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*Research Article*

# Global Uniqueness Results for Fractional Order Partial Hyperbolic Functional Differential Equations

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We investigate the global existence and uniqueness of solutions for some classes of partial hyperbolic differential equations involving the Caputo fractional derivative with finite and infinite delays. The existence results are obtained by applying some suitable fixed point theorems.

## 1. Introduction

In this paper, we provide sufficient conditions for the global existence and uniqueness of some classes of fractional order partial hyperbolic differential equations. As a first problem, we discuss the global existence and uniqueness of solutions for an initial value problem (IVP for short) of a system of fractional order partial differential equations given by

$$({}^cD_0^r u)(x,y) = f(x,y,u_{(x,y)}); \quad \text{if } (x,y) \in J, \tag{1.1}$$

$$u(x,y) = \phi(x,y); \quad \text{if } (x,y) \in \tilde{J}, \tag{1.2}$$

$$u(x,0) = \varphi(x), \quad u(0,y) = \psi(y); \quad x,y \in [0,\infty), \tag{1.3}$$

where $J = [0,\infty) \times [0,\infty)$, $\tilde{J} := [-\alpha,\infty) \times [-\beta,\infty) \setminus (0,\infty) \times (0,\infty)$; $\alpha,\beta > 0$, $\phi \in C(\tilde{J},\mathbb{R}^n)$, ${}^cD_0^r$ is the Caputo's fractional derivative of order $r = (r_1,r_2) \in (0,1] \times (0,1]$, $f : J \times \mathcal{C} \to \mathbb{R}^n$, is a given function $\varphi : [0,\infty) \to \mathbb{R}^n$, $\psi : [0,\infty) \to \mathbb{R}^n$ are given absolutely continuous functions with $\varphi(x) = \phi(x,0)$, $\psi(y) = \phi(0,y)$ for each $x, y \in [0,\infty)$, and $\mathcal{C} := C([-\alpha,0] \times [-\beta,0], \mathbb{R}^n)$ is the space of continuous functions on $[-\alpha,0] \times [-\beta,0]$.

If $u \in C([-\alpha,\infty) \times [-\beta,\infty), \mathbb{R}^n)$, then for any $(x,y) \in J$ define $u_{(x,y)}$ by

$$u_{(x,y)}(s,t) = u(x+s, y+t), \quad \text{for } (s,t) \in [-\alpha,0] \times [-\beta,0]. \tag{1.4}$$

Next we consider the following initial value problem for partial neutral functional differential equations with finite delay of the form

$$^cD_0^r\left(u(x,y) - g\left(x,y,u_{(x,y)}\right)\right) = f\left(x,y,u_{(x,y)}\right); \quad \text{if}(x,y) \in J, \tag{1.5}$$

$$u(x,y) = \phi(x,y); \quad \text{if}(x,y) \in \tilde{J}, \tag{1.6}$$

$$u(x,0) = \varphi(x), \quad u(0,y) = \psi(y); \quad x,y \in [0,\infty), \tag{1.7}$$

where $f, \phi, \varphi, \psi$ are as in problem (1.1)–(1.3), and $g : J \times \mathcal{C} \to \mathbb{R}^n$ is a given function.

The third result deals with the existence of solutions to fractional order partial hyperbolic functional differential equations with infinite delay of the form

$$\left(^cD_0^r u\right)(x,y) = f\left(x,y,u_{(x,y)}\right); \quad \text{if } (x,y) \in J, \tag{1.8}$$

$$u(x,y) = \phi(x,y); \quad \text{if } (x,y) \in \tilde{J}', \tag{1.9}$$

$$u(x,0) = \varphi(x), \quad u(0,y) = \psi(y); \quad x,y \in [0,\infty), \tag{1.10}$$

where $\varphi$, $\psi$ are as in problem (1.1)–(1.3) and $\tilde{J}' = \mathbb{R}^2 \setminus (0,\infty) \times (0,\infty)$, $f : J \times B \to \mathbb{R}^n$, $\phi \in C(\tilde{J}', \mathbb{R}^n)$, and $B$ is called a phase space that will be specified in Section 4.

We denote by $u_{(x,y)}$ the element of $B$ defined by

$$u_{(x,y)}(s,t) = u(x+s, y+t); \quad (s,t) \in (-\infty,0] \times (-\infty,0]. \tag{1.11}$$

Finally we consider the following initial value problem for partial neutral functional differential equations with infinite delay

$$^cD_0^r\left(u(x,y) - g\left(x,y,u_{(x,y)}\right)\right) = f\left(x,y,u_{(x,y)}\right); \quad \text{if } (x,y) \in J, \tag{1.12}$$

$$u(x,y) = \phi(x,y); \quad \text{if } (x,y) \in \tilde{J}', \tag{1.13}$$

$$u(x,0) = \varphi(x), \quad u(0,y) = \psi(y); \quad x,y \in [0,\infty), \tag{1.14}$$

where $f$, $\phi$, $\varphi$, $\psi$ are as in problem (1.8)–(1.10) and $g : J \times B \to \mathbb{R}^n$ is a given continuous function.

In this paper, we present global existence and uniqueness results for the above-cited problems. We make use of the nonlinear alternative of Leray-Schauder type for contraction maps on Fréchet spaces.

The problem of existence of solutions of Cauchy-type problems for ordinary differential equations of fractional order without delay in spaces of integrable functions was studied in numerous works (see [1, 2]), a similar problem in spaces of continuous functions was studied in [3]. We can find numerous applications of differential equations of fractional order in viscoelasticity, electrochemistry, control, porous media, electromagnetic, theory of neolithic transition, and so forth, (see [4–11]). There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Kilbas et al. [12], Lakshmikantham et al. [13], Miller and Ross [14], Samko et al. [15], the papers of Abbas and Benchohra [16–18], Agarwal et al. [19, 20], Ahmad and Nieto [21–23], Belarbi et al. [24], Benchohra et al. [25–27], Chang and Nieto [28], Diethelm et al. [4, 29], Heinsalu et al. [30], Jumarie [31], Kilbas and Marzan [32], Luchko et al. [33], Magdziarz et al. [34], Mainardi [9], Rossikhin and Shitikova [35], Vityuk and Golushkov [36], Yu and Gao [37], and Zhang [38] and the references therein.

For integer order derivative, various classes of hyperbolic differential equations were considered on bounded domain; see, for instance, the book by Kamont [39], the papers by Człapiński [40], Dawidowski and Kubiaczyk [41], Kamont, and Kropielnicka [42], Lakshmikantham and Pandit [43], and Pandit [44].

## 2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. Let $p \in \mathbb{N}$ and $J_0 := [0,p] \times [0,p]$. Let $C(J_0, \mathbb{R}^n)$ be the Banach space of all continuous functions from $J_0$ into $\mathbb{R}^n$ with the norm

$$\|z\|_\infty = \sup_{(x,y)\in J_0} \|z(x,y)\|, \tag{2.1}$$

where $\|\cdot\|$ denotes a suitable complete norm on $\mathbb{R}^n$. As usual, by $\mathrm{AC}(J_0, \mathbb{R}^n)$ we denote the space of absolutely continuous functions from $J_0$ into $\mathbb{R}^n$ and $L^1(J_0, \mathbb{R}^n)$ is the space of Lebegue-integrable functions $w : J_0 \to \mathbb{R}^n$ with the norm

$$\|w\|_1 = \int_0^p \int_0^p \|w(x,y)\| dy\, dx. \tag{2.2}$$

Let $r_1, r_2 > 0$ and $r = (r_1, r_2)$. For $z \in L^1(J_0, \mathbb{R}^n)$, the expression

$$(I_0^r z)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} z(s,t) dt\, ds, \tag{2.3}$$

where $\Gamma(\cdot)$ is the Euler gamma function, is called the left-sided mixed Riemann-Liouville integral of order $r$.

