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Research Article

A New Approach to q-Bernoulli Numbers and q-Bernoulli Polynomials Related to q-Bernstein Polynomials

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We present a new generating function related to the q-Bernoulli numbers and q-Bernoulli polynomials. We give a new construction of these numbers and polynomials related to the second-kind Stirling numbers and q-Bernstein polynomials. We also consider the generalized q-Bernoulli polynomials attached to Dirichlet's character χ and have their generating function. We obtain distribution relations for the q-Bernoulli polynomials and have some identities involving q-Bernoulli numbers and polynomials related to the second kind Stirling numbers and q-Bernstein polynomials. Finally, we derive the q-extensions of zeta functions from the Mellin transformation of this generating function which interpolates the q-Bernoulli polynomials at negative integers and is associated with q-Bernstein polynomials.

1. Introduction, Definitions, and Notations

Let $\mathbb{C}$ be the complex number field. We assume that $q \in \mathbb{C}$ with $|q| < 1$ and that the q-number is defined by $[x]_q = (q^x - 1)/(q - 1)$ in this paper.

Many mathematicians have studied q-Bernoulli, q-Euler polynomials, and related topics (see [1–23]). It is known that the Bernoulli polynomials are defined by

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad \text{for } |t| < 2\pi, \tag{1.1}$$

and that $B_n = B_n(0)$ are called the nth Bernoulli numbers.

The recurrence formula for the classical Bernoulli numbers B_n is as follows,

$$B_0 = 1, \quad (B+1)^n - B_n = 0, \quad \text{if } n > 0 \tag{1.2}$$

(see [1, 3, 23]). The q-extension of the following recurrence formula for the Bernoulli numbers is

$$B_{0,q} = 1, \qquad q(qB+1)^n - B_{n,q} = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases} \tag{1.3}$$

with the usual convention of replacing B^n by $B_{n,q}$ (see [5, 7, 14]).

Now, by introducing the following well-known identities

$$[x+y]_q = [x]_q + q^x [y]_q, \qquad [-x]_q = -\frac{1}{q^x}[x]_q, \qquad [xy]_q = [x]_q [y]_{q^x} \tag{1.4}$$

(see [6]).

The generating functions of the second kind Stirling numbers and q-Bernstein polynomials, respectively, can be defined as follows,

$$\frac{(e^t-1)^k}{k!} = \sum_{n=0}^{\infty} S(n,k)\frac{t^n}{n!}, \tag{1.5}$$

$$F_k(x,t;q) = \frac{\left(t[x]_q\right)^k}{k!} e^{t[1-x]_q} = \sum_{n=0}^{\infty} B_{k,n}(x;q)\frac{t^n}{n!}, \quad t \in \mathbb{C},\ k = 0,1,\ldots,n \tag{1.6}$$

(see [2]), where $\lim_{q\to 1} F_k(x,t;q) = F_k(t,x) = ((tx)^k/k!)e^{t(1-x)}$ (see [4]).

Throughout this paper, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will respectively denote the ring of rational integers, the field of rational numbers, the ring p-adic rational integers, the field of p-adic rational numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ such that $|p|_p = p^{-v_p(p)} = 1/p$. If $q \in \mathbb{C}_p$, we normally assume $|q-1|_p < p^{-1/(p-1)}$ or $|1-q|_p < 1$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$ (see [7–19]).

In this study, we present a new generating function related to the q-Bernoulli numbers and q-Bernoulli polynomials and give a new construction of these numbers and polynomials related to the second kind Stirling numbers and q-Bernstein polynomials. We also consider the generalized q-Bernoulli polynomials attached to Dirichlet's character χ and have their generating function. We obtain distribution relations for the q-Bernoulli polynomials and have some identities involving q-Bernoulli numbers and polynomials related to the second kind Stirling numbers and q-Bernstein polynomials. Finally, we derive the q-extensions of zeta functions from the Mellin transformation of this generating function which interpolates the q-Bernoulli polynomials at negative integers and are associated with q-Bernstein polynomials.

