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Research Article

Generalized Probability Functions

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From the integration of nonsymmetrical hyperboles, a one-parameter generalization of the logarithmic function is obtained. Inverting this function, one obtains the generalized exponential function. Motivated by the mathematical curiosity, we show that these generalized functions are suitable to generalize some probability density functions (pdfs). A very reliable rank distribution can be conveniently described by the generalized exponential function. Finally, we turn the attention to the generalization of one- and two-tail stretched exponential functions. We obtain, as particular cases, the generalized error function, the Zipf-Mandelbrot pdf, the generalized Gaussian and Laplace pdf. Their cumulative functions and moments were also obtained analytically.

1. Introduction

The convenience of generalizing the logarithmic function has attracted the attention of researchers since long ago [1] and particularly in the last years [2–7]. In Physics, several one-parameter generalizations of the logarithmic function have been proposed in different contexts such as nonextensive statistical mechanics [8–13], relativistic statistical mechanics [14, 15], and quantum group theory [16]. Also, more sophisticated such as two-parameter [17] and three-parameter [18] generalizations have been proposed, each one including previous situations as particular cases. Examples of the convenience of these generalizations have been seen in different fields, for instance, psychophysics [19], neuroeconomics [20, 21], econophysics [22, 23], complex networks [24, 25], population dynamics [26, 27], and so forth.

Here, our main objective is to show that the generalized stretched exponential function, written as probability density function (pdf), is suitable to generalize a wide range of one- and two-tail pdfs. This approach, which stresses the emergence of several probability distributions, is motivated by a mathematical curiosity. In Section 2, we show that from the integration of nonsymmetrical hyperboles, one obtains a one-parameter generalization of the logarithmic function, which we call $\tilde{q}$-logarithm. This generalization coincides with the one obtained in the context of nonextensive thermostatistics [9, 10]. Inverting the $\tilde{q}$-logarithm, one obtains the generalized exponential ($\tilde{q}$-exponential) function. Some properties of these generalized functions are presented. In Section 3, the very reliable rank distribution obtained by Naumis and Cocho [28] is conveniently described by the $\tilde{q}$-exponential function, which permits us to detect the effect of finite sample size in the description. In Section 4, we first show that the Zipf-Mandelbrot function, which is a fingerprint of complex systems, can be conveniently written in terms of the $\tilde{q}$-exponential. Raising the $\tilde{q}$-exponential argument to a given power, one obtains a function that generalizes the stretched exponential function. Its generating differential equation is then presented. In Section 5, we consider the pdfs for continuous variables. First we consider the one-tail stretched exponential generalization and obtain analytically its cumulative function and moments. One obtains the generalized error function as a particular case and the Zipf-Mandelbrot pdf as another. Next, we consider the two-tail generalized stretched exponential pdf and obtain analytically its cumulative function and moments. One has the generalized Gaussian and the generalized Laplace pdf as particular cases. The characteristic function is analytically calculated. Our final remarks are drawn in Section 6.

2. The $\tilde{q}$-Generalized Functions

From the integration of nonsymmetrical hyperboles, we obtain a one-parameter generalization of the logarithmic function, which coincides with the one obtained in the context of nonextensive thermostatistics [9, 10]. Inverting this function, one obtains the generalized exponential function. Some properties of these generalized functions are presented.

2.1. $\tilde{q}$-Generalized Logarithm Function

In the one-parameter generalization we address here, the $\tilde{q}$-logarithm function $\ln_{\tilde{q}}(x)$ is defined as the value of the area underneath $1/t^{1-\tilde{q}}$, in the interval $t \in [1, x]$ [29]

$$\ln_{\tilde{q}}(x) = \int_1^x \frac{dt}{t^{1-\tilde{q}}} = \lim_{\tilde{q}' \to \tilde{q}} \frac{x^{\tilde{q}'} - 1}{\tilde{q}'}, \tag{2.1}$$

which is defined for all real values of $\tilde{q}$, but only for positive x. This is exactly the same function obtained from the nonextensive statistical mechanics context [8, 9], but here, only simple geometrical arguments have been used in the derivation.