Denote by $D_{xy}^2 := \partial^2/\partial x \partial y$, the mixed second-order partial derivative.

*Definition 2.1* (see [36]). For $z \in L^1(J_0, \mathbb{R}^n)$, the Caputo fractional-order derivative of order $r \in (0,1] \times (0,1]$ of $z$ is defined by the expression $({}^cD_0^r z)(x,y) = (I_0^{1-r} D_{xy}^2 z)(x,y)$.

In the definition above by $1-r$ we mean $(1-r_1, 1-r_2) \in (0,1] \times (0,1]$.

If $z$ is an absolutely continuous function, then its Caputo fractional derivative $({}^cD_0^r z)(x,y)$ exists for each $(x,y) \in J_0$.

Let $X$ be a Fréchet space with a family of seminorms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$. We assume that the family of seminorms $\{\|\cdot\|_n\}$ verifies:

$$\|x\|_1 \le \|x\|_2 \le \|x\|_3 \le \cdots \quad \text{for every } x \in X. \tag{2.4}$$

Let $Y \subset X$, we say that $Y$ is bounded if for every $n \in \mathbb{N}$, there exists $\overline{M}_n > 0$ such that

$$\|y\|_n \le \overline{M}_n, \quad \forall y \in Y. \tag{2.5}$$

To $X$ we associate a sequence of Banach spaces $\{(X^n, \|\cdot\|_n)\}$ as follows. For every $n \in \mathbb{N}$, we consider the equivalence relation $\sim_n$ defined by: $x\sim_n y$ if and only if $\|x-y\|_n = 0$ for $x, y \in X$. We denote $X^n = (X|_{\sim_n}, \|\cdot\|_n)$ the quotient space, the completion of $X^n$ with respect to $\|\cdot\|_n$. To every $Y \subset X$, we associate a sequence $\{Y^n\}$ of subsets $Y^n \subset X^n$ as follows. For every $x \in X$, we denote $[x]_n$ the equivalence class of $x$ of subset $X^n$ and we defined $Y^n = \{[x]_n : x \in Y\}$. We denote $\overline{Y^n}$, $\mathrm{int}_n(Y^n)$ and $\partial_n Y^n$, respectively, the closure, the interior and the boundary of $Y^n$ with respect to $\|\cdot\|_n$ in $X^n$. For more information about this subject see [45].

*Definition 2.2.* Let $X$ be a Fréchet space. A function $N : X \to X$ is said to be a contraction if for each $n \in \mathbb{N}$ there exists $k_n \in (0,1)$ such that

$$\|N(u) - N(v)\|_n \le k_n \|u-v\|_n, \quad \forall u, v \in X. \tag{2.6}$$

**Theorem 2.3** (see [45]). *Let $X$ be a Fréchet space and $Y \subset X$ a closed subset in $X$. Let $N : Y \to X$ be a contraction such that $N(Y)$ is bounded. Then one of the following statements holds:*

(a) *the operator $N$ has a unique fixed point;*

(b) *there exists $\lambda \in [0,1)$, $n \in \mathbb{N}$ and $u \in \partial_n Y^n$ such that $\|u - \lambda N(u)\|_n = 0$.*

In the sequel we will make use of the following generalization of Gronwall's lemma for two independent variables and singular kernel.

**Lemma 2.4** (see [46]). *Let $\upsilon : J_0 \to [0,\infty)$ be a real function and $\omega(\cdot,\cdot)$ be a nonnegative, locally integrable function on $J$. If there are constants $c > 0$ and $0 < l_1, l_2 < 1$ such that*

$$\upsilon(x,y) \le \omega(x,y) + c\int_0^x\int_0^y \frac{\upsilon(s,t)}{(x-s)^{l_1}(y-t)^{l_2}}\,dt\,ds, \tag{2.7}$$

*then there exists a constant $k = k(l_1, l_2)$ such that*

$$\upsilon(x,y) \le \omega(x,y) + kc\int_0^x\int_0^y \frac{\omega(s,t)}{(x-s)^{l_1}(y-t)^{l_2}}\,dt\,ds, \tag{2.8}$$

*for every $(x,y) \in J_0$.*

## 3. Global Result for Finite Delay

Let us start by defining what we mean by a global solution of the problem (1.1)–(1.3).

*Definition 3.1.* A function $u \in C_0 := C([-\alpha,\infty)\times[-\beta,\infty),\mathbb{R}^n)$ such that its mixed derivative $D^2_{xy}$ exists and is integrable on $J$ is said to be a global solution of (1.1)–(1.3) if $u$ satisfies (1.1) and (1.3) on $J$ and the condition (1.2) on $\tilde{J}$.

Let $h \in L^1(J_0,\mathbb{R}^n)$ and consider the following problem

$$\begin{gathered}({}^cD^r_0 u)(x,y) = h(x,y); \quad (x,y)\in J_0,\\ u(x,0)=\varphi(x), \quad u(0,y)=\psi(y); \quad x,y\in[0,p],\\ \varphi(0)=\psi(0).\end{gathered} \tag{3.1}$$

For the existence of global solutions for the problem (1.1)–(1.3), we need the following known lemma.

**Lemma 3.2** (see [16, 17]). *A function $u \in \mathrm{AC}(J_0,\mathbb{R}^n)$ is a global solution of problem* (3.1) *if and only if $u(x,y)$ satisfies*

$$u(x,y) = \mu(x,y) + (I^r_0 h)(x,y), \quad (x,y)\in J_0, \tag{3.2}$$

*where*

$$\mu(x,y) = \varphi(x) + \psi(y) - \varphi(0). \tag{3.3}$$

As a consequence of Lemma 3.2, we have the following result.

**Lemma 3.3.** *A function $u \in \mathrm{AC}(J_0,\mathbb{R}^n)$ is a global solution of problem* (1.1)–(1.3) *if and only if $u(x,y)=\phi(x,y)$, $(x,y)\in\tilde{J}$ and $u(x,y)$ satisfies*

$$u(x,y) = \mu(x,y) + (I^r_0 f)(x,y), \quad (x,y)\in J_0, \tag{3.4}$$

*where*

$$\mu(x,y) = \varphi(x) + \psi(y) - \varphi(0). \tag{3.5}$$

For each $p\in\mathbb{N}$, we consider following set:

$$C_p = C([-\alpha,p]\times[-\beta,p],\mathbb{R}^n), \tag{3.6}$$

and we define in $C_0$ the seminorms by

$$\|u\|_p = \sup\{\|u(x,y)\| : -\alpha\le x\le p,\ -\beta\le y\le p\}. \tag{3.7}$$

Then $C_0$ is a Fréchet space with the family of seminorms $\{\|u\|_p\}$.

Further, we present conditions for the existence and uniqueness of a global solution of problem (1.1)–(1.3).

