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Research Article

Oscillation of Solutions of a Linear Second-Order Discrete-Delayed Equation

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A linear second-order discrete-delayed equation $\Delta x(n) = -p(n)x(n-1)$ with a positive coefficient p is considered for $n \to \infty$. This equation is known to have a positive solution if p fulfils an inequality. The goal of the paper is to show that, in the case of the opposite inequality for p, all solutions of the equation considered are oscillating for $n \to \infty$.

1. Introduction

The existence of a positive solution of difference equations is often encountered when analysing mathematical models describing various processes. This is a motivation for an intensive study of the conditions for the existence of positive solutions of discrete or continuous equations. Such analysis is related to an investigation of the case of all solutions being oscillating (for relevant investigation in both directions, we refer, e.g., to [1–15] and to the references therein). In this paper, sharp conditions are derived for all the solutions being oscillating for a class of linear second-order delayed-discrete equations.

We consider the delayed second-order linear discrete equation

$$\Delta x(n) = -p(n)x(n-1), \tag{1.1}$$

where $n \in \mathbb{Z}_a^\infty := \{a, a+1, \ldots\}$, $a \in \mathbb{N}$ is fixed, $\Delta x(n) = x(n+1) - x(n)$, and $p : \mathbb{Z}_a^\infty \to \mathbb{R}^+ := (0, \infty)$. A solution $x = x(n) : \mathbb{Z}_a^\infty \to \mathbb{R}$ of (1.1) is positive (negative) on $\mathbb{Z}_a^\infty$ if $x(n) > 0$ ($x(n) < 0$) for every $n \in \mathbb{Z}_a^\infty$. A solution $x = x(n) : \mathbb{Z}_a^\infty \to \mathbb{R}$ of (1.1) is oscillating on $\mathbb{Z}_a^\infty$ if it is not positive or negative on $\mathbb{Z}_{a_1}^\infty$ for arbitrary $a_1 \in \mathbb{Z}_a^\infty$.

Definition 1.1. Let us define the expression $\ln_q t$, $q \geq 1$, by $\ln_q t = \ln(\ln_{q-1} t)$, $\ln_0 t \equiv t$ where $t > \exp_{q-2} 1$ and $\exp_s t = \exp(\exp_{s-1} t)$, $s \geq 1$, $\exp_0 t \equiv t$ and $\exp_{-1} t \equiv 0$ (instead of $\ln_0 t$, $\ln_1 t$, we will only write t and $\ln t$).

In [2] a delayed linear difference equation of higher order is considered and the following result related to (1.1) on the existence of a positive solution is proved.

Theorem 1.2. *Let $a \in \mathbb{N}$ be sufficiently large and $q \in \mathbb{N}$. If the function $p : \mathbb{Z}_a^\infty \to \mathbb{R}^+$ satisfies*

$$p(n) \leq \frac{1}{4} + \frac{1}{16n^2} + \frac{1}{16(n\ln n)^2} + \frac{1}{16(n\ln n\ln_2)^2} + \frac{1}{16(n\ln n \ln_2 n \ln_3 n)^2} + \cdots + \frac{1}{16\left(n\ln n \ln_2 n \cdots \ln_q n\right)^2} \tag{1.2}$$

for every $n \in \mathbb{Z}_a^\infty$, then there exist a positive integer $a_1 \geq a$ and a solution $x = x(n)$, $n \in \mathbb{Z}_{a_1}^\infty$ of (1.1) *such that $x(n) > 0$ holds for every $n \in \mathbb{Z}_{a_1}^\infty$.*

Our goal is to answer the open question whether all solutions of (1.1) are oscillating if inequality (1.2) is replaced by the opposite inequality

$$p(n) \geq \frac{1}{4} + \frac{1}{16n^2} + \frac{1}{16(n\ln n)^2} + \frac{1}{16(n\ln n\ln_2 n)^2} + \frac{1}{16(n\ln n \ln_2 n \ln_3 n)^2} + \cdots + \frac{1}{16\left(n\ln n \ln_2 n \cdots \ln_{q-1} n\right)^2} + \frac{\kappa}{16\left(n\ln n \ln_2 n \cdots \ln_q n\right)^2} \tag{1.3}$$

assuming $\kappa > 1$ and n is sufficiently large. Below we prove that if (1.3) holds and $\kappa > 1$, then all solutions of (1.1) are oscillatory. The proof of our main result will use a consequence of one of Domshlak's results [8, Corollary 4.2, page 69].

Lemma 1.3. *Let q and r be fixed natural numbers such that $r - q > 1$. Let $\{\varphi(n)\}_1^\infty$ be a given sequence of positive numbers and ν_0 a positive number such that there exists a number $\nu \in (0, \nu_0)$ satisfying*

$$\sum_{q+1}^{r} \varphi(n) \leq \frac{\pi}{\nu}, \quad \frac{\pi}{\nu} \leq \sum_{q+1}^{r+1} \varphi(n) \leq \frac{2\pi}{\nu}. \tag{1.4}$$

Then, if $p(q+1) \geq 0$ and for $n \in \mathbb{Z}_{q+2}^r$

$$p(n) \geq \frac{\sin \nu\varphi(n-1) \cdot \sin \nu\varphi(n+1)}{\sin \nu\left[\varphi(n-1) + \varphi(n)\right] \cdot \sin \nu\left[\varphi(n) + \varphi(n+1)\right]} \tag{1.5}$$

holds, then any solution of the equation

$$x(n+1) - x(n) + p(n)x(n-1) = 0 \tag{1.6}$$

has at least one change of sign on $\mathbb{Z}_{q-1}^{r+1}$.

