Put

$$x(t)(\xi) = u(t,\xi), \quad t \in (-\infty, T], \ \xi \in [0,\pi]. \tag{4.4}$$

Then we get

$$D^{\alpha} x(t) = Ax(t) + f(t, x_t), \tag{4.5}$$

where $f(t,x)$ is obviously Lipschitzian in x uniformly in t. Thus we can state what follows.

Theorem 4.1. *Under the above assumptions* (4.1) *has a unique mild solution.*

References

[1] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional-Differential Equations*, vol. 99 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1993.

[2] J. Wu, *Theory and Applications of Partial Functional-Differential Equations*, vol. 119 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1996.

[3] J. Liang, F. Huang, and T. Xiao, "Exponential stability for abstract linear autonomous functional-differential equations with infinite delay," *International Journal of Mathematics and Mathematical Sciences*, vol. 21, no. 2, pp. 255–259, 1998.

[4] J. Liang and T. J. Xiao, "Functional-differential equations with infinite delay in Banach spaces," *International Journal of Mathematics and Mathematical Sciences*, vol. 14, no. 3, pp. 497–508, 1991.

[5] J. Liang and T.-J. Xiao, "The Cauchy problem for nonlinear abstract functional differential equations with infinite delay," *Computers & Mathematics with Applications*, vol. 40, no. 6-7, pp. 693–703, 2000.

[6] J. Liang and T.-J. Xiao, "Solvability of the Cauchy problem for infinite delay equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 58, no. 3-4, pp. 271–297, 2004.

[7] J. Liang and T.-J. Xiao, "Solutions to nonautonomous abstract functional equations with infinite delay," *Taiwanese Journal of Mathematics*, vol. 10, no. 1, pp. 163–172, 2006.

[8] J. Liang, T.-J. Xiao, and J. van Casteren, "A note on semilinear abstract functional differential and integrodifferential equations with infinite delay," *Applied Mathematics Letters*, vol. 17, no. 4, pp. 473–477, 2004.

[9] T.-J. Xiao and J. Liang, "Blow-up and global existence of solutions to integral equations with infinite delay in Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 12, pp. e1442–e1447, 2009.

[10] V. Lakshmikantham, S. Leela, and J. Vasundhara, *Theory of Fractional Dynamic Systems*, Cambridge Academic, Cambridge, UK, 2009.

[11] V. Lakshmikantham, "Theory of fractional functional differential equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 10, pp. 3337–3343, 2008.

[12] V. Lakshmikantham and A. S. Vatsala, "Basic theory of fractional differential equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 8, pp. 2677–2682, 2008.

[13] V. Lakshmikantham and A. S. Vatsala, "Theory of fractional differential inequalities and applications," *Communications in Applied Analysis*, vol. 11, no. 3-4, pp. 395–402, 2007.

[14] I. Podlubny, *Fractional Differential Equations*, vol. 198 of *Mathematics in Science and Engineering*, Academic Press, San Diego, Calif, USA, 1999.

[15] R. P. Agarwal, M. Benchohra, and B. A. Slimani, "Existence results for differential equations with fractional order and impulses," *Memoirs on Differential Equations and Mathematical Physics*, vol. 44, pp. 1–21, 2008.

[16] M. Benchohra, J. Henderson, S. K. Ntouyas, and A. Ouahab, "Existence results for fractional order functional differential equations with infinite delay," *Journal of Mathematical Analysis and Applications*, vol. 338, no. 2, pp. 1340–1350, 2008.

[17] A. Anguraj, P. Karthikeyan, and G. M. N'Guérékata, "Nonlocal Cauchy problem for some fractional abstract integro-differential equations in Banach spaces," *Communications in Mathematical Analysis*, vol. 6, no. 1, pp. 31–35, 2009.

[18] G. M. Mophou and G. M. N'Guérékata, "Mild solutions for semilinear fractional differential equations," *Electronic Journal of Differential Equations*, vol. 2009, no. 21, pp. 1–9, 2009.

[19] G. M. Mophou, O. Nakoulima, and G. M. N'Guérékata, "Existence results for some fractional differential equations with nonlocal conditions," *Nonlinear Studies*, vol. 17, no. 1, pp. 15–22, 2010.

[20] G. M. Mophou and G. M. N'Guérékata, "Existence of the mild solution for some fractional differential equations with nonlocal conditions," *Semigroup Forum*, vol. 79, no. 2, pp. 315–322, 2009.

[21] J. K. Hale and J. Kato, "Phase space for retarded equations with infinite delay," *Funkcialaj Ekvacioj*, vol. 21, no. 1, pp. 11–41, 1978.

[22] M. M. El-Borai, "Some probability densities and fundamental solutions of fractional evolution equations," *Chaos, Solitons and Fractals*, vol. 14, no. 3, pp. 433–440, 2002.

[23] M. M. El-Borai, "On some stochastic fractional integro-differential equations," *Advances in Dynamical Systems and Applications*, vol. 1, no. 1, pp. 49–57, 2006.