Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2011, Article ID 571804, 13 pages
doi:10.1155/2011/571804

Research Article

Positive Solutions of m-Point Boundary Value Problems for Fractional Differential Equations

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Received 28 November 2010; Accepted 19 January 2011

Academic Editor: Toka Diagana

We discuss the existence of minimal and maximal positive solutions for fractional differential equations with multipoint boundary value conditions, and new results are given. An example is also given to illustrate the abstract results.

1. Introduction

Recently, [1] discussed the existence of positive solutions for the following boundary value problem of fractional order differential equation

$$\begin{gathered} D_{0+}^{\alpha}u(t) + f(t,u(t)) = 0, \quad 0 < t < 1, \\ u(0) = 0, \quad D_{0+}^{\beta}u(1) = aD_{0+}^{\beta}u(\xi), \end{gathered} \tag{1.1}$$

where D_{0+}^{α} is the standard Riemann-Liouville fractional derivative of order $1 < \alpha \le 2$, $0 \le \beta \le 1$, $0 \le a \le 1$, $\xi \in (0,1)$, $a\xi^{\alpha-\beta-2} \le 1-\beta$, $0 \le \alpha - \beta - 1$ and $f : [0,1] \times [0,\infty) \to [0,\infty)$ satisfies Carathéodory-type conditions. Moreover, [2] considered the following nonlinear m-point boundary value problem of fractional type:

$$\begin{gathered} D^{\alpha}x(t) + q(t)f(t,x(t)) = 0, \quad \text{a.e. on } [0,1],\ \alpha \in (n-1,n],\ n \ge 2, \\ x(0) = x'(0) = x''(0) = \cdots = x^{(n-2)}(0) = 0, \qquad x(1) = \sum_{i=1}^{m-2} \zeta_i x(\eta_i), \end{gathered} \tag{1.2}$$

where x takes values in a reflexive Banach space E,

$$0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1, \tag{1.3}$$

$\zeta_i > 0$ with $\sum_{i=1}^{m-2} \zeta_i \eta_i^{\alpha-1} < 1$ and $x^{(k)}$ denotes the kth Pseudo-derivative of x, D^α denotes the Pseudo fractional differential operator of order α, $q(\cdot)$ is a continuous real-valued function on $[0,1]$, and f is a vector-valued Pettis-integrable function.

In this paper, we consider the existence of minimal and maximal positive solutions for the following multiple-point boundary value problem:

$$\begin{gathered} D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = 0, \quad D_{0+}^{\beta} u(1) = \sum_{i=1}^{m-2} \xi_i D_{0+}^{\beta} u(\eta_i), \end{gathered} \tag{1.4}$$

where D_{0+}^{α} is the standard Riemann-Liouville fractional derivative,

$$D_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} u(s) ds, \tag{1.5}$$

$n = [\alpha] + 1$, $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous, $1 < \alpha \leq 2$, $0 \leq \beta \leq 1$, $0 \leq \alpha - \beta - 1$, $0 < \xi_i, \eta_i < 1$, $i = 1, 2, \ldots, m-2$, and

$$\sum_{i=1}^{m-2} \xi_i \eta_i^{\alpha-\beta-1} < 1. \tag{1.6}$$

New results on the problem will be obtained.

Recall the following well-known definition and lemma (for more details on cone theory, see [3]).

Definition 1.1. Let E be a real Banach space. Then,

(a) a nonempty convex closed set $P \subset E$ is called a cone if it satisfied the following two conditions:

(i) $x \in P$, $\lambda \geq 0$ implies $\lambda x \in P$,

(ii) $x \in P$, $-x \in P$ implies $x = \theta$, where θ denotes the zero element of E.

(b) a cone P is said to be normal if there exists a constant $N > 0$ such that $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$.

Lemma 1.2. *Assume that $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L(0,1)$. Then,*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N}, \quad \textit{for some } C_i \in R, \ i = 1, 2, \ldots, N, \tag{1.7}$$

where N is the smallest integer greater than or equal to α.