2. New Approach to q-Bernoulli Numbers and Polynomials

Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}^* = \mathbb{N} \cup \{0\}$. For $q \in \mathbb{C}$ with $|q| < 1$, let us define the q-Bernoulli polynomials $B_{n,q}(x)$ as follows,

$$D_q(t,x) = -t\sum_{y=0}^{\infty} q^y e^{[x+y]t} = \sum_{n=0}^{\infty} B_{n,q}(x)\frac{t^n}{n!}. \tag{2.1}$$

Note that

$$\lim_{q\to 1} D_q(t,x) = \frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \quad |t| < 2\pi, \tag{2.2}$$

where $B_n(x)$ are classical Bernoulli polynomials. In the special case $x = 0$, $B_{n,q} = B_{n,q}(0)$ are called the nth q-Bernoulli numbers. That is,

$$D_q(t) = D_q(t,0) = -t\sum_{y=0}^{\infty} q^y e^{[y]t} = \sum_{n=0}^{\infty} B_{n,q}\frac{t^n}{n!}. \tag{2.3}$$

From (2.1) and (2.3), we note that

$$\begin{aligned} qD_q(t,1) - D_q(t) &= qe^t D_q(qt) - D_q(t) \\ &= q\left(\sum_{l=0}^{\infty}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty} q^m B_{m,q}\frac{t^m}{m!}\right) - \sum_{n=0}^{\infty} B_{n,q}\frac{t^n}{n!} \\ &= q\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l} q^l B_{l,q}\right)\frac{t^n}{n!} - \sum_{n=0}^{\infty} B_{n,q}\frac{t^n}{n!}. \end{aligned} \tag{2.4}$$

From (2.1) and (2.3), we can easily derive the following equation:

$$qD_q(t,1) - D_q(t) = 1. \tag{2.5}$$

Equations (2.4) and (2.5), we see that $B_{0,q} = 1$ and

$$\sum_{l=0}^{n}\binom{n}{l} q^{l+1} B_{l,q} - B_{n,q} = \begin{cases} 1, & \text{if } n = 0 \\ 0, & \text{if } n > 0. \end{cases} \tag{2.6}$$

Therefore, we obtain the following theorem.

Theorem 2.1. *For $n \in \mathbb{N}^*$, one has*

$$B_{0,q} = 1, \qquad q(qB+1)^n - B_{n,q} = \begin{cases} 1, & \text{if } n = 0 \\ 0, & \text{if } n > 0. \end{cases} \tag{2.7}$$

with the usual convention of replacing B^i and $B_{i,q}$.

From (2.1), *one notes that*

$$\begin{aligned} D_q(t,x) &= e^{[x]_q t} D_q(q^x t) \\ &= \left(\sum_{n=0}^{\infty} [x]_q^n \frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} q^{nx} B_{n,q} \frac{t^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l} q^{lx} B_{l,q} [x]_q^{n-l}\right)\frac{t^n}{n!}. \end{aligned} \tag{2.8}$$

Therefore, one obtains the following theorem.

Theorem 2.2. *For $n \in \mathbb{N}^*$, one has*

$$B_{n,q}(x) = \sum_{l=0}^{n}\binom{n}{l} q^{lx} B_{l,q} [x]_q^{n-l}. \tag{2.9}$$

By (2.1), one sees that

$$\begin{aligned} D_q(t,x) &= \sum_{n=0}^{\infty}\left(-t\sum_{m=0}^{\infty} q^m [x+m]_q^n\right)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{lx}\frac{l+1}{[l+1]_q}\right)\frac{t^n}{n!}. \end{aligned} \tag{2.10}$$

By (2.1) and (2.10), one obtains the following theorem.

Theorem 2.3. *For $n \in \mathbb{N}^*$, one has*

$$B_{n,q}(x) = \frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{lx}\frac{l+1}{[l+1]_q}. \tag{2.11}$$

From (2.11) one can derive that, for $s \in \mathbb{N}$,

$$D_q(t,x) = \sum_{a=0}^{s-1} q^a D_{q^s}\left(t[s]_q, \frac{x+a}{s}\right). \tag{2.12}$$

By (2.12), one sees that, for $s \in \mathbb{N}$,

$$\sum_{n=0}^{\infty} B_{n,q}(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left([s]_q^n \sum_{a=0}^{s-1} q^a B_{n,q^s}\left(\frac{x+a}{s}\right)\right)\frac{t^n}{n!}. \tag{2.13}$$

Therefore, one obtains the following theorem.