**Theorem 3.4.** *Assume that*

$(H1)$ *the function* $f : J \times C \to \mathbb{R}^n$ *is continuous,*

$(H2)$ *for each* $p \in \mathbb{N}$, *there exists* $l_p \in C(J_0, \mathbb{R}^n)$ *such that for each* $(x, y) \in J_0$

$$\|f(x,y,u) - f(x,y,v)\| \le l_p(x,y)\|u - v\|_C, \quad \textit{for each } u, v \in C. \tag{3.8}$$

*If*

$$\frac{l_p^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} < 1, \tag{3.9}$$

*where*

$$l_p^* = \sup_{(x,y)\in J_0} l_p(x,y), \tag{3.10}$$

*then, there exists a unique solution for* IVP (1.1)–(1.3) *on* $[-\alpha, \infty) \times [-\beta, \infty)$.

*Proof.* Transform the problem (1.1)–(1.3) into a fixed point problem. Consider the operator $N : C_0 \to C_0$ defined by,

$$N(u)(x,y) = \begin{cases} \phi(x,y), & (x,y) \in \tilde{J}, \\ \mu(x,y) + \dfrac{1}{\Gamma(r_1)\Gamma(r_2)} \displaystyle\int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,u_{(s,t)})\,dt\,ds, & (x,y) \in J. \end{cases} \tag{3.11}$$

Clearly, from Lemma 3.3, the fixed points of $N$ are solutions of (1.1)–(1.3). Let $u$ be a possible solution of the problem $u = \lambda N(u)$ for some $0 < \lambda < 1$. This implies that for each $(x, y) \in J_0$, we have

$$u(x,y) = \lambda\mu(x,y) + \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,u_{(s,t)})\,dt\,ds. \tag{3.12}$$

Introducing $f(s,t,0) - f(s,t,0)$, it follows by $(H2)$ that

$$\begin{aligned} \|u(x,y)\| &\le \|\mu(x,y)\| + \frac{f^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} \\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} l_p(s,t)\|u_{(s,t)}\|_C\,dt\,ds, \end{aligned} \tag{3.13}$$

where

$$f^* = \sup_{(x,y)\in J_0} \|f(x,y,0)\|. \tag{3.14}$$

We consider the function $\tau$ defined by

$$\tau(x,y) = \sup\{\|u(s,t)\| : -\alpha \le s \le x,\ -\beta \le t \le y;\ x,y \in [0,p]\}. \tag{3.15}$$

Let $(x^*,y^*) \in [-\alpha,x]\times[-\beta,y]$ be such that $\tau(x,y) = \|u(x^*,y^*)\|$. If $(x^*,y^*) \in J_0$, then by the previous inequality, we have for $(x,y)\in J_0$,

$$\begin{aligned}\|u(x,y)\| &\le \|\mu(x,y)\| + \frac{f^*p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}\\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} l_p(s,t)\tau(s,t)\,dt\,ds.\end{aligned} \tag{3.16}$$

If $(x^*,y^*) \in \widetilde{J}$, then $\tau(x,y) = \|\phi\|_C$ and the previous inequality holds.

By (3.16) we obtain that

$$\begin{aligned}\tau(x,y) &\le \|\mu(x,y)\| + \frac{f^*p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}\\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} l_p(s,t)\tau(s,t)\,dt\,ds\\ &\le \|\mu(x,y)\| + \frac{f^*p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}\\ &\quad + \frac{l_p^*}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\tau(s,t)\,dt\,ds,\end{aligned} \tag{3.17}$$

and Lemma 2.4 implies that there exists a constant $k = k(r_1,r_2)$ such that

$$\tau(x,y) \le \left(\|\mu\|_p + \frac{f^*p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}\right)\left(1 + \frac{kl_p^*}{\Gamma(r_1+1)\Gamma(r_2+1)}\right) := M_p. \tag{3.18}$$

Then from (3.16), we have

$$\|u\|_p \le \|\mu\|_p + \frac{f^*p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} + \frac{M_p l_p^*}{\Gamma(r_1+1)\Gamma(r_2+1)} := M_p^*. \tag{3.19}$$

Since for every $(x,y)\in J_0$, $\|u_{(x,y)}\|_C \le \tau(x,y)$, we have

$$\|u\|_p \le \max\left(\|\phi\|_C, M_p^*\right) := R_p. \tag{3.20}$$

Set

$$U = \left\{ u \in C_0 : \|u\|_p \le R_p + 1 \ \forall p \in \mathbb{N} \right\}. \tag{3.21}$$

We will show that $N : U \to C_p$ is a contraction map. Indeed, consider $v, w \in U$. Then for each $x, y \in [0, p]$, we have

$$\begin{aligned}
&\|N(v)(x,y) - N(w)(x,y)\| \\
&\quad \le \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y \left|(x-s)^{r_1-1}\right| \left|(y-t)^{r_2-1}\right| \left\| f\left(s,t,v_{(s,t)}\right) - f\left(s,t,w_{(s,t)}\right)\right\| dt\, ds \\
&\quad \le \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} l_{(p,q)}(s,t) \left\|v_{(s,t)} - w_{(s,t)}\right\|_C dt\, ds \\
&\quad \le \frac{l_p^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} \|v - w\|_p.
\end{aligned} \tag{3.22}$$

Thus,

$$\|N(v) - N(w)\|_p \le \frac{l_p^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} \|v - w\|_p. \tag{3.23}$$

Hence by (3.9), $N : U \to C_p$ is a contraction. By our choice of $U$, there is no $u \in \partial_n U^n$ such that $u = \lambda N(u)$, for $\lambda \in (0,1)$. As a consequence of Theorem 2.3, we deduce that $N$ has a unique fixed point $u$ in $U$ which is a solution to problem (1.1)–(1.3). □

Now we present a global existence and uniqueness result for the problem (1.5)–(1.7).

*Definition 3.5.* A function $u \in C_0$ such that the mixed derivative $D^2_{xy}(u(x,y) - g(x,y,u_{(x,y)}))$ exists and is integrable on $J$ is said to be a global solution of (1.5)–(1.7) if $u$ satisfies equations (1.5) and (1.7) on $J$ and the condition (1.6) on $\tilde{J}$.

Let $f \in L^1(J_0, \mathbb{R}^n)$, $g \in \mathrm{AC}(J_0, \mathbb{R}^n)$ and consider the following linear problem

$$\begin{aligned}
&{}^c D_0^r (u(x,y) - g(x,y)) = f(x,y); \quad (x,y) \in J_0, \\
&u(x,0) = \varphi(x), \quad u(0,y) = \psi(y); \quad x, y \in [0,p],
\end{aligned} \tag{3.24}$$

with $\varphi(0) = \psi(0)$.

For the existence of solutions for the problem (1.5)–(1.7), we need the following lemma.

**Lemma 3.6.** *A function $u \in \mathrm{AC}(J_0, \mathbb{R}^n)$ is a global solution of problem* (3.24) *if and only if $u(x,y)$ satisfies*

$$u(x,y) = \mu(x,y) + g(x,y) - g(x,0) - g(0,y) + g(0,0) + I_0^r(f)(x,y); \quad (x,y) \in J_0. \tag{3.25}$$

*Proof.* Let $u(x,y)$ be a solution of problem (3.24). Then, taking into account the definition of the fractional Caputo derivative, we have

$$I_0^{1-r} D_{xy}^2 (u(x,y) - g(x,y)) = f(x,y). \tag{3.26}$$

Hence, we obtain

$$I_0^r I_0^{1-r} D_{xy}^2 (u(x,y) - g(x,y)) = (I_0^r f)(x,y), \tag{3.27}$$

then,

$$I_0^1 D_{xy}^2 (u(x,y) - g(x,y)) = (I_0^r f)(x,y). \tag{3.28}$$

Since

$$\begin{aligned} I_0^1 D_{xy}^2 (u(x,y) - g(x,y)) &= (u(x,y) - g(x,y)) - (u(x,0) - g(x,0)) \\ &\quad - (u(0,y) - g(0,y)) + (u(0,0) - g(0,0)), \end{aligned} \tag{3.29}$$

we have

$$u(x,y) = \mu(x,y) + g(x,y) - g(x,0) - g(0,y) + g(0,0) + I_0^r(f)(x,y). \tag{3.30}$$

Now, let $u(x,y)$ satisfy (3.25). It is clear that $u(x,y)$ satisfies (3.24). □

As a consequence of Lemma 3.6 we have the following result.