2. Main Results

Let $E = C[0,1]$ and $P = \{u \in E : u(t) \geq 0, t \in [0,1]\}$. Then, E is the Banach space endowed with the norm $\|u\| = \sup_{0\leq t\leq 1}|u(t)|$ and P is normal cone.

We list the following assumptions to be used in this paper.

(H_1) there exist two nonnegative real-valued functions $p, q \in L[0,1]$, such that

$$f(s, x(s)) \leq p(s) + q(s)x(s), \quad s \in [0,1],\ x(s) \in [0,\infty). \tag{2.1}$$

(H_2) for $s \in [0,1]$, $0 \leq v_1 \leq v_2$ implies $f(s, v_1) \leq f(s, v_2)$.

In the following, we will prove our main results.

Lemma 2.1. *Let $y \in C[0,1]$. Then, the fractional differential equation*

$$\begin{gathered} D_{0+}^{\alpha}u(t) + y(t) = 0, \quad 0 < t < 1,\ 1 < \alpha \leq 2, \\ u(0) = 0, \quad D_{0+}^{\beta}u(1) = \sum_{i=1}^{m-2}\xi_i D_{0+}^{\beta}u(\eta_i), \end{gathered} \tag{2.2}$$

has a unique solution which is given by

$$u(t) = \int_0^1 G(t,s)y(s)ds, \tag{2.3}$$

where

$$G(t,s) = G_1(t,s) + G_2(t,s), \tag{2.4}$$

in which

$$\begin{gathered} G_1(t,s) = \begin{cases} \dfrac{t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \dfrac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1, \end{cases} \\ G_2(t,s) = \begin{cases} \dfrac{1}{A\Gamma(\alpha)}\left[\displaystyle\sum_{0\leq s\leq \eta_i}\left(\xi_i {\eta_i}^{\alpha-\beta-1}t^{\alpha-1}(1-s)^{\alpha-\beta-1} - \xi_i t^{\alpha-1}(\eta_i - s)^{\alpha-\beta-1}\right)\right], & t \in [0,1], \\ \dfrac{1}{A\Gamma(\alpha)}\left(\displaystyle\sum_{\eta_i\leq s\leq 1}\xi_i {\eta_i}^{\alpha-\beta-1}t^{\alpha-1}(1-s)^{\alpha-\beta-1}\right), & t \in [0,1], \end{cases} \end{gathered} \tag{2.5}$$

where

$$A = 1 - \sum_{i=1}^{m-2} \xi_i \eta_i^{\alpha-\beta-1}. \tag{2.6}$$

Proof. Using Lemma 1.2, we have

$$u(t) = -I_{0+}^{\alpha} y(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2}. \tag{2.7}$$

It follows from the condition $u(0) = 0$ that $C_2 = 0$.

Thus,

$$u(t) = -I_{0+}^{\alpha} y(t) + C_1 t^{\alpha-1}. \tag{2.8}$$

This, together with the relation $D_{0+}^{\alpha} t^{\gamma} = (\Gamma(\gamma+1)/\Gamma(\gamma-\alpha+1)) t^{\gamma-\alpha}$, yields

$$\begin{aligned} D_{0+}^{\beta} u(t) &= -I_{0+}^{\alpha-\beta} y(t) + C_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} t^{\alpha-\beta-1} \\ &= -\frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} y(s) ds + C_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} t^{\alpha-\beta-1}. \end{aligned} \tag{2.9}$$

From the boundary value condition $D_{0+}^{\beta} u(1) = \sum_{i=1}^{m-2} \xi_i D_{0+}^{\beta} u(\eta_i)$, we deduce that

$$\begin{aligned} C_1 &= \frac{1}{A\Gamma(\alpha)} \left(\int_0^1 (1-s)^{\alpha-\beta-1} y(s) ds - \sum_{i=1}^{m-2} \xi_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-\beta-1} y(s) ds \right) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\beta-1} y(s) ds + \frac{1}{A\Gamma(\alpha)} \sum_{i=1}^{m-2} \xi_i \eta_i^{\alpha-\beta-1} \int_0^1 (1-s)^{\alpha-\beta-1} y(s) ds \\ &\quad - \frac{1}{A\Gamma(\alpha)} \sum_{i=1}^{m-2} \xi_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-\beta-1} y(s) ds. \end{aligned} \tag{2.10}$$