Moreover, we will use an auxiliary result giving the asymptotic decomposition of the iterative logarithm [7]. The symbols "o" and "O" used below stand for the Landau order symbols.

Lemma 1.4. *For fixed $r, \sigma \in \mathbb{R} \setminus \{0\}$ and fixed integer $s \geq 1$, the asymptotic representation*

$$\begin{aligned}\frac{\ln_s^\sigma(n-r)}{\ln_s^\sigma n} = 1 &- \frac{r\sigma}{n\ln n\cdots\ln_s n} - \frac{r^2\sigma}{2n^2\ln n\cdots\ln_s n} \\ &- \frac{r^2\sigma}{2(n\ln n)^2\ln_2 n\cdots\ln_s n} - \cdots - \frac{r^2\sigma}{2(n\ln n\cdots\ln_{s-1}n)^2\ln_s n} \\ &+ \frac{r^2\sigma(\sigma-1)}{2(n\ln n\cdots\ln_s n)^2} - \frac{r^3\sigma(1+o(1))}{3n^3\ln n\cdots\ln_s n}\end{aligned} \tag{1.7}$$

holds for $n \to \infty$.

2. Main Result

In this part, we give sufficient conditions for all solutions of (1.1) to be oscillatory as $n \to \infty$.

Theorem 2.1. *Let $a \in \mathbb{N}$ be sufficiently large, $q \in \mathbb{N}$, and $\kappa > 1$. Assuming that the function $p : \mathbb{Z}_a^\infty \to \mathbb{R}^+$ satisfies inequality (1.3) for every $n \in \mathbb{Z}_a^\infty$, all solutions of (1.1) are oscillating as $n \to \infty$.*

Proof. We set

$$\varphi(n) := \frac{1}{n\ln n\ln_2 n\ln_3 n\cdots\ln_q n} \tag{2.1}$$

and consider the asymptotic decomposition of $\varphi(n-1)$ when n is sufficiently large. Applying Lemma 1.4 (for $\sigma = -1$, $r = 1$, and $s = 1, 2, \ldots, q$), we get

$$\begin{aligned}\varphi(n-1) &= \frac{1}{(n-1)\ln(n-1)\ln_2(n-1)\ln_3(n-1)\cdots\ln_q(n-1)} \\ &= \frac{1}{n(1-1/n)\ln(n-1)\ln_2(n-1)\ln_3(n-1)\cdots\ln_q(n-1)} \\ &= \varphi(n)\cdot\frac{1}{1-1/n}\cdot\frac{\ln n}{\ln(n-1)}\cdot\frac{\ln_2 n}{\ln_2(n-1)}\cdot\frac{\ln_3 n}{\ln_3(n-1)}\cdots\frac{\ln_q n}{\ln_q(n-1)} \\ &= \varphi(n)\left(1+\frac{1}{n}+\frac{1}{n^2}+O\left(\frac{1}{n^3}\right)\right) \\ &\quad\times\left(1+\frac{1}{n\ln n}+\frac{1}{2n^2\ln n}+\frac{1}{(n\ln n)^2}+O\left(\frac{1}{n^3}\right)\right)\end{aligned}$$

$$\times\left(1+\frac{1}{n\ln n\ln_2 n}+\frac{1}{2n^2\ln n\ln_2 n}+\frac{1}{2(n\ln n)^2\ln_2 n}+\frac{1}{(n\ln n\ln_2 n)^2}+O\left(\frac{1}{n^3}\right)\right)$$
$$\times\left(1+\frac{1}{n\ln n\ln_2 n\ln_3 n}+\frac{1}{2n^2\ln n\ln_2 n\ln_3 n}+\frac{1}{2(n\ln n)^2\ln_2 n\ln_3 n}\right.$$
$$\left.+\frac{1}{2(n\ln n\ln_2 n)^2\ln_3 n}+\frac{1}{(n\ln n\ln_2 n\ln_3 n)^2}+O\left(\frac{1}{n^3}\right)\right)$$
$$\times\cdots\times\left(1+\frac{1}{n\ln n\ln_2 n\ln_3 n\cdots\ln_q n}+\frac{1}{2n^2\ln n\cdots\ln_q n}+\frac{1}{2(n\ln n)^2\ln_2\cdots n\ln_q n}\right.$$
$$\left.+\cdots+\frac{1}{2(n\ln n\cdots\ln_{q-1} n)^2\ln_q n}+\frac{1}{(n\ln n\cdots\ln_q n)^2}+O\left(\frac{1}{n^3}\right)\right). \tag{2.2}$$