The usual natural logarithm ($\ln x$) is retrieved for $\tilde{q} = 0$ and a linear function for $\tilde{q} = 1$. Scaling and deformation of the variable x are given by $\ln_{\tilde{q}}(\alpha x^\beta) = \beta \ln_{\beta\tilde{q}}(\alpha^{1/\beta} x)$, so that for $\beta = -1$, one has $\ln_{\tilde{q}}(\alpha/x) = -\ln_{-\tilde{q}}(x/\alpha)$ and for the particular case $\alpha = 1$, $\ln_{\tilde{q}}(x^{-1}) = -\ln_{-\tilde{q}}(x)$. Notice that using $\tilde{q}$ instead of $q = 1 - \tilde{q}$, as used in [8, 9], makes simpler to handle the scaling and deformation operations.

2.2. $\tilde{q}$-Generalized Exponential Function

The $\tilde{q}$-exponential function $e_{\tilde{q}}(x)$ is defined as the t-value, in such a way that the area underneath $f_{\tilde{q}}(t) = 1/t^{1-\tilde{q}}$, in the interval $t \in [1, e_{\tilde{q}}(x)]$, is x. This is the inverse of the

$\tilde{q}$-logarithm function $e_{\tilde{q}}[\ln_{\tilde{q}}(x)] = x = \ln_{\tilde{q}}[e_{\tilde{q}}(x)]$ and it is given by

$$e_{\tilde{q}}(x) = \begin{cases} 0, & \text{for } \tilde{q}x < -1, \\ \lim\limits_{\tilde{q}' \to \tilde{q}} (1 + \tilde{q}' x)^{1/\tilde{q}'}, & \text{for } \tilde{q}x \geq -1. \end{cases} \tag{2.2}$$

Notice that $(1 + \tilde{q}x)^{1/\tilde{q}}$ is real only if $\tilde{q}x \geq -1$.

This is a nonnegative function ($e_{\tilde{q}}(x) \geq 0$, for all $\tilde{q}$ and x) and $e_{\tilde{q}}(0) = 1$, independently of the $\tilde{q}$ value. For $\tilde{q} = 0$, one retrieves the usual exponential function e^x and for $\tilde{q} = 1$, a linear function. Notice that taking the surface underneath $1/t^{1-\tilde{q}}$ to be unitary, one generalizes the Euler's number: $e_{\tilde{q}} = e_{\tilde{q}}(1) = (1 + \tilde{q})^{1/\tilde{q}}$. An interesting property is that $[e_{\tilde{q}}(x)]^a = e_{\tilde{q}/a}(ax)$, meaning that the $\tilde{q}$-exponential argument scaling corresponds to a power of a different $\tilde{q}$-exponential function. For $a = -1$, one has that $e_{-\tilde{q}}(-x) = 1/e_{\tilde{q}}(x)$.

The derivative of the $\tilde{q}$-exponential function with respect to x is $\mathrm{d}e_{\tilde{q}}(kx)/\mathrm{d}x = k\,[e_{\tilde{q}}(kx)]^{1-\tilde{q}} = k\, e_{\tilde{q}/(1-\tilde{q})}[(1-\tilde{q})kx]$, so that it is the solution of the following nonlinear first-order differential equation: $\mathrm{d}y(x)/\mathrm{d}x = ky^{1-\tilde{q}}(x)$, which is a particular case of Bernoulli's differential equation: $[\mathrm{d}/\mathrm{d}x + p(x)]y(x) = q(x)y^{1-\tilde{q}}(x)$, with $p(x) = 0$ and $q(x) = k$. Notice that k has the dimension of $y^{\tilde{q}}$ over the dimension of x. This means that it sets up a scale (inversion dimension of x) to the problem only if $\tilde{q} = 0$. One important application of the above differential equation concerns reaction kinetics [30].

3. Beta-Like Distribution

Let us turn our attention to discrete random variables, the rank distribution in particular. We show that the rank distribution obtained by Naumis and Cocho [28] is conveniently described by the $\tilde{q}$-exponential function. In this way, we are able to quantify the finite size effects. This rank distribution is very reliable since it has an underlying microscopic model and has been validated by a wide range of experimental data [28].