Thus,

$$
\begin{aligned}
u(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1}(1-s)^{\alpha-\beta-1}y(s)ds \\
&\quad + \frac{1}{A\Gamma(\alpha)}\sum_{i=1}^{m-2}\xi_i\eta_i^{\alpha-\beta-1}\int_0^1 t^{\alpha-1}(1-s)^{\alpha-\beta-1}y(s)ds \\
&\quad - \frac{1}{A\Gamma(\alpha)}\sum_{i=1}^{m-2}\xi_i\int_0^{\eta_i} t^{\alpha-1}(\eta_i-s)^{\alpha-\beta-1}y(s)ds \\
&= \frac{1}{\Gamma(\alpha)}\int_0^t\Big(t^{\alpha-1}(1-s)^{\alpha-\beta-1}-(t-s)^{\alpha-1}\Big)y(s)ds + \frac{1}{\Gamma(\alpha)}\int_t^1 t^{\alpha-1}(1-s)^{\alpha-\beta-1}y(s)ds \\
&\quad + \frac{1}{A\Gamma(\alpha)}\xi_1\eta_1^{\alpha-\beta-1}\int_0^{\eta_1} t^{\alpha-1}(1-s)^{\alpha-\beta-1}y(s)ds + \frac{1}{A\Gamma(\alpha)}\xi_1\eta_1^{\alpha-\beta-1}\int_{\eta_1}^1 t^{\alpha-1}(1-s)^{\alpha-\beta-1}y(s)ds \\
&\quad - \frac{1}{A\Gamma(\alpha)}\xi_1\int_0^{\eta_1} t^{\alpha-1}(\eta_1-s)^{\alpha-\beta-1}y(s)ds \\
&\quad + \cdots \\
&\quad + \frac{1}{A\Gamma(\alpha)}\xi_{m-2}\eta_{m-2}^{\alpha-\beta-1}\int_0^{\eta_{m-2}} t^{\alpha-1}(1-s)^{\alpha-\beta-1}y(s)ds \\
&\quad + \frac{1}{A\Gamma(\alpha)}\xi_{m-2}\eta_{m-2}^{\alpha-\beta-1}\int_{\eta_{m-2}}^1 t^{\alpha-1}(1-s)^{\alpha-\beta-1}y(s)ds \\
&\quad - \frac{1}{A\Gamma(\alpha)}\xi_{m-2}\int_0^{\eta_{m-2}} t^{\alpha-1}(\eta_{m-2}-s)^{\alpha-\beta-1}y(s)ds \\
&= \int_0^1 G_1(t,s)y(s)ds + \int_0^1 G_2(t,s)y(s)ds \\
&= \int_0^1 G(t,s)y(s)ds.
\end{aligned}
\tag{2.11}
$$

The proof is complete. □

Lemma 2.2. *If $\sum_{i=1}^{m-2}\xi_i\eta_i^{\alpha-\beta-1} < 1$, then function $G(t,s)$ in Lemma 2.1 satisfies the following conditions:*

(i) $G(t,s) > 0$, *for* $s,t \in (0,1)$,

(ii) $G(t,s) \le \overline{G}(t,s) \le G_*(s,s)$, *for* $s,t \in [0,1]$,

where

$$
\overline{G}(t,s) = \overline{G}_1(t,s) + \overline{G}_2(t,s),
\tag{2.12}
$$

in which

$$\begin{gathered}
\overline{G}_1(t,s) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1}(1-s)^{\alpha-\beta-1}, \\
\overline{G}_2(t,s) = \frac{1}{A\Gamma(\alpha)} \sum_{i=1}^{m-2} \xi_i \eta_i^{\alpha-\beta-1} t^{\alpha-1}(1-s)^{\alpha-\beta-1}, \\
G_*(s,s) = \max_{t\in[0,1]} \overline{G}_1(t,s) + \max_{t\in[0,1]} \overline{G}_2(t,s).
\end{gathered} \tag{2.13}$$