Finally, we obtain

$$\begin{aligned}
&\varphi(n-1)\\
&=\varphi(n)\left(1+\frac{1}{n}+\frac{1}{n\ln n}+\frac{1}{n\ln n\ln_2 n}+\frac{1}{n\ln n\ln_2 n\ln_3 n}+\cdots+\frac{1}{n\ln n\ln_2 n\cdots\ln_q n}\right.\\
&\quad+\frac{1}{n^2}+\frac{3}{2n^2\ln n}+\frac{3}{2n^2\ln n\ln_2 n}+\cdots+\frac{3}{2n^2\ln n\ln_2 n\cdots\ln_q n}\\
&\quad+\frac{1}{(n\ln n)^2}+\frac{3}{2(n\ln n)^2\ln_2 n}+\frac{3}{2(n\ln n)^2\ln_3 n}+\cdots+\frac{3}{2(n\ln n)^2\ln_3 n\cdots\ln_q n}\\
&\quad+\frac{1}{(n\ln n\ln_2 n)^2}+\frac{3}{2(n\ln n\ln_2 n)^2\ln_3 n}+\cdots+\frac{3}{2(n\ln n\ln_2 n)^2\ln_3 n\cdots\ln_q n}\\
&\quad+\frac{1}{(n\ln n\ln_2 n\ln_3 n)^2}+\frac{3}{2(n\ln n\ln_2 n\ln_3 n)^2\ln_4 n}\\
&\quad+\cdots+\frac{3}{2(n\ln n\ln_2 n\ln_3 n)^2\ln_4 n\cdots\ln_q n}\\
&\quad+\cdots+\frac{1}{(n\ln n\ln_2 n\cdots\ln_{q-1} n)^2}+\frac{3}{2(n\ln n\ln_2 n\cdots\ln_{q-1} n)^2\ln_q n}\\
&\quad\left.+\frac{1}{(n\ln n\ln_2 n\cdots\ln_q n)^2}+O\left(\frac{1}{n^3}\right)\right).
\end{aligned} \tag{2.3}$$

Similarly, applying Lemma 1.4 (for $\sigma = -1$, $r = -1$, and $s = 1, 2, \dots, q$), we get

$$\begin{aligned}
\varphi(n+1) &= \frac{1}{(n+1)\ln(n+1)\ln_2(n+1)\cdots\ln_q(n+1)} \\
&= \frac{1}{n(1+(1/n))\ln(n+1)\ln_2(n+1)\cdots\ln_q(n+1)} \\
&= \varphi(n)\cdot\frac{1}{1+1/n}\cdot\frac{\ln n}{\ln(n+1)}\cdot\frac{\ln_2 n}{\ln_2(n+1)}\cdot\frac{\ln_3 n}{\ln_3(n+1)}\cdots\frac{\ln_q n}{\ln_q(n+1)} \\
&= \varphi(n)\left(1-\frac{1}{n}+\frac{1}{n^2}+O\left(\frac{1}{n^3}\right)\right) \\
&\quad\times\left(1-\frac{1}{n\ln n}+\frac{1}{2n^2\ln n}+\frac{1}{(n\ln n)^2}+O\left(\frac{1}{n^3}\right)\right) \\
&\quad\times\left(1-\frac{1}{n\ln n\ln_2 n}+\frac{1}{2n^2\ln n\ln_2 n}+\frac{1}{2(n\ln n)^2\ln_2 n}+\frac{1}{(n\ln n\ln_2 n)^2}+O\left(\frac{1}{n^3}\right)\right) \\
&\quad\times\Bigg(1-\frac{1}{n\ln n\ln_2 n\ln_3 n}+\frac{1}{2n^2\ln n\ln_2 n\ln_3 n}+\frac{1}{2(n\ln n)^2\ln_2 n\ln_3 n} \\
&\qquad+\frac{1}{2(n\ln n\ln_2 n)^2\ln_3 n}+\frac{1}{(n\ln n\ln_2 n\ln_3 n)^2}+O\left(\frac{1}{n^3}\right)\Bigg) \\
&\quad\times\cdots\times\Bigg(1-\frac{1}{n\ln n\ln_2 n\cdots\ln_q n}+\frac{1}{2n^2\ln n\ln_2 n\cdots\ln_q n}+\frac{1}{2(n\ln n)^2\ln_2 n\cdots\ln_q n} \\
&\qquad+\cdots+\frac{1}{2(n\ln n\cdots\ln_{q-1} n)^2\ln_q n}+\frac{1}{(n\ln n\ln_2 n\cdots\ln_q n)^2}+O\left(\frac{1}{n^3}\right)\Bigg) \\
&= \varphi(n)\Bigg(1-\frac{1}{n}-\frac{1}{n\ln n}-\frac{1}{n\ln n\ln_2 n}-\cdots-\frac{1}{n\ln n\ln_2 n\cdots\ln_q n} \\
&\qquad+\frac{1}{n^2}+\frac{3}{2n^2\ln n}+\frac{3}{2n^2\ln n\ln_2 n}+\cdots+\frac{3}{2n^2\ln n\ln_2 n\cdots\ln_q n} \\
&\qquad+\frac{1}{(n\ln n)^2}+\frac{3}{2(n\ln n)^2\ln_2 n}+\cdots+\frac{3}{2(n\ln n)^2\ln_2 n\cdots\ln_q n} \\
&\qquad+\frac{1}{(n\ln n\ln_2 n)^2}+\frac{3}{2(n\ln n\ln_2 n)^2\ln_3 n}+\cdots+\frac{3}{2(n\ln n\ln_2 n)^2\ln_3 n\cdots\ln_q n} \\
&\qquad+\cdots+\frac{1}{(n\ln n\ln_2 n\cdots\ln_{q-1} n)^2}+\frac{1}{(n\ln n\ln_2 n\cdots\ln_{q-1} n)^2\ln_q n} \\
&\qquad+\frac{1}{(n\ln n\ln_2 n\cdots\ln_q n)^2}+O\left(\frac{1}{n^3}\right)\Bigg).
\end{aligned} \tag{2.4}$$