Theorem 2.4. *For $s \in \mathbb{N}^*$, one has*

$$B_{n,q}(x) = [s]_q^n \sum_{a=0}^{s-1} q^a B_{n,q^s}\left(\frac{x+a}{s}\right). \tag{2.14}$$

In (2.9), substitute $1-x$ instead of x, one obtains

$$\begin{aligned} B_{n,q}(1-x) &= \sum_{v=0}^{n}\binom{n}{v} B_{v,q} q^{v(1-x)} [1-x]_q^{n-v} \\ &= \sum_{v=0}^{n}\binom{n}{v} [x]_q^v [1-x]_q^{n-v} B_{v,q} \cdot q^{v(1-x)} [\mathrm{x}]_q^{-v} \\ &= \sum_{m=0}^{\infty}\sum_{v=0}^{n} B_{v,n}(x;q)\binom{v+m-1}{m} q^v (1-q)^m [x]_q^{m-v} B_{v,q}, \end{aligned} \tag{2.15}$$

which is the relation between q-Bernoulli polynomials, q-Bernoulli numbers, and q-Bernstein polynomials. In (1.5), substitute $(x \log q)$ instead of t, one gets

$$[x]_q^k = \frac{k!}{(q-1)^k}\sum_{y=0}^{\infty}\frac{S(y,k)(x\log q)^y}{y!}. \tag{2.16}$$

In (2.16), substitute $m-v$ instead of k, and putting the result in (2.15), one has the following theorem.

Theorem 2.5. *For $n \in \mathbb{N}^*$ and $|q| < 1$, one has*

$$\begin{aligned} B_{n,q}(x) = \sum_{m,y=0}^{\infty}\sum_{v=0}^{n}\sum_{j=0}^{v}\binom{v+m-1}{m}\binom{v}{j}\frac{(-1)^{m-v+j}(m-v)!q^{v+j}}{y!} \\ \times S(y,m-v) B_{n-v,n}(x;q) B_{v,q}(x\log q)^y, \end{aligned} \tag{2.17}$$

where $S(k,n)$and $B_{k,n}(x;q)$ are the second kind Stirling numbers and q-Bernstein polynomials, respectively.

Let χ be Dirichlet's character with $s \in \mathbb{N}$. Then, one defines the generalized q-Bernoulli polynomials attached to χ as follows,

$$D_{q,\chi}(t,x) = -t\sum_{d=0}^{\infty}\chi(d)q^{d}e^{[d+x]_{q}t} = \sum_{n=0}^{\infty}B_{n,\chi,q}(x)\frac{t^{n}}{n!}. \tag{2.18}$$

In the special case $x = 0$, $B_{\mathrm{n},\chi,q} = B_{n,\chi,q}(0)$ are called the nth generalized q-Bernoulli numbers attached to χ. Thus, the generating function of the generalized q-Bernoulli numbers attached to χ are as follows,

$$\begin{aligned} D_{q,\chi}(t,x) &= -t\sum_{d=0}^{\infty}\chi(d)q^{d}e^{[d]_{q}t} \\ &= \sum_{n=0}^{\infty}B_{n,\chi,q}\frac{t^{n}}{n!}. \end{aligned} \tag{2.19}$$

By (2.1) and (2.18), one sees that

$$\begin{aligned} D_{q,\chi}(t,x) &= \sum_{a=0}^{s-1}q^{a}\chi(a)D_{q^{s}}\left(t[s]_{q},\frac{x+a}{s}\right) \\ &= \sum_{n=0}^{\infty}\left([s]_{q}^{n}\sum_{a=0}^{s-1}q^{a}\chi(a)B_{n,q^{s}}\left(\frac{x+a}{s}\right)\right)\frac{t^{n}}{n!}. \end{aligned} \tag{2.20}$$

Therefore, one obtains the following theorem.

Theorem 2.6. *For $n \in \mathbb{N}^{*}$ and $s \in \mathbb{N}$, one has*

$$B_{n,\chi,q}(x) = [s]_{q}^{n}\sum_{a=0}^{s-1}q^{a}\chi(a)B_{n,q^{s}}\left(\frac{x+a}{s}\right). \tag{2.21}$$

By (2.18) and (2.19), one sees that

$$D_{q,\chi}(t,x) = e^{[x]_{q}t}D_{q,\chi}(q^{x}t) = \sum_{n=0}^{\infty}\left(\sum_{d=0}^{n}\binom{n}{d}q^{dx}[x]_{q}^{n-d}B_{d,\chi,q}\right)\frac{t^{n}}{n!}. \tag{2.22}$$

Hence,

$$B_{n,\chi,q}(x) = \sum_{d=0}^{n}\binom{n}{d}q^{dx}[x]_{q}^{n-d}B_{d,\chi,q}. \tag{2.23}$$

For $s \in \mathbb{C}$, one now considers the Mellin transformation for the generating function of $D_q(t, x)$. That is,

$$\frac{1}{\Gamma(s)} \int_0^\infty D_q(-t, x) t^{s-2} dt = \sum_{n=0}^{\infty} \frac{q^n}{[x+n]_q^s}, \tag{2.24}$$

for $s \in \mathbb{C}$, and $x \neq 0, -1, -2, \ldots$.