To simultaneously fit the beginning, body and tail of experimental rank distributions of complex systems, Naumis and Cocho [28] consider N independent subsystems with a large number of internal states. The rank r of a system property dependent on the internal states of the subsystems decays as a two-free-parameter beta-like function: $f(r) = K[1 - r/(R+1)]^{\beta} r^{-\alpha}$, where $K = 1/\sum_{r=1}^{N}[1 - r/(R+1)]^{\beta} r^{-\alpha}$, with $R \leq N$ being the maximal value of r, K is the normalization factor, and the two free parameters are α and β. As noticed by the authors, if $R \gg 1$, then $K \approx 1/\mathrm{B}(1-\alpha, 1+\beta)$, where $\mathrm{B}(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ is the beta function [31] and $\Gamma(x) = \int_0^\infty dt\, t^{x-1} e^{-t}$, $x > 0$ is the gamma function [31].

Finite size effects are described by factor $[1 - r/(R+1)]^{\beta} \approx 1 - \beta r/R \approx e^{-r/r_0}$, for $R \gg r$ with $r_0 = R/\beta$. In this way, we see that the rank distribution of $f(r)$ is in fact a generalization of the standard technique of multiplying the power-law ($r^{-\alpha}$) by an exponential cutoff: $f(r) \propto r^{-\alpha} e^{-r/r_0}$.

If one writes $[1 - r/(R+1)]^{\beta} = [1 - (1/\beta)\beta r/(R+1)]^{\beta} = e_{1/\beta}[-\beta r/(R+1)]$ then

$$f(r) = \frac{K}{r^{\alpha}} e_{1/\beta}\left[\frac{-\beta r}{R+1}\right] = \frac{K}{r^{\alpha}}\left[e_1\left(\frac{-r}{R+1}\right)\right]^{\beta}, \tag{3.1}$$

meaning that the $\tilde{q}$-exponential function can properly take into account finite size effects. The exponential cutoff is retrieved where R and β grow, but R must grow faster than β, to have $R \to \infty$ and $\beta \to \infty$, with $\beta/R \to 0$.

4. Special Functions and Processes

In what follows, we first show that the Zipf-Mandelbrot function, which is a fingerprint of complex systems, can be written in terms of the $\tilde{q}$-exponential. Next, raising the $\tilde{q}$-exponential argument to a given power, one obtains a function that generalizes the stretched exponential function. Finally, we obtain the process (differential equation) of which the generalized stretched exponential is the solution.

4.1. Zipf-Mandelbrot Function

The envelope of a typical rank distribution of complex systems can be well described by the Zipf-Mandelbrot function [32], which may be written in terms of $\tilde{q}$-exponential function:

$$P_{\tilde{q},\alpha,A}(x) = \frac{d}{(c+x)^{\gamma}} = \frac{A}{e_{\tilde{q}}(x/\alpha)} = Ae_{-\tilde{q}}\left(\frac{-x}{\alpha}\right), \tag{4.1}$$

with $\tilde{q} = 1/\gamma$, $\alpha = c/\gamma$, and $A = d/c^{\gamma}$.

We remark that (4.1) has another interesting application in time-dependent luminescence spectroscopy. In this case the relaxation processes are known as Becquerel decay function [33].

4.2. Generalized Stretched Exponential Function

A deformation in the argument of (4.1) leads to the generalized stretched exponential function

$$P_{\tilde{q},\alpha,\beta,A}(x) = P_{\tilde{q},\alpha,A}\left(x^{1/\beta}\right) = \frac{d}{\left(d + x^{1/\beta}\right)^{\gamma}} = Ae_{-\tilde{q}}\left(\frac{-x^{1/\beta}}{\alpha}\right). \tag{4.2}$$

The usual stretched exponential function, also known as the Kohlrausch function [34–36], is obtained from (4.2) in the limit $\tilde{q} \to 0$. Although the (usual) stretched exponential function has been used to describe relaxation processes in time-dependent luminescence spectroscopy [36], a generalization of the form $e^{\beta/a}e^{-e_{1/\beta}(x/\alpha)/a}$ seems to be more convenient to fit experimetal data [37]. In this case we stress the $\tilde{q}$-exponential figures as the argument of the usual exponential function.