Using the previous decompositions, we have

$$
\begin{aligned}
\varphi(n-1)\varphi(n+1) = \varphi^2(n)\Bigg(1 &+ \frac{1}{n^2} + \frac{1}{n^2\ln n} + \frac{1}{n^2\ln n\ln_2 n} + \cdots + \frac{1}{n^2\ln n\ln_2 n\cdots\ln_q n}\\
&+ \frac{1}{(n\ln n)^2} + \frac{1}{(n\ln n)^2\ln_2 n} + \cdots + \frac{1}{(n\ln n)^2\ln_2 n\cdots\ln_q n}\\
&+ \frac{1}{(n\ln n\ln_2 n)^2} + \frac{1}{(n\ln n\ln_2 n)^2\ln_3 n}\\
&+ \cdots + \frac{1}{(n\ln n\ln_2 n)^2\ln_3 n\cdots\ln_q n}\\
&+ \cdots + \frac{1}{(n\ln n\cdots\ln_{q-1} n)^2} + \frac{1}{(n\ln n\cdots\ln_{q-1} n)^2\ln_q}\\
&+ \frac{1}{(n\ln n\ln_2 n\cdots\ln_q n)^2} + O\left(\frac{1}{n^3}\right)\Bigg).
\end{aligned}
\tag{2.5}
$$

Recalling the asymptotical decomposition of $\sin x$ when $x \to 0$: $\sin x = x + O(x^3)$, we get (since $\lim_{n\to\infty}\varphi(n) = \lim_{n\to\infty}\varphi(n-1) = \lim_{n\to\infty}\varphi(n+1) = 0$)

$$
\begin{aligned}
\sin\nu\varphi(n-1) &= \nu\varphi(n-1) + O\left(\nu^3\varphi^3(n-1)\right),\\
\sin\nu\varphi(n+1) &= \nu\varphi(n+1) + O\left(\nu^3\varphi^3(n+1)\right),\\
\sin\nu\left[\varphi(n-1)+\varphi(n)\right] &= \nu\left[\varphi(n-1)+\varphi(n)\right] + O\left(\nu^3\left[\varphi(n-1)+\varphi(n)\right]^3\right),\\
\sin\nu\left[\varphi(n)+\varphi(n+1)\right] &= \nu\left[\varphi(n)+\varphi(n+1)\right] + O\left(\nu^3\left[\varphi(n)+\varphi(n+1)\right]^3\right)
\end{aligned}
\tag{2.6}
$$

as $n \to \infty$. Due to (2.3) and (2.4) we have $\varphi(n+1) = O(\varphi(n))$ and $\varphi(n-1) = O(\varphi(n))$ as $n \to \infty$. Then it is easy to see that, for the right-hand side of the inequality (1.5), we have

$$
\mathcal{R} := \frac{\sin\nu\varphi(n-1)\cdot\sin\nu\varphi(n+1)}{\sin\nu\left[\varphi(n-1)+\varphi(n)\right]\cdot\sin\nu\left[\varphi(n)+\varphi(n+1)\right]} = \mathcal{R}_1\cdot\left(1+O\left(\nu^2\varphi^2(n)\right)\right), \quad n\longrightarrow\infty,
\tag{2.7}
$$

where

$$
\mathcal{R}_1 := \frac{\varphi(n-1)\varphi(n+1)}{\varphi^2(n)+\varphi(n)\varphi(n-1)+\varphi(n)\varphi(n+1)+\varphi(n-1)\varphi(n+1)}.
\tag{2.8}
$$

Moreover, for $\mathcal{R}_1$, we will get an asymptotical decomposition as $n \to \infty$. We represent $\mathcal{R}_1$ in the form

$$\mathcal{R}_1 = \frac{\varphi(n-1)\varphi(n+1)/\varphi^2(n)}{1 + (\varphi(n-1)/\varphi(n)) + (\varphi(n+1)/\varphi(n)) + (\varphi(n-1)\varphi(n+1)/\varphi^2(n))}. \tag{2.9}$$

As the asymptotical decompositions for

$$\frac{\varphi(n-1)\varphi(n+1)}{\varphi^2(n)}, \quad \frac{\varphi(n-1)}{\varphi(n)}, \quad \frac{\varphi(n+1)}{\varphi(n)} \tag{2.10}$$