**Lemma 3.7.** *The function $u \in \mathrm{AC}(J_0, \mathbb{R}^n)$ is a global solution of problem* (1.5)–(1.7) *if and only if $u$ satisfies the equation*

$$\begin{aligned} u(x,y) = {} & \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1} (y-t)^{r_2-1} f(s,t,u_{(s,t)})\, ds\, dt \\ & + \mu(x,y) + g(x,y,u_{(x,y)}) - g(x,0,u_{(x,0)}) \\ & - g(0,y,u_{(0,y)}) + g(0,0,u_{(0,0)}), \end{aligned} \tag{3.31}$$

*for all $(x,y) \in J_0$ and the condition* (1.6) *on $\tilde{J}$.*

**Theorem 3.8.** *Assume that $(H1)$, $(H2)$, and the following condition holds*

$(H3)$ *For each $p = 1, 2, \ldots$, there exists a constant $c_p$ with $0 < c_p < 1/4$ such that for each $(x,y) \in J_0$, one has*

$$\| g(x,y,u) - g(x,y,v) \| \le c_p \|u - v\|_C, \quad \text{for each } u, v \in C. \tag{3.32}$$

*If*

$$4c_p + \frac{l_p^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} < 1, \tag{3.33}$$

*then there exists a unique solution for* IVP (1.5)–(1.7) *on* $[-\alpha, \infty) \times [-\beta, \infty)$.

*Proof.* Transform the problem (1.5)–(1.7) into a fixed point problem. Consider the operator $N_1 : C_0 \to C_0$ defined by,

$$N_1(u)(x,y) = \begin{cases} \phi(x,y), & (x,y) \in \tilde{J}, \\ \mu(x,y) + g(x,y,u_{(x,y)}) - g(x,0,u_{(x,0)}) & \\ \quad -g(0,y,u_{(0,y)}) + g(0,0,u_{(0,0)}) & \\ \quad +\dfrac{1}{\Gamma(r_1)\Gamma(r_2)}\displaystyle\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,u_{(s,t)})dt\,ds, & (x,y) \in J. \end{cases} \tag{3.34}$$

From Lemma 3.7, the fixed points of $N_1$ are solutions to problem (1.5)–(1.7). In order to use the nonlinear alternative, we will obtain a priori estimates for the solutions of the integral equation

$$\begin{aligned} u(x,y) &= \lambda(\mu(x,y) + g(x,y,u_{(x,y)}) - g(x,0,u_{(x,0)}) - g(0,y,u_{(0,y)}) + g(0,0,u_{(0,0)})) \\ &\quad + \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,u_{(s,t)})dt\,ds, \end{aligned} \tag{3.35}$$

for some $\lambda \in (0,1)$. Then, using $(H1)$–$(H3)$ and (3.16) we get for each $(x,y) \in J_0$,

$$\begin{aligned} \|u(x,y)\| &\le \|\mu(x,y)\| + \frac{f^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} \\ &\quad + \|g(x,y,u_{(x,y)})\| + \|g(x,0,u_{(x,0)})\| + \|g(0,y,u_{(0,y)})\| + \|g(0,0,u_{(0,0)})\| \\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} l_p(s,t)\tau(s,t)dt\,ds, \end{aligned} \tag{3.36}$$

then, we obtain

$$\begin{aligned} \|u(x,y)\| &\le \|\mu(x,y)\| + \frac{f^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} \\ &\quad + 4c_p\tau(x,y) + \|g(x,y,0)\| + \|g(x,0,0)\| + \|g(0,y,0)\| + \|g(0,0,0)\| \\ &\quad + \frac{l_p^*}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \tau(s,t)dt\,ds. \end{aligned} \tag{3.37}$$

Replacing (3.37) in the definition of $\tau(x,y)$ we get

$$\begin{aligned}\tau(x,y) \leq{}& \frac{1}{1-4c_p}\left[\|\mu(x,y)\| + \frac{f^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} + 4g^*\right]\\ &+ \frac{\widetilde{l}_p^*}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\tau(s,t)\,dt\,ds,\end{aligned}\tag{3.38}$$

where $\widetilde{l}_p^* = l_p^*/(1-4c_p)$ and $g_p^* = \sup_{(x,y)\in J_0}\|g(x,y,0)\|$.

By Lemma 2.4, there exists a constant $\delta = \delta(r_1,r_2)$ such that

$$\begin{aligned}\|\tau\|_p \leq{}& \frac{1}{1-4c_p}\left[\|\mu\|_p + \frac{f^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} + 4g_p^*\right]\\ &\times\left[1 + \frac{\delta\widetilde{l}_p^*}{\Gamma(r_1+1)\Gamma(r_2+1)}\right] := D_p.\end{aligned}\tag{3.39}$$

Then, from (3.37) and (3.39), we get

$$\begin{aligned}\|u\|_p \leq{}& \|\mu\|_p + \frac{f^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} + 4g_p^*\\ &+ 4c_p D_p + \frac{D_p l_p^*}{\Gamma(r_1+1)\Gamma(r_2+1)} := D_p^*.\end{aligned}\tag{3.40}$$

Since for every $(x,y)\in J_0$, $\|u_{(x,y)}\|_C \leq \tau(x,y)$, we have

$$\|u\|_p \leq \max\left(\|\phi\|_C, D_p^*\right) := R_p^*.\tag{3.41}$$

Set

$$U_1 = \left\{u\in C_0 : \|u\|_p \leq R_p^* + 1\ \forall p = 1,2,\ldots\right\}.\tag{3.42}$$

Clearly, $U_1$ is a closed subset of $C_0$. As in Theorem 3.4, we can show that $N_1 : U_1 \to C_0$ is a contraction operator. Indeed

$$\|N_1(v) - N_1(w)\|_p \leq \left(4c_p + \frac{l_p^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}\right)\|v-w\|_p\tag{3.43}$$

for each $v,w\in U_1$ and $(x,y)\in J_0$. From the choice of $U_1$, there is no $u\in\partial_n U_1^n$ such that $u = \lambda N_1(u)$, for some $\lambda\in(0,1)$. As a consequence of Theorem 2.3, we deduce that $N_1$ has a unique fixed point $u$ in $U_1$ which is a solution to problem (1.5)–(1.7). □

## 4. The Phase Space $B$

The notation of the phase space $B$ plays an important role in the study of both qualitative and quantitative theory for functional differential equations. A usual choice is a seminormed space satisfying suitable axioms, which was introduced by Hale and Kato (see [47]). For further applications see, for instance, the books [48–50] and their references.

Inspired by [47], Człapiński [40] introduced the following construction of the phase space. For any $(x,y)\in J_0$ denote $E_{(x,y)}:=[0,x]\times\{0\}\cup\{0\}\times[0,y]$, furthermore in case $x=y=p$ we write simply $E$. Consider the space $(B,\|(\cdot,\cdot)\|_B)$ is a seminormed linear space of functions mapping $(-\infty,0]\times(-\infty,0]$ into $\mathbb{R}^n$, and satisfying the following fundamental axioms which were adapted from those introduced by Hale and Kato for ordinary differential functional equations.