The stretched exponential function can also be obtained as the solution of the following differential equation: $dy(x)/dx = -x^{1/\beta-1}y(x)/(\alpha\beta)$, which can be written in terms of the logarithmic function (relative variation) as $d\ln y(x)/dx = -x^{1/\beta-1}/(\alpha\beta)$. If one replaces the logarithmic function in the differential equation obtained above by the $\tilde{q}$-logarithm, one obtains

$$\frac{d\ln_{-\tilde{q}} y(x)}{dx} = -\frac{x^{1/\beta-1}}{\alpha\beta}, \tag{4.3}$$

or equivalently

$$\frac{dy(x)}{dx} = -\frac{x^{1/\beta-1}}{\alpha\beta} y^{1+\tilde{q}}(x), \tag{4.4}$$

which is a particular case of Bernoulli's equation: $dy(x)/dx + p(x)y(x) = q(x)y^{1+\tilde{q}}(x)$, with $p(x) = 0$ and $q(x) = -x^{1/\beta-1}/(\alpha\beta)$ and which solution is precisely the generalized stretched exponential function of (4.2).

5. Probability Functions

Considering the factor $A > 0$ and using the nonnegativeness of (4.2), we write the generalization of the stretched exponential pdf and study its properties. We consider one- and two-tail distributions and obtain some known pdfs (generalized Gaussian) and new ones (generalized error function and generalized Laplace pdf) as particular cases.

5.1. One-Tail PDF

If the considered independent variable x is constrained to nonnegative (or eventually nonpositive) values, then one uses the one-tail pdf. We consider the generalization of the stretched exponential pdf and analytically obtain its cumulative function and moments. From this pdf, one obtains the generalized error function as a particular case and the Zipf-Mandelbrot pdf as another.

The normalization factor A of (4.2) is

$$\frac{1}{A} = \int_0^\infty dt\, e_{-\tilde{q}}\left(\frac{-t^{1/\beta}}{\alpha}\right) = \beta\left(\frac{\alpha}{\tilde{q}}\right)^{\beta} \mathrm{B}\left(\beta, \frac{1}{\tilde{q}} - \beta\right). \tag{5.1}$$

The integral of (5.1) does not diverge only if $0 < \tilde{q} < 1$, $0 < \beta < 1/\tilde{q}$, $\alpha > 0$, and one has the *generalized stretched exponential pdf*:

$$P^{(1)}_{\tilde{q},\alpha,\beta}(x) = \frac{(\tilde{q}/\alpha)^{\beta}}{\beta \mathrm{B}(\beta, 1/\tilde{q} - \beta)} e_{-\tilde{q}}\left[-\frac{x^{1/\beta}}{\alpha}\right]. \tag{5.2}$$

For our purposes, it is more convenient to write $a = (\alpha\beta)^{\beta} > 0$:

$$P^{(1)}_{\tilde{q},a,\beta}(x) = \frac{(\tilde{q}\beta)^{\beta}}{a\beta \mathrm{B}(\beta, 1/\tilde{q} - \beta)} e_{-\tilde{q}}\left[-\beta\left(\frac{x}{a}\right)^{1/\beta}\right], \tag{5.3}$$

which is depicted in Figures 1 and 2.