have been derived above (see (2.3)–(2.5)), after some computation, we obtain

$$\begin{aligned}
\mathcal{R}_1 = \Bigg(& 1 + \frac{1}{n^2} + \frac{1}{n^2 \ln n} + \frac{1}{n^2 \ln n \ln_2 n} + \cdots + \frac{1}{n^2 \ln n \ln_2 n \cdots \ln_q n} \\
& + \frac{1}{(n \ln n)^2} + \frac{1}{(n \ln n)^2 \ln_2 n} + \cdots + \frac{1}{(n \ln n)^2 \ln_2 n \cdots \ln_q n} \\
& + \frac{1}{(n \ln n \ln_2 n)^2} + \frac{1}{(n \ln n \ln_2 n)^2 \ln_3 n} + \cdots + \frac{1}{(n \ln n \ln_2 n)^2 \ln_3 n \cdots \ln_q n} \\
& + \cdots + \frac{1}{(n \ln n \cdots \ln_{q-1} n)^2} + \frac{1}{(n \ln n \cdots \ln_{q-1} n)^2 \ln_q} + \frac{1}{(n \ln n \ln_2 n \cdots \ln_q n)^2} + O\left(\frac{1}{n^3}\right)\Bigg) \\
\times \Bigg[1 + \Bigg(& 1 + \frac{1}{n} + \frac{1}{n \ln n} + \frac{1}{n \ln n \ln_2 n} + \frac{1}{n \ln n \ln_2 n \ln_3 n} + \cdots + \frac{1}{n \ln n \ln_2 n \cdots \ln_q n} \\
& + \frac{1}{n^2} + \frac{3}{2n^2 \ln n} + \frac{3}{2n^2 \ln n \ln_2 n} + \cdots + \frac{3}{2n^2 \ln n \ln_2 n \cdots \ln_q n} \\
& + \frac{1}{(n \ln n)^2} + \frac{3}{2(n \ln n)^2 \ln_2 n} + \frac{3}{2(n \ln n)^2 \ln_3 n} + \cdots + \frac{3}{2(n \ln n)^2 \ln_3 n \cdots \ln_q n} \\
& + \frac{1}{(n \ln n \ln_2 n)^2} + \frac{3}{2(n \ln n \ln_2 n)^2 \ln_3 n} + \cdots + \frac{3}{2(n \ln n \ln_2 n)^2 \ln_3 n \cdots \ln_q n} \\
& + \frac{1}{(n \ln n \ln_2 n \ln_3 n)^2} + \frac{3}{2(n \ln n \ln_2 n \ln_3 n)^2 \ln_4 n} \\
& + \cdots + \frac{3}{2(n \ln n \ln_2 n \ln_3 n)^2 \ln_4 n \cdots \ln_q n}
\end{aligned}$$

$$+\cdots+\frac{1}{(n\ln n\ln_2 n\cdots\ln_{q-1}n)^2}+\frac{3}{2(n\ln n\ln_2 n\cdots\ln_{q-1}n)^2\ln_q n}$$

$$+\frac{1}{(n\ln n\ln_2 n\cdots\ln_q n)^2}+O\left(\frac{1}{n^3}\right)\Bigg)$$

$$+\Bigg(1-\frac{1}{n}-\frac{1}{n\ln n}-\frac{1}{n\ln n\ln_2 n}-\cdots-\frac{1}{n\ln n\ln_2 n\cdots\ln_q n}$$

$$+\frac{1}{n^2}+\frac{3}{2n^2\ln n}+\frac{3}{2n^2\ln n\ln_2 n}+\cdots+\frac{3}{2n^2\ln n\ln_2 n\cdots\ln_q n}$$

$$+\frac{1}{(n\ln n)^2}+\frac{3}{2(n\ln n)^2\ln_2 n}+\cdots+\frac{3}{2(n\ln n)^2\ln_2 n\cdots\ln_q n}$$

$$+\frac{1}{(n\ln n\ln_2 n)^2}+\frac{3}{2(n\ln n\ln_2 n)^2\ln_3 n}+\cdots+\frac{3}{2(n\ln n\ln_2 n)^2\ln_3 n\cdots\ln_q n}$$

$$+\cdots+\frac{1}{(n\ln n\ln_2 n\cdots\ln_{q-1}n)^2}+\frac{1}{(n\ln n\ln_2 n\cdots\ln_{q-1}n)^2\ln_q n}$$

$$+\frac{1}{(n\ln n\ln_2 n\cdots\ln_q n)^2}+O\left(\frac{1}{n^3}\right)\Bigg)$$

$$+\Bigg(1+\frac{1}{n^2}+\frac{1}{n^2\ln n}+\frac{1}{n^2\ln n\ln_2 n}+\cdots+\frac{1}{n^2\ln n\ln_2 n\cdots\ln_q n}$$

$$+\frac{1}{(n\ln n)^2}+\frac{1}{(n\ln n)^2\ln_2 n}+\cdots+\frac{1}{(n\ln n)^2\ln_2 n\cdots\ln_q n}$$

$$+\frac{1}{(n\ln n\ln_2 n)^2}+\frac{1}{(n\ln n\ln_2 n)^2\ln_3 n}+\cdots+\frac{1}{(n\ln n\ln_2 n)^2\ln_3 n\cdots\ln_q n}$$

$$+\cdots+\frac{1}{(n\ln n\cdots\ln_{q-1}n)^2}$$

$$+\frac{1}{(n\ln n\cdots\ln_{q-1}n)^2\ln_q}+\frac{1}{(n\ln n\ln_2 n\cdots\ln_q n)^2}+O\left(\frac{1}{n^3}\right)\Bigg)\Bigg]^{-1}$$

$$=\Bigg(1+\frac{1}{n^2}+\frac{1}{n^2\ln n}+\frac{1}{n^2\ln n\ln_2 n}+\cdots+\frac{1}{n^2\ln n\ln_2 n\cdots\ln_q n}$$

$$+\frac{1}{(n\ln n)^2}+\frac{1}{(n\ln n)^2\ln_2 n}+\cdots+\frac{1}{(n\ln n)^2\ln_2 n\cdots\ln_q n}$$