Proof. When $0 < s \le t < 1$, we have

$$t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1} = (t-ts)^{\alpha-1}(1-s)^{-\beta} - (t-s)^{\alpha-1} > 0. \tag{2.14}$$

Thus, $G_1(t,s) > 0$ for $s,t \in (0,1)$.

Furthermore, we conclude that

$$\begin{aligned}
\xi_i t^{\alpha-1}(\eta_i - s)^{\alpha-\beta-1} &= \xi_i \eta_i^{\alpha-\beta-1} t^{\alpha-1}\left(1 - \frac{s}{\eta_i}\right)^{\alpha-\beta-1} \\
&\le \xi_i \eta_i^{\alpha-\beta-1} t^{\alpha-1}(1-s)^{\alpha-\beta-1}.
\end{aligned} \tag{2.15}$$

So, $G_2(t,s) \ge 0$ for $s,t \in [0,1]$. This, together with $G_1(t,s) > 0$ for $s,t \in (0,1)$, yields $G(t,s) > 0$ for $t,s \in (0,1)$.

Observing the express of $G(t,s)$, $\overline{G}(t,s)$, and $G_*(t,s)$, we see that (ii) holds.

The proof is complete. □

Remark 2.3. From the express of $\overline{G}_1(t,s)$ and $\overline{G}_2(t,s)$, we see that

$$\begin{gathered}
\max_{t\in[0,1]} \overline{G}_1(t,s) = \frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-\beta-1}, \\
\max_{t\in[0,1]} \overline{G}_2(t,s) = \frac{1}{A\Gamma(\alpha)} \sum_{i=1}^{m-2} \xi_i \eta_i^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1}.
\end{gathered} \tag{2.16}$$

Thus,

$$G_*(s,s) = \frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-\beta-1} + \frac{1}{A\Gamma(\alpha)} \sum_{i=1}^{m-2} \xi_i \eta_i^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1}. \tag{2.17}$$

Now, we define an operator $T : P \to P$ by

$$(Tu)(t) = \int_0^1 G(t,s) f(s,u(s)) ds. \tag{2.18}$$

Theorem 2.4. *Let condition* (H_1) *be satisfied. Suppose that* $\int_0^1 G_*(s,s)q(s)ds < 1$. *Then, problem* (1.4) *has at least one positive solution.*

Proof. Let $B_r = \{u \in E, \|u\| \le r\}$, where

$$r = \frac{\int_0^1 G_*(s,s)p(s)ds}{1 - \int_0^1 G_*(s,s)q(s)ds}. \tag{2.19}$$

Step 1. $T : B_r \to B_r$, for any $u \in B_r$

$$\begin{aligned}
|(Tu)(t)| &= \left| \int_0^1 G(t,s) f(s,u(s))ds \right| \\
&\le \int_0^1 G_*(s,s)\big(p(s) + q(s)u(s)\big)ds \\
&\le \int_0^1 G_*(s,s)p(s)ds + \int_0^1 G_*(s,s)q(s)ds\|u\| \\
&\le \int_0^1 G_*(s,s)p(s)ds + r\int_0^1 G_*(s,s)q(s)ds \\
&= r,
\end{aligned} \tag{2.20}$$

which implies that $\|Tu\| \le r$.

Step 2. $T : B_r \to B_r$ is continuous.
It is obvious from $f \in C([0,1] \times [0,\infty), [0,\infty))$.