$(A_1)$ If $z:(-\infty,p]\times(-\infty,p]\to\mathbb{R}^n$ continuous on $J_0$ and $z_{(x,y)}\in B$, for all $(x,y)\in E$, then there are constants $H,K,M>0$ such that for any $(x,y)\in J_0$ the following conditions hold:

  (i) $z_{(x,y)}$ is in $B$;

  (ii) $\|z(x,y)\|\le H\|z_{(x,y)}\|_B$, and

  (iii) $\|z_{(x,y)}\|_B\le K\sup_{(s,t)\in[0,x]\times[0,y]}\|z(s,t)\|+M\sup_{(s,t)\in E_{(x,y)}}\|z_{(s,t)}\|_B$.

$(A_2)$ For the function $z(\cdot,\cdot)$ in $(A_1)$, $z_{(x,y)}$ is a $B$-valued continuous function on $J_0$.

$(A_3)$ The space $B$ is complete.

Now, we present some examples of phase spaces (see [40]).

*Example 4.1.* Let $B$ be the set of all functions $\phi:(-\infty,0]\times(-\infty,0]\to\mathbb{R}^n$ which are continuous on $[-\alpha,0]\times[-\beta,0]$, $\alpha,\beta\ge 0$, with the seminorm

$$\|\phi\|_B=\sup_{(s,t)\in[-\alpha,0]\times[-\beta,0]}\|\phi(s,t)\|. \tag{4.1}$$

Then, we have $H=K=M=1$. The quotient space $\widehat{B}=B/\|\cdot\|_B$ is isometric to the space $C([-\alpha,0]\times[-\beta,0],\mathbb{R}^n)$ of all continuous functions from $[-\alpha,0]\times[-\beta,0]$ into $\mathbb{R}^n$ with the supremum norm, this means that partial differential functional equations with finite delay are included in our axiomatic model.

*Example 4.2.* Let $C_\gamma$ be the set of all continuous functions $\phi:(-\infty,0]\times(-\infty,0]\to\mathbb{R}^n$ for which a limit $\lim_{\|(s,t)\|\to\infty}e^{\gamma(s+t)}\phi(s,t)$ exists, with the norm

$$\|\phi\|_{C_\gamma}=\sup_{(s,t)\in(-\infty,0]\times(-\infty,0]}e^{\gamma(s+t)}\|\phi(s,t)\|. \tag{4.2}$$

Then we have $H=K=M=1$.

*Example 4.3.* Let $\alpha,\beta,\gamma\ge 0$ and let

$$\|\phi\|_{\mathrm{CL}_\gamma}=\sup_{(s,t)\in[-\alpha,0]\times[-\beta,0]}\|\phi(s,t)\|+\iint_{-\infty}^{0}e^{\gamma(s+t)}\|\phi(s,t)\|dt\,ds \tag{4.3}$$

be the seminorm for the space $\mathrm{CL}_\gamma$ of all functions $\phi : (-\infty,0]\times(-\infty,0] \to \mathbb{R}^n$ which are continuous on $[-\alpha,0]\times[-\beta,0]$ measurable on $(-\infty,-\alpha]\times(-\infty,0]\cup(-\infty,0]\times(-\infty,-\beta]$, and such that $\|\phi\|_{CL_\gamma}<\infty$. Then,

$$H=1, \qquad K=\int_{-\alpha}^{0}\int_{-\beta}^{0}e^{\gamma(s+t)}dt\,ds, \qquad M=2. \tag{4.4}$$

## 5. Global Result for Infinite Delay

In this section we present a global existence and uniqueness result for the problems (1.8)–(1.10) and (1.12)–(1.14). Let us define the space

$$\Omega := \left\{u:\mathbb{R}^2 \longrightarrow \mathbb{R}^n : u_{(x,y)}\in B \text{ for } (x,y)\in E_0,\ u|_J\in C(J,\mathbb{R}^n)\right\}, \tag{5.1}$$

where $E_0 := [0,\infty)\times\{0\}\cup\{0\}\times[0,\infty)$.

*Definition 5.1.* A function $u\in\Omega$ such that its mixed derivative $D^2_{xy}$ exists and is integrable on $J$ is said to be a global solutionis of (1.8)–(1.10) if $u$ satisfies equations (1.8) and (1.10) on $J$ and the condition (1.9) on $\widetilde{J}'$.

For each $p\in\mathbb{N}$, we consider following set,

$$C'_p=\{u:(-\infty,p]\times(-\infty,p]\longrightarrow\mathbb{R}^n : u\in B\cap C(J_0,\mathbb{R}^n),\ u_{(x,y)}=0 \text{ for } (x,y)\in E\}, \tag{5.2}$$

and we define in

$$C'_0 := \left\{u:\mathbb{R}^2\longrightarrow\mathbb{R}^n : u\in B\cap C([0,\infty)\times[0,\infty),\mathbb{R}^n),\ u_{(x,y)}=0 \text{ for } (x,y)\in E_0\right\} \tag{5.3}$$

the seminorms by

$$\begin{aligned}\|u\|_{p'} &= \sup_{(x,y)\in E}\left\|u_{(x,y)}\right\|_B+\sup_{(x,y)\in J_0}\left\|u(x,y)\right\|\\ &= \sup_{(x,y)\in J_0}\left\|u(x,y)\right\|, \quad u\in C'_p.\end{aligned} \tag{5.4}$$

Then, $C'_0$ is a Fréchet space with the family of seminorms $\{\|u\|_{p'}\}$.

**Theorem 5.2.** *Assume that*

($H'1$) *the function $f : J\times B \to \mathbb{R}^n$ is continuous and*

($H'2$) *for each $p\in\mathbb{N}$, there exists $l'_p\in C(J_0,\mathbb{R}^n)$ such that for and $(x,y)\in J_0$*

$$\|f(x,y,u)-f(x,y,v)\|\le l'_p(x,y)\|u-v\|_B, \quad \textit{for each } u,v\in B. \tag{5.5}$$

*If*

$$\frac{K l_p'^{*} p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} < 1, \tag{5.6}$$

*where*

$$l_p'^{*} = \sup_{(x,y)\in J_0} l_p'(x,y), \tag{5.7}$$

*then, there exists a unique solution for* IVP (1.8)–(1.10) *on* $\mathbb{R}^2$.