$$+\frac{1}{(n\ln n\ln_2 n)^2}+\frac{1}{(n\ln n\ln_2 n)^2\ln_3 n}+\cdots+\frac{1}{(n\ln n\ln_2 n)^2\ln_3 n\cdots\ln_q n}$$

$$
\begin{aligned}
&+\cdots+\frac{1}{(n\ln n\cdots\ln_{q-1}n)^2}+\frac{1}{(n\ln n\cdots\ln_{q-1}n)^2\ln_q}\\
&+\frac{1}{(n\ln n\ln_2 n\cdots\ln_q n)^2}+O\left(\frac{1}{n^3}\right)\Bigg)\\
&\times\Bigg[4+\frac{3}{n^2}+\frac{4}{n^2\ln n}+\frac{4}{n^2\ln n\ln_2 n}+\frac{4}{n^2\ln n\ln_2 n\ln_3 n}+\cdots+\frac{4}{n^2\ln n\ln_2 n\cdots\ln_q n}\\
&+\frac{3}{(n\ln n)^2}+\frac{4}{(n\ln n)^2\ln_2 n}+\frac{4}{(n\ln n)^2\ln_2 n\ln_3 n}+\cdots+\frac{4}{(n\ln n)^2\ln_2 n\ln_3 n\cdots\ln_q n}\\
&+\frac{3}{(n\ln n\ln_2 n)^2}+\frac{4}{(n\ln n\ln_2 n)^2\ln_3 n}+\cdots+\frac{4}{(n\ln n\ln_2 n)^2\ln_3 n\cdots\ln_q n}\\
&+\cdots+\frac{3}{(n\ln n\ln_2 n\cdots\ln_{q-1}n)^2}+\frac{4}{(n\ln n\ln_2 n\cdots\ln_{q-1}n)^2\ln_q}\\
&+\frac{3}{(n\ln n\ln_2 n\ln_3 n\cdots\ln_q n)^2}+O\left(\frac{1}{n^3}\right)\Bigg]^{-1}\\
=\ &\frac{1}{4}\Bigg(1+\frac{1}{n^2}+\frac{1}{n^2\ln n}+\frac{1}{n^2\ln n\ln_2 n}+\cdots+\frac{1}{n^2\ln n\ln_2 n\cdots\ln_q n}\\
&+\frac{1}{(n\ln n)^2}+\frac{1}{(n\ln n)^2\ln_2 n}+\cdots+\frac{1}{(n\ln n)^2\ln_2 n\cdots\ln_q n}\\
&+\frac{1}{(n\ln n\ln_2 n)^2}+\frac{1}{(n\ln n\ln_2 n)^2\ln_3 n}+\cdots+\frac{1}{(n\ln n\ln_2 n)^2\ln_3 n\cdots\ln_q n}\\
&+\cdots+\frac{1}{(n\ln n\cdots\ln_{q-1}n)^2}+\frac{1}{(n\ln n\cdots\ln_{q-1}n)^2\ln_q}\\
&+\frac{1}{(n\ln n\ln_2 n\cdots\ln_q n)^2}+O\left(\frac{1}{n^3}\right)\Bigg)\\
&\times\Bigg[1+\frac{3}{4n^2}+\frac{1}{n^2\ln n}+\frac{1}{n^2\ln n\ln_2 n}+\frac{1}{n^2\ln n\ln_2 n\ln_3 n}+\cdots+\frac{1}{n^2\ln n\ln_2 n\cdots\ln_q n}\\
&+\frac{3}{4(n\ln n)^2}+\frac{1}{(n\ln n)^2\ln_2 n}+\frac{1}{(n\ln n)^2\ln_2 n\ln_3 n}+\cdots+\frac{1}{(n\ln n)^2\ln_2 n\ln_3 n\cdots\ln_q n}\\
&+\frac{3}{4(n\ln n\ln_2 n)^2}+\frac{1}{(n\ln n\ln_2 n)^2\ln_3 n}+\cdots+\frac{1}{(n\ln n\ln_2 n)^2\ln_3 n\cdots\ln_q n}\\
&+\cdots+\frac{3}{4(n\ln n\ln_2 n\cdots\ln_{q-1}n)^2}+\frac{1}{(n\ln n\ln_2 n\cdots\ln_{q-1}n)^2\ln_q}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{3}{4(n\ln n\ln_2 n\ln_3 n\cdots\ln_q n)^2}+O\left(\frac{1}{n^3}\right)\Bigg]^{-1}\\
={}&\frac{1}{4}\Bigg(1+\frac{1}{n^2}+\frac{1}{n^2\ln n}+\frac{1}{n^2\ln n\ln_2 n}+\cdots+\frac{1}{n^2\ln n\ln_2 n\cdots\ln_q n}\\
&+\frac{1}{(n\ln n)^2}+\frac{1}{(n\ln n)^2\ln_2 n}+\cdots+\frac{1}{(n\ln n)^2\ln_2 n\cdots\ln_q n}\\
&+\frac{1}{(n\ln n\ln_2 n)^2}+\frac{1}{(n\ln n\ln_2 n)^2\ln_3 n}+\cdots+\frac{1}{(n\ln n\ln_2 n)^2\ln_3 n\cdots\ln_q n}\\
&+\cdots+\frac{1}{(n\ln n\cdots\ln_{q-1} n)^2}\\
&+\frac{1}{(n\ln n\cdots\ln_{q-1} n)^2\ln_q}+\frac{1}{(n\ln n\ln_2 n\cdots\ln_q n)^2}+O\left(\frac{1}{n^3}\right)\Bigg)\\
&\times\Bigg[1-\frac{3}{4n^2}-\frac{1}{n^2\ln n}-\frac{1}{n^2\ln n\ln_2 n}-\frac{1}{n^2\ln n\ln_2 n\ln_3 n}-\cdots-\frac{1}{n^2\ln n\ln_2 n\cdots\ln_q n}\\
&-\frac{3}{4(n\ln n)^2}-\frac{1}{(n\ln n)^2\ln_2 n}-\frac{1}{(n\ln n)^2\ln_2 n\ln_3 n}-\cdots-\frac{1}{(n\ln n)^2\ln_2 n\ln_3 n\cdots\ln_q n}\\
&-\frac{3}{4(n\ln n\ln_2 n)^2}-\frac{1}{(n\ln n\ln_2 n)^2\ln_3 n}-\cdots-\frac{1}{(n\ln n\ln_2 n)^2\ln_3 n\cdots\ln_q n}\\
&-\cdots-\frac{3}{4(n\ln n\ln_2 n\cdots\ln_{q-1} n)^2}-\frac{1}{(n\ln n\ln_2 n\cdots\ln_{q-1} n)^2\ln_q}\\
&-\frac{3}{4(n\ln n\ln_2 n\ln_3 n\cdots\ln_q n)^2}+O\left(\frac{1}{n^3}\right)\Bigg]\\
={}&\frac{1}{4}\Bigg(1+\frac{1}{4n^2}+\frac{1}{4(n\ln n)^2}+\frac{1}{4(n\ln n\ln_2 n)^2}+\frac{1}{4(n\ln n\ln_2 n\ln_3 n)^2}\\
&+\cdots+\frac{1}{4(n\ln n\ln_2 n\ln_3 n\cdots\ln_q n)^2}+O\left(\frac{1}{n^3}\right)\Bigg).
\end{aligned}
\tag{2.11}
$$