From (2.24), one defines the zeta type function as follows,

$$\zeta_q^\star(s, x) = \sum_{n=0}^{\infty} \frac{q^n}{[x+n]_q^s}, \quad s \in \mathbb{C},\ x \neq 0, -1, -2, \ldots. \tag{2.25}$$

Note that $\zeta_q^\star(s, x)$ is an analytic function in the whole complex s-plane. Using the Laurent series and the Cauchy residue theorem, one has

$$-n\zeta_q^\star(1-n, x) = B_{n,q}(x), \quad \text{for } n \in \mathbb{N}^*. \tag{2.26}$$

By the same method, one can also obtain the following equations:

$$\frac{1}{\Gamma(s)} \int_0^\infty D_{q,\chi}(-t, x) t^{s-2} dt = \sum_{n=0}^{\infty} \frac{\chi(n) q^n}{[n+x]_q^s}. \tag{2.27}$$

For $s \in \mathbb{C}$,one defines Dirichlet type q-l-function as

$$l_q(s, x \mid \chi) = \sum_{n=0}^{\infty} \frac{\chi(n) q^n}{[n+x]_q^s}, \tag{2.28}$$

where $x \neq 0, -1, -2, \ldots$. Note that $l_q(s, x \mid \chi)$ is also a holomorphic function in the whole complex s-plane. From the Laurent series and the Cauchy residue theorem, one can also derive the following equation:

$$-nl_q(1-n, x \mid \chi) = B_{n,\chi,q}(x). \tag{2.29}$$

In (2.23), substitute $1-x$ instead of x, one obtains

$$\begin{aligned}
B_{n,\chi,q}(1-x) &= \sum_{v=0}^{n} \binom{n}{v} B_{v,\chi,q} q^{v(1-x)} [1-x]_q^{n-v} \\
&= \sum_{v=0}^{n} \binom{n}{v} [x]_q^v [1-x]_q^{n-v} B_{v,\chi,q} \cdot q^{v(1-x)} [x]_q^{-v} \\
&= \sum_{m=0}^{\infty} \sum_{v=0}^{n} B_{v,n}(x;q) \binom{v+m-1}{m} q^v (1-q)^m [x]_q^{m-v} B_{v,\chi,q},
\end{aligned} \tag{2.30}$$

which is the relation between the nth generalized q-Bernoulli numbers and q-Bernoulli polynomials attached to χ and q-Bernstein polynomials. From (2.16), one has the following theorem.

Theorem 2.7. *For $n \in \mathbb{N}^*$ and $|q| < 1$, one has*

$$B_{n,\chi,q}(x) = \sum_{m,y=0}^{\infty}\sum_{v=0}^{n}\sum_{j=0}^{v}\binom{v+m-1}{m}\binom{v}{j}\frac{(-1)^{m-v+j}(m-v)!q^{v+j}}{y!} \times S(y,m-v)B_{n-v,n}(x;q)B_{v,\chi,q}(x\log q)^{y}. \tag{2.31}$$

One now defines particular q-zeta function as follows,

$$H_q(s,a \mid F) = \sum_{m\equiv a(\bmod F)}\frac{q^m}{[m]_q^s}. \tag{2.32}$$

From (2.32), one has

$$H_q(s,a \mid F) = \frac{q^a}{[F]_q^s}\zeta^*_{q^F}\left(s,\frac{a}{F}\right), \tag{2.33}$$

where $\zeta^*_{q^F}(s,a/F)$ is given by (2.25). By (2.26), one has

$$H_q(1-n,a \mid F) = -\frac{q^a[F]_q^{n-1}B_{n,q^F}(a/F)}{n}, \quad n \in \mathbb{N}. \tag{2.34}$$

Therefore, one obtains the following theorem.

Theorem 2.8. *For $n \in \mathbb{N}$, we have*

$$B_{n,q^F}\left(\frac{a}{F}\right) = -\frac{nH_q(1-n,a \mid F)}{q^a[F]_q^{n-1}}. \tag{2.35}$$

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