Step 3. $T(B_r)$ is equicontinuous.
From (2.11) and (2.18), for any $t_1, t_2 \in [0,1]$, $t_1 < t_2$, $u \in B_r$, we conclude that

$$\begin{aligned}
|(Tu)(t_2) - (Tu)(t_1)| &= \left| \int_0^1 (G(t_2,s) - G(t_1,s)) f(s,u(s))ds \right| \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left| (t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right| \big(p(s) + rq(s)\big)ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \big(p(s) + rq(s)\big)ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^1 \left| t_2^{\alpha-1} - t_1^{\alpha-1} \right| (1 - s)^{\alpha-\beta-1} \big(p(s) + rq(s)\big)ds
\end{aligned}$$

$$+\frac{1}{A\Gamma(\alpha)}\sum_{i=1}^{m-2}\xi_i\eta_i^{\alpha-\beta-1}\int_0^1\left|t_2^{\alpha-1}-t_1^{\alpha-1}\right|(1-s)^{\alpha-\beta-1}(p(s)+rq(s))ds$$

$$+\frac{1}{A\Gamma(\alpha)}\sum_{i=1}^{m-2}\xi_i\int_0^{\eta_i}\left|t_2^{\alpha-1}-t_1^{\alpha-1}\right|(\eta_i-s)^{\alpha-\beta-1}(p(s)+rq(s))ds. \tag{2.21}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero, so, $T(B_r)$ is equicontinuous.

By the Arzelá-Ascoli theorem, we conclude that the operator $T: B_r \to B_r$ is completely continuous. Thus, our conclusion follows from Schauder fixed point theorem, and the proof is complete.

□

Theorem 2.5. *Besides the hypotheses of Theorem 2.4, we suppose that (H_2) holds. Then, BVP (1.4) has minimal positive solution $\overline{u}$ in B_r and maximal positive solution $\overline{w}$ in B_r; Moreover, $v_m(t) \to \overline{u}(t)$, $w_m(t) \to \overline{w}(t)$ as $m \to \infty$ uniformly on $[0,1]$, where*

$$v_m(t) = \int_0^1 G(t,s)f(s,v_{m-1}(s))ds, \tag{2.22}$$

$$w_m(t) = \int_0^1 G(t,s)f(s,w_{m-1}(s))ds. \tag{2.23}$$

Proof. By Theorem 2.4, we know that BVP (1.4) has at least one positive solution in B_r.

Step 1. BVP (1.4) has a positive solution in B_r, which is minimal positive solution.

From (2.18) and (2.22), one can see that

$$v_m(t) = (Tv_{m-1})(t), \quad t \in [0,1],\ m = 1,2,3,\ldots. \tag{2.24}$$

This, together with (H_2), yields that

$$0 = v_0(t) \le v_1(t) \le \cdots \le v_m(t) \le \cdots, \quad t \in [0,1]. \tag{2.25}$$

From $v_0 \in B_r$ and the proof of Theorem 2.4, it may be concluded that $v_m \in B_r$ and $Tv_m \in B_r$.

Let

$$W = \{v_m : m = 0,1,2,\ldots\}, \quad TW = \{Tv_m : m = 0,1,2,\ldots\}. \tag{2.26}$$

Thus,

$$W = \{v_0\} \cup TW, \quad W \subset B_r,\ T: W \longrightarrow W. \tag{2.27}$$

By the complete community of T, we know that TW is relatively compact. So, there exists a $\overline{u} \in E$ and a subsequence

$$\left\{v_{m_j} : j = 1,2,3,\ldots\right\} \subset W \tag{2.28}$$

such that $\{v_{m_j} : j = 1,2,3,\ldots\}$ converges to $\overline{u}$ uniformly on $[0,1]$. Since P is normal and $\{v_m(t) : m = 1,2,\ldots\}$ is nondecreasing, it is easily seen that the entire sequence $\{v_m(t) : m = 1,2,\ldots\}$ converges to $\overline{u}(t)$ uniformly on $[0,1]$. B_r being closed convex set in E and $v_m \in B_r$ imply that $\overline{u} \in B_r$.