Thus we have

$$
\begin{aligned}
\mathcal{R}_1 ={}& \frac{1}{4}+\frac{1}{16n^2}+\frac{1}{16(n\ln n)^2}+\frac{1}{16(n\ln n\ln_2 n)^2}+\frac{1}{16(n\ln n\ln_2 n\ln_3 n)^2}\\
&+\cdots+\frac{1}{16(n\ln n\ln_2 n\ln_3 n\cdots\ln_q n)^2}+O\left(\frac{1}{n^3}\right).
\end{aligned}
\tag{2.12}
$$

Finalizing our decompositions, we see that

$$
\begin{aligned}
\mathcal{R} &= \mathcal{R}_1 \cdot \left(1 + O\left(\nu^2\varphi^2(n)\right)\right) \\
&= \left(\frac{1}{4} + \frac{1}{16n^2} + \frac{1}{16(n\ln n)^2} + \frac{1}{16(n\ln n\ln_2 n)^2} + \frac{1}{16(n\ln n\ln_2 n\ln_3 n)^2}\right. \\
&\qquad \left. + \cdots + \frac{1}{16\left(n\ln n\ln_2 n\ln_3 n\cdots\ln_q n\right)^2} + O\left(\frac{1}{n^3}\right)\right)\left(1 + O\left(\nu^2\varphi^2(n)\right)\right) \\
&= \frac{1}{4} + \frac{1}{16n^2} + \frac{1}{16(n\ln n)^2} + \frac{1}{16(n\ln n\ln_2 n)^2} + \frac{1}{16(n\ln n\ln_2 n\ln_3 n)^2} \\
&\qquad + \cdots + \frac{1}{16\left(n\ln n\ln_2 n\ln_3 n\cdots\ln_q n\right)^2} + O\left(\frac{\nu^2}{\left(n\ln n\ln_2 n\ln_3 n\cdots\ln_q n\right)^2}\right).
\end{aligned}
\tag{2.13}
$$

It is easy to see that inequality (1.5) becomes

$$
\begin{aligned}
p(n) \geq\ & \frac{1}{4} + \frac{1}{16n^2} + \frac{1}{16(n\ln n)^2} + \frac{1}{16(n\ln n\ln_2 n)^2} + \frac{1}{16(n\ln n\ln_2 n\ln_3 n)^2} \\
&+ \cdots + \frac{1}{16\left(n\ln n\ln_2 n\ln_3 n\cdots\ln_q n\right)^2} + O\left(\frac{\nu^2}{\left(n\ln n\ln_2 n\ln_3 n\cdots\ln_q n\right)^2}\right)
\end{aligned}
\tag{2.14}
$$

and will be valid if (see (1.3))