*Proof.* Transform the problem (1.8)–(1.10) into a fixed point problem. Consider the operator $N' : \Omega \to \Omega$ defined by

$$N'(u)(x,y) = \begin{cases} \phi(x,y), & (x,y) \in \tilde{J}', \\ \mu(x,y) + \dfrac{1}{\Gamma(r_1)\Gamma(r_2)} \displaystyle\int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,u_{(s,t)})\,dt\,ds; & (x,y) \in J. \end{cases} \tag{5.8}$$

Let $v(\cdot,\cdot) : \mathbb{R}^2 \to \mathbb{R}^n$ be a function defined by

$$v(x,y) = \begin{cases} \phi(x,y), & (x,y) \in \tilde{J}', \\ \mu(x,y), & (x,y) \in J. \end{cases} \tag{5.9}$$

Then, $v_{(x,y)} = \phi$ for all $(x,y) \in E_0$. For each $w \in C(J,\mathbb{R}^n)$ with $w(x,y) = 0$; for all $(x,y) \in E_0$, we denote by $\overline{w}$ the function defined by

$$\overline{w}(x,y) = \begin{cases} 0, & (x,y) \in \tilde{J}', \\ w(x,y), & (x,y) \in J. \end{cases} \tag{5.10}$$

If $u(\cdot,\cdot)$ satisfies the integral equation,

$$u(x,y) = \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,u_{(s,t)})\,dt\,ds, \tag{5.11}$$

we can decompose $u(\cdot,\cdot)$ as $u(x,y) = \overline{w}(x,y) + v(x,y)$; $x,y \ge 0$, which implies that $u_{(x,y)} = \overline{w}_{(x,y)} + v_{(x,y)}$, for every $x,y \ge 0$, and the function $w(\cdot,\cdot)$ satisfies

$$w(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,\overline{w}_{(s,t)} + v_{(s,t)})\,dt\,ds. \tag{5.12}$$

Let the operator $P' : C'_0 \to C'_0$ be defined by

$$(P'w)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \times f\left(s,t,\overline{w}_{(s,t)} + v_{(s,t)}\right) dt\, ds; \quad (x,y) \in J. \tag{5.13}$$

Obviously, the operator $N'$ has a fixed point is equivalent to $P'$ having a fixed point, and so we turn to prove that $P'$ has a fixed point. We will use the alternative to prove that $P'$ has a fixed point. Let $w$ be a possible solution of the problem $w = P'(w)$ for some $0 < \lambda < 1$. This implies that for each $(x,y) \in J_0$, we have

$$w(x,y) = \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f\left(s,t,\overline{w}_{(s,t)} + v_{(s,t)}\right) dt\, ds. \tag{5.14}$$

This implies by $(H'1)$ that

$$\|w(x,y)\| \leq \frac{f_p^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} l'_p(s,t) \left\|\overline{w}_{(s,t)} + v_{(s,t)}\right\|_B dt\, ds, \tag{5.15}$$

where

$$f_p^* = \sup\{\|f(x,y,0)\| : (x,y) \in J_0\}. \tag{5.16}$$

But

$$\begin{aligned}\left\|\overline{w}_{(s,t)} + v_{(s,t)}\right\|_B &\leq \left\|\overline{w}_{(s,t)}\right\|_B + \left\|v_{(s,t)}\right\|_B \\ &\leq K \sup\left\{u\left(\tilde{s},\tilde{t}\right) : \left(\tilde{s},\tilde{t}\right) \in [0,s] \times [0,t]\right\} \\ &\quad + M\|\phi\|_B + K\|\phi(0,0)\|.\end{aligned} \tag{5.17}$$

If we name $z(s,t)$ the right-hand side of (5.17), then we have

$$\left\|\overline{w}_{(s,t)} + v_{(s,t)}\right\|_B \leq z(s,t). \tag{5.18}$$

Therefore, from (5.15) and (5.18) we get

$$\|w(x,y)\| \leq \frac{f_p^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} l'_p(s,t) z(s,t) dt\, ds. \tag{5.19}$$

Replacing (5.19) in the definition of $w$, we have that

$$\begin{aligned}\|z(x,y)\| &\leq \frac{Kf_p^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} + M\|\phi\|_B \\ &\quad + \frac{Kl_p'^*}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}z(s,t)\,dt\,ds.\end{aligned} \tag{5.20}$$

By Lemma 2.4, there exists a constant $\delta = \delta(r_1, r_2)$ such that

$$\begin{aligned}\|z\|_{p'} &\leq \left(\frac{Kf_p^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} + M\|\phi\|_B\right) \\ &\quad \times \left(1 + \frac{\delta K l_p'^*}{\Gamma(r_1+1)\Gamma(r_2+1)}\right) \\ &:= \widetilde{M}.\end{aligned} \tag{5.21}$$

Then, from (5.19), we have

$$\|w\|_{p'} \leq \widetilde{M}\frac{l_p'^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} + \frac{f_p^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} := \widetilde{M}^*. \tag{5.22}$$

Since for every $(x,y) \in J_0$, $\|w_{(x,y)}\|_B \leq z(x,y)$, we have

$$\|w\|_{p'} \leq \max\left(\|\phi\|_B, \widetilde{M}^*\right) := \widetilde{R}^*. \tag{5.23}$$

Set

$$U' = \left\{w \in C_0' : \|w\|_{p'} \leq \widetilde{R}^* + 1 \ \forall p \in \mathbb{N}\right\}. \tag{5.24}$$

We will show that $P' : U' \to C_p'$ is a contraction operator. Indeed, consider $w, w^* \in U'$. Then for each $(x,y) \in J_0$, we have

$$\begin{aligned}&\|P'(w)(x,y) - P'(w^*)(x,y)\| \\ &\quad \leq \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \\ &\qquad \times \left\|f\left(s,t,\overline{w}_{(s,t)} + v_{(s,t)}\right) - f\left(s,t,\overline{w^*}_{(s,t)} + v_{(s,t)}\right)\right\|dt\,ds \\ &\quad \leq \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} l_p'(s,t)\left\|\overline{w}_{(s,t)} - \overline{w^*}_{(s,t)}\right\|_B dt\,ds \\ &\quad \leq K\frac{l_p'^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}\|w - w^*\|_{p'}.\end{aligned} \tag{5.25}$$

Thus,

$$\left\|P'(w)-P'(w^*)\right\|_{p'} \leq \frac{Kl_p'^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}\|w-w^*\|_{p'}. \tag{5.26}$$

Hence by (5.6), $P' : U' \to C_p'$ is a contraction. By our choice of $U'$, there is no $w \in \partial_n(U')^n$ such that $w = \lambda P'(w)$, for $\lambda \in (0,1)$. As a consequence of Theorem 2.3, we deduce that $N'$ has a unique fixed point which is a solution to problem (1.8)–(1.10). □

Now, we present an existence result for the problem (1.12)–(1.14).

*Definition 5.3.* A function $u \in \Omega$ such that the mixed derivative $D_{xy}^2(u(x,y) - g(x,y,u_{(x,y)}))$ exists and is integrable on $J$ is said to be a global solutionis of (1.12)–(1.14) if $u$ satisfies equations (1.12) and (1.14) on $J$ and the condition (1.13) on $\tilde{J}'$.

**Theorem 5.4.** *Let $f, g : J \times B \to \mathbb{R}^n$ be continuous functions. Assume that $(H'1)$, $(H'2)$, and the following condition hold.*

$(H'3)$ *For each $p = 1, 2, \ldots$, there exists a constant $c_p'$ with $0 < Kc_p' < 1/4$ such that for any $(x,y) \in J_0$, one has*

$$\left\|g(x,y,u) - g(x,y,v)\right\| \leq c_p'\|u-v\|_B, \quad \text{for any } u, v \in B. \tag{5.27}$$

*If*

$$4c_p' + \frac{Kl_p'^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} < 1, \quad \text{for each } p \in \mathbb{N}, \tag{5.28}$$

*then, there exists a unique solution for* IVP (1.12)–(1.14) *on* $\mathbb{R}^2$.