From

$$f \in C([0,1] \times [0,\infty), [0,\infty)) \tag{2.29}$$

and (H_1), we see that

$$\begin{aligned} f(s, v_m(s)) \longrightarrow f(s, \overline{u}(s)) \quad &\text{as } m \longrightarrow \infty, \text{ for } s \in [0,1], \\ G(t,s) f(s, v_m(s)) &\le G_*(s,s) f(s, v_m(s)) \\ &\le \left(\frac{1}{\Gamma(\alpha)} + \frac{1}{A\Gamma(\alpha)} \sum_{i=1}^{m-2} \xi_i \eta_i^{\alpha-\beta-1}\right) f(s, v_m(s)) \\ &\le \left(\frac{1}{\Gamma(\alpha)} + \frac{1}{A\Gamma(\alpha)} \sum_{i=1}^{m-2} \xi_i \eta_i^{\alpha-\beta-1}\right) (p(s) + rq(s)) \in L[0,1]. \end{aligned} \tag{2.30}$$

By (2.30), (2.22), and Lebesgue's dominated convergence theorem, we get

$$\overline{u}(t) = \int_0^1 G(t,s) f(s, \overline{u}(s)) ds. \tag{2.31}$$

Let $u(t)$ be any positive solution of BVP (1.4) in B_r. It is obvious that $0 = v_0(t) \le u(t) = (Tu)(t)$.

Thus,

$$v_m(t) \le u(t) \quad (m = 0,1,2,3,\ldots). \tag{2.32}$$

Taking limits as $m \to \infty$ in (2.32), we get $\overline{u}(t) \le u(t)$ *for* $t \in [0,1]$.

Step 2. BVP (1.4) has a positive solution in B_r, which is maximal positive solution.

Let

$$w_0(t) = \int_0^1 \overline{G}(t,s)(p(s) + rq(s)) ds. \tag{2.33}$$

It is obvious that

$$|w_0(t)| \leq \int_0^1 G_*(s,s)\big(p(s)+rq(s)\big)ds = r. \tag{2.34}$$

Thus, $\|w_0\| \leq r$ and $w_0 \in B_r$.

By (2.18), (2.23), and (H_1), we have

$$\begin{aligned} w_1(t) = (Tw_0)(t) &= \int_0^1 G(t,s)f(s,w_0(s))ds \\ &\leq \int_0^1 \overline{G}(t,s)\big(p(s)+q(s)w_0(s)\big)ds \\ &\leq \int_0^1 \overline{G}(t,s)\big(p(s)+rq(s)\big)ds \\ &= w_0(t). \end{aligned} \tag{2.35}$$

This, together with (H_2), yields that

$$\cdots \leq w_m(t) \leq \cdots \leq w_1(t) \leq w_0(t), \quad t \in [0,1]. \tag{2.36}$$

Using a proof similar to that of Step 1, we can show that

$$\begin{gathered} w_m(t) \longrightarrow \overline{w}(t) \quad (m \longrightarrow \infty), \\ \overline{w}(t) = \int_0^1 G(t,s)f(s,\overline{w}(s))ds. \end{gathered} \tag{2.37}$$

Let $u(t)$ be any positive solution of BVP (1.4) in B_r.

Obviously,

$$u(t) = (Tu)(t) = \int_0^1 G(t,s)f(s,u(s))ds \leq \int_0^1 \overline{G}(t,s)\big(p(s)+rq(s)\big)ds = w_0(t). \tag{2.38}$$

This, together with (H_2), implies

$$u(t) \leq w_m(t). \tag{2.39}$$

Taking limits as $m \longrightarrow \infty$ in (2.39), we obtain $u(t) \leq \overline{w}(t)\ \ for\ t \in [0,1]$.

The proof is complete. □

On the other hand, we note that in these years, going with the significant developments of various differential equations in abstract spaces (cf., e.g., [3–17] and references therein), fractional differential equations in Banach spaces have also been investigated by many authors (cf. e.g., [1, 2, 18–26] and references therein). In our coming papers, we will present more results on fractional differential equations in Banach spaces.