$$
\begin{aligned}
&\frac{1}{4} + \frac{1}{16n^2} + \frac{1}{16(n\ln n)^2} + \frac{1}{16(n\ln n\ln_2 n)^2} \\
&\quad + \cdots + \frac{1}{16\left(n\ln n\ln_2 n\ln_3 n\cdots\ln_{q-1} n\right)^2} + \frac{\kappa}{16\left(n\ln n\ln_2 n\ln_3 n\cdots\ln_q n\right)^2} \\
&\geq \frac{1}{4} + \frac{1}{16n^2} + \frac{1}{16(n\ln n)^2} + \frac{1}{16(n\ln n\ln_2 n)^2} + \cdots + \frac{1}{16\left(n\ln n\ln_2 n\ln_3 n\cdots\ln_{q-1} n\right)^2} \\
&\quad + \cdots + \frac{1}{16\left(n\ln n\ln_2 n\ln_3 n\cdots\ln_q n\right)^2} + O\left(\frac{\nu^2}{\left(n\ln n\ln_2 n\ln_3 n\cdots\ln_q n\right)^2}\right)
\end{aligned}
\tag{2.15}
$$

or

$$
\kappa \geq 1 + O\left(\nu^2\right) \tag{2.16}
$$

for $n \to \infty$. If $n \geq n_0$ where n_0 is sufficiently large, then (2.16) holds for sufficiently small $\nu \in (0, \nu_0]$ with ν_0 fixed because $\kappa > 1$. Consequently, (2.14) is satisfied and the assumption (1.5) of Lemma 1.3 holds for $n \in \mathbb{Z}_{n_0}^{\infty}$. Let $q \geq n_0$ in Lemma 1.3 be fixed and let $r > q + 1$

be so large that inequalities (1.4) hold. This is always possible since the series $\sum_{n=q+1}^{\infty}\varphi(n)$ is divergent. Then Lemma 1.3 holds and any solution of (1.1) has at least one change of sign on $\mathbb{Z}_{q-1}^{r+1}$. Obviously, inequalities (1.4) can be satisfied for another couple of (p, r), say (p_1, r_1) with $p_1 > r$ and $r_1 > q_1 + 1$ sufficiently large, and by Lemma 1.3 any solution of (1.1) has at least one change of sign on $\mathbb{Z}_{q_1-1}^{r_1+1}$. Continuing this process, we get a sequence of intervals (p_n, r_n) with $\lim_{n\to\infty} p_n = \infty$ such that any solution of (1.1) has at least one change of sign on $\mathbb{Z}_{q_n-1}^{r_n+1}$. This fact concludes the proof. □

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References

[1] R. P. Agarwal and A. Zafer, "Oscillation criteria for second-order forced dynamic equations with mixed nonlinearities," *Advances in Difference Equations*, vol. 2009, Article ID 938706, 20 pages, 2009.

[2] J. Baštinec, J. Diblík, and Z. Šmarda, "Existence of positive solutions of discrete linear equations with a single delay," *Journal of Difference Equations and Applications*, vol. 16, no. 5, pp. 1165–1177, 2010.

[3] L. Berezansky and E. Braverman, "On existence of positive solutions for linear difference equations with several delays," *Advances in Dynamical Systems and Applications*, vol. 1, no. 1, pp. 29–47, 2006.

[4] L. Berezansky and E. Braverman, "Oscillation of a logistic difference equation with several delays," *Advances in Difference Equations*, vol. 2006, Article ID 82143, 12 pages, 2006.

[5] M. Bohner, B. Karpuz, and Ö. Öcalan, "Iterated oscillation criteria for delay dynamic equations of first order," *Advances in Difference Equations*, vol. 2008, Article ID 458687, 12 pages, 2008.

[6] G. E. Chatzarakis, R. Koplatadze, and I. P. Stavroulakis, "Oscillation criteria of first order linear difference equations with delay argument," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 68, no. 4, pp. 994–1005, 2008.

[7] J. Diblík and N. Koksch, "Positive solutions of the equation $\dot{x}(t) = -c(t)x(t-\tau)$ in the critical case," *Journal of Mathematical Analysis and Applications*, vol. 250, no. 2, pp. 635–659, 2000.

[8] Y. Domshlak, "Oscillation properties of discrete difference inequalities and equations: the new approach," in *Functional-Differential Equations*, vol. 1 of *Functional Differential Equations Israel Sem.*, pp. 60–82, Coll. Judea Samaria, Ariel, Israel, 1993.

[9] Y. Domshlak and I. P. Stavroulakis, "Oscillations of first-order delay differential equations in a critical state," *Applicable Analysis*, vol. 61, no. 3-4, pp. 359–371, 1996.

[10] B. Dorociaková and R. Olach, "Existence of positive solutions of delay differential equations," *Tatra Mountains Mathematical Publications*, vol. 43, pp. 63–70, 2009.

[11] I. Györi and G. Ladas, *Oscillation Theory of Delay Differential Equations*, Oxford Mathematical Monographs, The Clarendon Press, New York, NY, USA, 1991.

[12] L. Hanuštiaková and R. Olach, "Nonoscillatory bounded solutions of neutral differential systems," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 68, no. 7, pp. 1816–1824, 2008.

[13] L. K. Kikina and I. P. Stavroulakis, "A survey on the oscillation of solutions of first order delay difference equations," *Cubo*, vol. 7, no. 2, pp. 223–236, 2005.

[14] R. Medina and M. Pituk, "Nonoscillatory solutions of a second-order difference equation of Poincaré type," *Applied Mathematics Letters*, vol. 22, no. 5, pp. 679–683, 2009.

[15] I. P. Stavroulakis, "Oscillation criteria for first order delay difference equations," *Mediterranean Journal of Mathematics*, vol. 1, no. 2, pp. 231–240, 2004.