*Proof.* Consider the operator $N_1' : \Omega \to \Omega$ defined by

$$N_1'(u)(x,y) = \begin{cases} \phi(x,y), & (x,y) \in \tilde{J}', \\ \mu(x,y) + g(x,y,u_{(x,y)}) - g(x,0,u_{(x,0)}) & \\ \quad -g(0,y,u_{(0,y)}) + g(0,0,u_{(0,0)}) & \\ \quad +\dfrac{1}{\Gamma(r_1)\Gamma(r_2)}\displaystyle\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} & \\ \quad \times f(s,t,u_{(s,t)})\,dt\,ds, & (x,y) \in J. \end{cases} \tag{5.29}$$

In analogy to Theorem 5.2, we consider the operator $P_1' : C_0' \to C_0'$ defined by

$$
\begin{aligned}
P_1'(w)(x,y) = {} & g\left(x,y,\overline{w}_{(x,y)} + v_{(x,y)}\right) - g\left(x,0,\overline{w}_{(x,0)} + v_{(x,0)}\right) \\
& - g\left(0,y,\overline{w}_{(0,y)} + v_{(0,y)}\right) + g\left(0,0,\overline{w}_{(0,0)} + v_{(0,0)}\right) \\
& + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \\
& \times f\left(s,t,\overline{w}_{(s,t)} + v_{(s,t)}\right) dt\, ds, \quad (x,y) \in J.
\end{aligned}
\tag{5.30}
$$

In order to use the nonlinear alternative, we will obtain a priori estimates for the solutions of the integral equation

$$
\begin{aligned}
w(x,y) = {} & \lambda\Big(g\left(x,y,\overline{w}_{(x,y)} + v_{(x,y)}\right) - g\left(x,0,\overline{w}_{(x,0)} + v_{(x,0)}\right) \\
& -g\left(0,y,\overline{w}_{(0,y)} + v_{(0,y)}\right) + g\left(0,0,\overline{w}_{(0,0)} + v_{(0,0)}\right)\Big) \\
& + \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f\left(s,t,\overline{w}_{(s,t)} + v_{(s,t)}\right) dt\, ds,
\end{aligned}
\tag{5.31}
$$

for some $\lambda \in (0,1)$. Then from $(H_1')$–$(H_3')$, (5.15), and (5.18) we get for each $(x,y) \in J_0$,

$$
\begin{aligned}
\|w(x,y)\| \le {} & \frac{f_p^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} + 4c_p' z(x,y) \\
& + \|g(x,y,0)\| + \|g(x,0,0)\| + \|g(0,y,0)\| + \|g(0,0,0)\| \\
& + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} l_p'(s,t) z(s,t)\, dt\, ds.
\end{aligned}
\tag{5.32}
$$

Replacing (5.32) in the definition of $z(x,y)$, we get

$$
\begin{aligned}
z(x,y) \le {} & \frac{1}{1-4Kc_p'} \Bigg[ M\|\phi\|_B + 4K\|\phi(0,0)\| + 4K\|g(0,0,\phi(0,0))\| \\
& \qquad + 4K g_p^* + \frac{K f_p^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} \Bigg] \\
& + \frac{\widetilde{l}_p'^*}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} z(s,t)\, dt\, ds,
\end{aligned}
\tag{5.33}
$$

where $\widetilde{l}_p'^*(x,y) = l_p'^*/(1-4Kc_p')$ and $g_p^* = \sup\{\|g(x,y,0)\| : (x,y) \in J_0\}$.

By (5.32) and Lemma 2.4, there exists a constant $\delta = \delta(r_1, r_2)$ such that

$$z(x,y) \leq \frac{1}{1-4Kc'_p}\left[M\|\phi\|_B + 4K\|\phi(0,0)\| + 4K\|g(0,0,\phi(0,0))\| + 4Kg^*_p + \frac{Kf^*_p p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}\right] \times \left[1 + \frac{\delta \tilde{l}'^*_p}{\Gamma(r_1+1)\Gamma(r_2+1)}\right] := D'. \tag{5.34}$$

Then, from (5.32) and (5.34), we get

$$\|w\|_{p'} \leq \frac{\left(D' l'^*_p + f^*_p\right)p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} + 4c'_p D' + 4g^*_p := D'^*. \tag{5.35}$$

Since for every $(x,y) \in J_0$, $\|w_{(x,y)}\|_B \leq z(x,y)$, we have

$$\|w\|_p \leq \max\left(\|\phi\|_B, D'^*\right) := R'^*. \tag{5.36}$$

Set

$$U'_1 = \left\{w \in C'_0 : \|w\|_{p'} \leq R'^* + 1\right\}. \tag{5.37}$$

Clearly, $U'_1$ is a closed subset of $C'_0$. As in Theorem 5.2, we can show that $P'_1 : U'_1 \to C'_0$ is a contraction operator. Indeed

$$\|N_1(v) - N_1(w)\|_{p'} \leq \left(4c'_p + \frac{Kl'^*_p p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}\right)\|v - w\|_{p'}, \tag{5.38}$$

for each $v, w \in U'_1$, and $(x,y) \in J_0$. From the choice of $U'_1$, there is no $w \in \partial_n(U'_1)^n$ such that $w = \lambda P'_1(w)$, for some $\lambda \in (0,1)$. As a consequence of Theorem 2.3, we deduce that $N'_1$ has a unique fixed point which is a solution to problem (1.12)–(1.14). □

## 6. Examples

*Example 6.1.* As an application of our results we consider the following partial hyperbolic functional differential equations with finite delay of the form

$$\begin{gathered} {}^{c}D_{0}^{r}u(x,y) = \frac{c_p}{e^{x+y+2}\left(1+\left|u(x-1,y-2)\right|\right)}; \quad \text{if } (x,y)\in[0,\infty)\times[0,\infty), \\ u(x,0)=x, \quad u(0,y)=y^2; \quad x,y\in[0,\infty), \\ u(x,y)=x+y^2; \quad (x,y)\in[-1,\infty)\times[-2,\infty)\setminus(0,\infty)\times(0,\infty), \end{gathered} \tag{6.1}$$

where

$$c_p = \frac{\Gamma(r_1+1)\Gamma(r_2+1)}{p^{r_1+r_2}}; \quad p\in\mathbb{N}^*. \tag{6.2}$$

Set

$$f\left(x,y,u_{(x,y)}\right) = \frac{c_p}{e^{x+y+2}\left(1+\left|u(x-1,y-2)\right|\right)}; \quad (x,y)\in[0,\infty)\times[0,\infty). \tag{6.3}$$

For each $p\in\mathbb{N}^*$ and $(x,y)\in[0,p]\times[0,p]$, we have

$$\left|f\left(x,y,u_{(x,y)}\right)-f\left(x,y,\overline{u}_{(x,y)}\right)\right| \le \frac{c_p}{e^2}\|u-\overline{u}\|_C. \tag{6.4}$$

Hence conditions $(H1)$ and $(H2)$ are satisfied with $l_p^* = c_p/e^2$. We will show that condition (3.9) holds for all $p\in\mathbb{N}^*$. Indeed

$$\frac{l_p^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} = \frac{1}{e^2} < 1, \tag{6.5}$$

which is satisfied for each $(r_1,r_2)\in(0,1]\times(0,1]$. Consequently Theorem 3.4 implies that problem (6.1) has a unique global solution defined on $[-1,\infty)\times[-2,\infty)$.