3. An Example

Example 3.1. Consider the following boundary value problem

$$\begin{gathered} D_{0+}^{3/2}u(t) + \frac{u}{1+u}t^{11} + e^t + 1 = 0, \quad 0 < t < 1, \\ u(0) = 0, \quad D_{0+}^{1/5}u(1) = \sum_{i=1}^{2}\xi_i D_{0+}^{1/5}u(\eta_i), \end{gathered} \tag{3.1}$$

where $\alpha = 3/2$, $\beta = 1/5$, $m = 4$, $\xi_1 = \eta_1 = 1/4$, $\xi_2 = \eta_2 = 1/2$,

$$f(t,u) = \frac{u}{1+u}t^{11} + e^t + 1, \tag{3.2}$$

$p(t) = e^t + 1$, $q(t) = t^{11}$. By computation, we deduce that

$$\begin{gathered} \sum_{i=1}^{2}\xi_i\eta_i^{\alpha-\beta-1} = \sum_{i=1}^{2}\xi_i\eta_i^{3/10} < \sum_{i=1}^{2}\xi_i = \frac{1}{4} + \frac{1}{2} = \frac{3}{4} < 1, \\ A = 1 - \sum_{i=1}^{2}\xi_i\eta_i^{\alpha-\beta-1} > \frac{1}{4}, \qquad \frac{1}{A} < 4, \qquad \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}. \end{gathered} \tag{3.3}$$

From Remark 2.3, we get

$$G_*(s,s) = \frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-\beta-1} + \frac{1}{A\Gamma(\alpha)}\sum_{i=1}^{2}\xi_i\eta_i^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1}. \tag{3.4}$$

Therefore,

$$\begin{aligned} \int_0^1 G_*(s,s)q(s)ds &= \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\beta-1}s^{11}ds + \frac{1}{A\Gamma(\alpha)}\sum_{i=1}^{2}\xi_i\eta_i^{\alpha-\beta-1}\int_0^1 (1-s)^{\alpha-\beta-1}s^{11}ds \\ &\le \frac{2}{\sqrt{\pi}}\int_0^1 s^{11}ds + 4 \times \frac{2}{\sqrt{\pi}} \times \frac{3}{4}\int_0^1 s^{11}ds \\ &= \frac{1}{6\sqrt{\pi}} + \frac{1}{2\sqrt{\pi}} \\ &= \frac{2}{3\sqrt{\pi}} < 1. \end{aligned} \tag{3.5}$$

On the one hand, it is obvious that $f(t,u) \le p(t) + q(t)u$. Thus, (H_1) is satisfied. For $u_1 \le u_2$, we see that $f(t,u_1) \le f(t,u_2)$, which implies that (H_2) holds.

Hence, by Theorem 2.5, BVP (3.1) has minimal and maximal positive solutions in B_r. Furthermore, we can conclude that

$$\begin{aligned}
\int_0^1 G_*(s,s)p(s)ds &= \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\beta-1}(e^s+1)ds + \frac{1}{A\Gamma(\alpha)}\sum_{i=1}^{2}\xi_i\eta_i^{\alpha-\beta-1}\int_0^1 (1-s)^{\alpha-\beta-1}(e^s+1)ds \\
&\le \frac{2}{\sqrt{\pi}}\int_0^1 (e^s+1)ds + 4\times\frac{2}{\sqrt{\pi}}\times\frac{3}{4}\int_0^1 (e^s+1)ds \\
&= \frac{2e}{\sqrt{\pi}} + \frac{6e}{\sqrt{\pi}} \\
&= \frac{8e}{\sqrt{\pi}}, \\
r &= \frac{\int_0^1 G_*(s,s)p(s)ds}{1-\int_0^1 G_*(s,s)q(s)ds} \le \frac{8e/\sqrt{\pi}}{1-2/3\sqrt{\pi}} = \frac{24e}{3\sqrt{\pi}-2}.
\end{aligned} \tag{3.6}$$

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