*Example 6.2.* We consider now the following partial hyperbolic functional differential equations with infinite delay of the form

$$\begin{aligned} \left({}^{c}D_{0}^{r}u\right)(x,y) &= \frac{4e^{x+y}}{c_p\pi^2\left(e^{x+y}+e^{-x-y}\right)} \\ &\quad\times\int_{-\infty}^{-x}\int_{-\infty}^{-y}\frac{e^{\gamma(\theta+\eta)}u(x+\theta,y+\eta)\,d\eta\, d\theta}{\left(1+(x+\theta)^2\right)\left(1+(y+\eta)^2\right)}; \quad \text{if } (x,y)\in[0,\infty)\times[0,\infty), \\ &\quad u(x,y)=x+y^2; \quad (x,y)\in\mathbb{R}^2\setminus(0,\infty)\times(0,\infty), \\ &\quad u(x,0)=x, \quad u(0,y)=y^2; \quad x,y\in[0,\infty), \end{aligned} \tag{6.6}$$

where $c_p = 3p^{r_1+r_2}/\Gamma(r_1+1)\Gamma(r_2+1)$, $p\in\mathbb{N}^*$ and $\gamma$ a positive real constant.

Let

$$B_\gamma = \left\{ u \in C((-\infty,0]\times(-\infty,0],\mathbb{R}) : \lim_{\|(\theta,\eta)\|\to\infty} e^{\gamma(\theta+\eta)} \left| u(\theta,\eta) \right| \text{ exists in } \mathbb{R} \right\}. \tag{6.7}$$

The norm of $B_\gamma$ is given by

$$\|u\|_\gamma = \sup_{-\infty<\theta,\,\eta\le 0} e^{\gamma(\theta+\eta)} \left| u(\theta,\eta) \right|. \tag{6.8}$$

Let $u : \mathbb{R}^2 \to \mathbb{R}$ such that $u_{(x,y)} \in B_\gamma$, $(x,y) \in E := [0,p]\times\{0\}\cup\{0\}\times[0,p]$, then

$$\begin{aligned} \lim_{\|(\theta,\eta)\|\to\infty} e^{\gamma(\theta+\eta)} u_{(x,y)}(\theta,\eta) &= \lim_{\|(\theta,\eta)\|\to\infty} e^{\gamma(\theta-x+\eta-y)} u(\theta,\eta) \\ &= e^{-\gamma(x+y)} \lim_{\|(\theta,\eta)\|\to\infty} e^{\gamma(\theta+\eta)} u(\theta,\eta) < \infty. \end{aligned} \tag{6.9}$$

Hence, $u_{(x,y)} \in B_\gamma$. Finally we prove that

$$\left\| u_{(x,y)} \right\|_\gamma = K \sup_{(s,t)\in[0,x]\times[0,y]} |u(s,t)| + M \sup_{(s,t)\in E_{(x,y)}} \left\| u_{(s,t)} \right\|_\gamma, \tag{6.10}$$

where $K = M = 1$ and $H = 1$.

If $x+\theta \le 0$, $y+\eta \le 0$ we get

$$\left\| u_{(x,y)} \right\|_\gamma = \sup\{|u(s,t)| : (s,t) \in (-\infty,0]\times(-\infty,0]\}, \tag{6.11}$$

and if $x+\theta \ge 0$, $y+\eta \ge 0$ then we have

$$\left\| u_{(x,y)} \right\|_\gamma = \sup\{|u(s,t)| : (s,t) \in [0,x]\times[0,y]\}. \tag{6.12}$$

Thus, for all $x+\theta, y+\eta \in [0,p]$, we get

$$\left\| u_{(x,y)} \right\|_\gamma = \sup_{(s,t)\in(-\infty,0]\times(-\infty,0]} |u(s,t)| + \sup_{(s,t)\in[0,x]\times[0,y]} |u(s,t)|. \tag{6.13}$$

Then,

$$\left\| u_{(x,y)} \right\|_\gamma = \sup_{(s,t)\in E} \left\| u_{(s,t)} \right\|_\gamma + \sup_{(s,t)\in[0,x]\times[0,y]} |u(s,t)|. \tag{6.14}$$

$(B_\gamma, \|\cdot\|_\gamma)$ is a Banach space. We conclude that $B_\gamma$ is a phase space.

Let

$$f(x,y,u) = \frac{4e^{x+y}}{c_p\pi^2(e^{x+y}+e^{-x-y})} \int_{-\infty}^{-x}\int_{-\infty}^{-y} \frac{e^{\gamma(\theta+\eta)} u(x+\theta,y+\eta)}{\left(1+(x+\theta)^2\right)\left(1+(y+\eta)^2\right)} d\eta\, d\theta; \tag{6.15}$$

for each $(x, y, u) \in J \times B_{\gamma}$. Then for each $p \in \mathbb{N}^*$, $(x, y) \in [0, p] \times [0, p]$ and $u, v \in B_{\gamma}$, we have

$$
\begin{aligned}
&\left|f(x,y,u) - f(x,y,v)\right| \\
&\quad = \left| \frac{4e^{x+y}}{c_p\pi^2(e^{x+y} + e^{-x-y})} \int_{-\infty}^{-x}\int_{-\infty}^{-y} \frac{e^{\gamma(\theta+\eta)}u(x+\theta, y+\eta)}{\left(1+(x+\theta)^2\right)\left(1+(y+\eta)^2\right)} d\eta\, d\theta \right. \\
&\qquad \left. - \frac{4e^{x+y}}{c_p\pi^2(e^{x+y} + e^{-x-y})} \int_{-\infty}^{-x}\int_{-\infty}^{-y} \frac{e^{\gamma(\theta+\eta)}v(x+\theta, y+\eta)}{\left(1+(x+\theta)^2\right)\left(1+(y+\eta)^2\right)} d\eta\, d\theta \right| \\
&\quad \leq \frac{4e^{x+y}}{c_p\pi^2(e^{x+y} + e^{-x-y})} \int_{-\infty}^{-x}\int_{-\infty}^{-y} \frac{e^{\gamma(\theta+\eta)}\left|u(x+\theta, y+\eta) - v(x+\theta, y+\eta)\right|}{\left(1+(x+\theta)^2\right)\left(1+(y+\eta)^2\right)} d\eta\, d\theta \qquad (6.16) \\
&\quad \leq \frac{4e^{x+y}}{c_p\pi^2(e^{x+y} + e^{-x-y})} \iint_{-\infty}^{0} \frac{e^{\gamma(\theta+\eta)}\left|u(\theta,\eta) - v(\theta,\eta)\right|}{(1+\theta^2)(1+\eta^2)} d\eta\, d\theta \\
&\quad \leq \frac{4e^{x+y}}{c_p\pi^2(e^{x+y} + e^{-x-y})} \iint_{0}^{\infty} \frac{1}{(1+\theta^2)(1+\eta^2)} d\eta\, d\theta \|u - v\|_{\gamma} \\
&\quad \leq \frac{e^{x+y}}{c_p(e^{x+y} + e^{-x-y})} \|u - v\|_{\gamma}.
\end{aligned}
$$

Hence, condition $(H'2)$ is satisfied with $l'_p(x, y) = e^{x+y}/c_p(e^{x+y} + e^{-x-y})$. Since

$$
l'^{*}_p = \sup\left\{ \frac{e^{x+y}}{c_p(e^{x+y} + e^{-x-y})} : (x, y) \in [0, \infty) \times [0, \infty) \right\} \leq \frac{1}{c_p} \qquad (6.17)
$$

and $K = 1$, we have

$$
\frac{K l'^{*}_p p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} = \frac{1}{3} < 1. \qquad (6.18)
$$

Hence, condition (5.6) holds for each $(r_1, r_2) \in (0, 1] \times (0, 1]$ and all $p \in \mathbb{N}^*$. Consequently Theorem 5.2 implies that problem (6.6) has a unique global solution defined on $\mathbb{R}^2$.

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