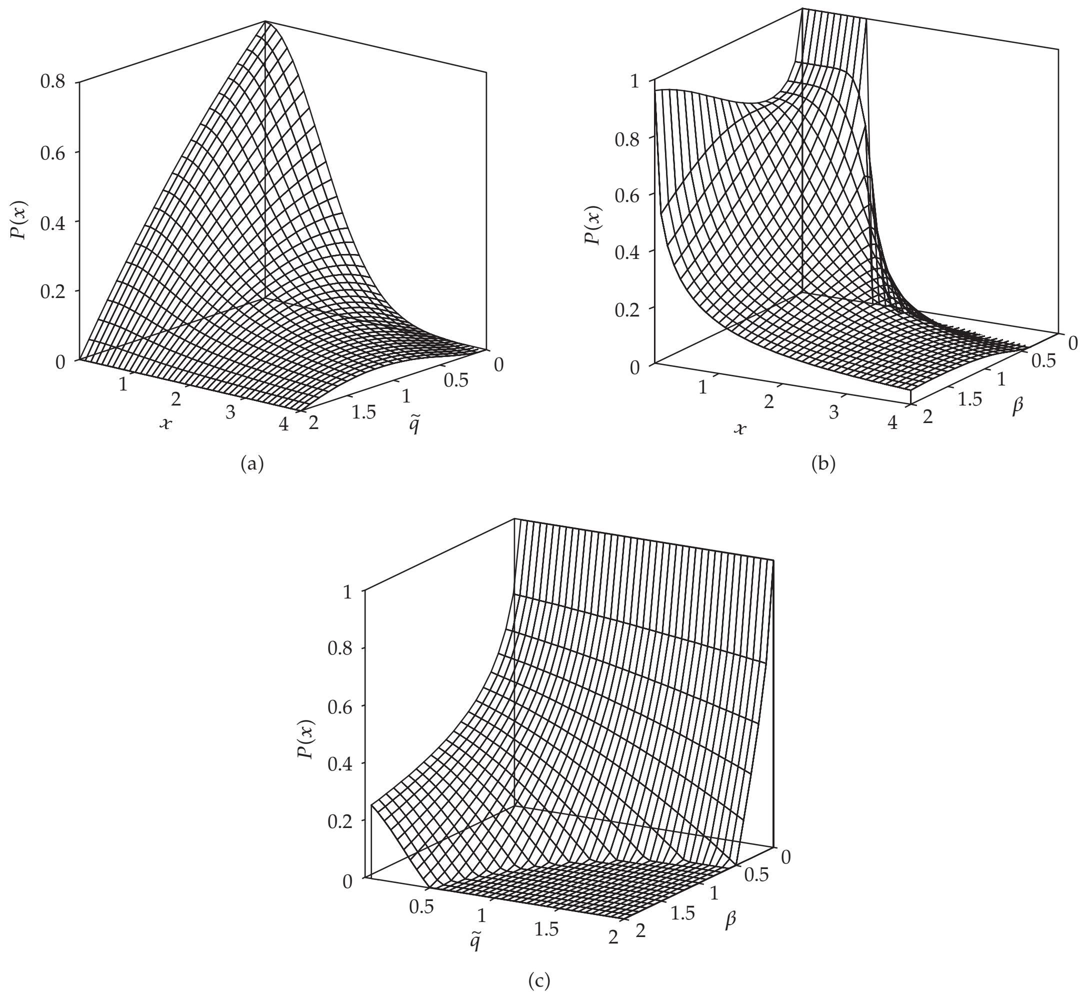

Figure 1: Behavior of (5.3) as a function of (a) x and $\tilde{q}$, with $\beta = 1/2$ and $a = 1$, (b) x and β, with $\tilde{q} = 1/5$ and $a = 1$, (c) $\tilde{q}$ and β, with $x = 1$ and $a = 1$.

As $\tilde{q} \to 0$, $\mathrm{B}(\beta, 1/\tilde{q} - \beta) \approx \tilde{q}^{\beta}\Gamma(\beta)$, since $1/\tilde{q} \gg \beta$, one retrieves the stretched exponential function $P^{(1)}_{0,a,\beta}(x) = \beta^{\beta} e^{-\beta(x/a)^{1/\beta}} / [a\beta\Gamma(\beta)]$ or $P^{(1)}_{0,\alpha,\beta}(x) = e^{-x^{1/\beta}/\alpha} / [\alpha^{\beta}\beta\Gamma(\beta)]$.

The cumulative function of (5.3) is

$$F^{(1)}_{\tilde{q},a,\beta}(x) = \int_0^x dt\, P^{(1)}_{\tilde{q},a,\beta}(t) = \frac{x}{A}\, {}_2\mathrm{F}_1\left(\beta, \frac{1}{q}; \beta + 1; -\frac{\tilde{q}x^{1/\beta}}{\alpha}\right), \tag{5.4}$$

where

$$ {}_2\mathrm{F}_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!} \tag{5.5}$$

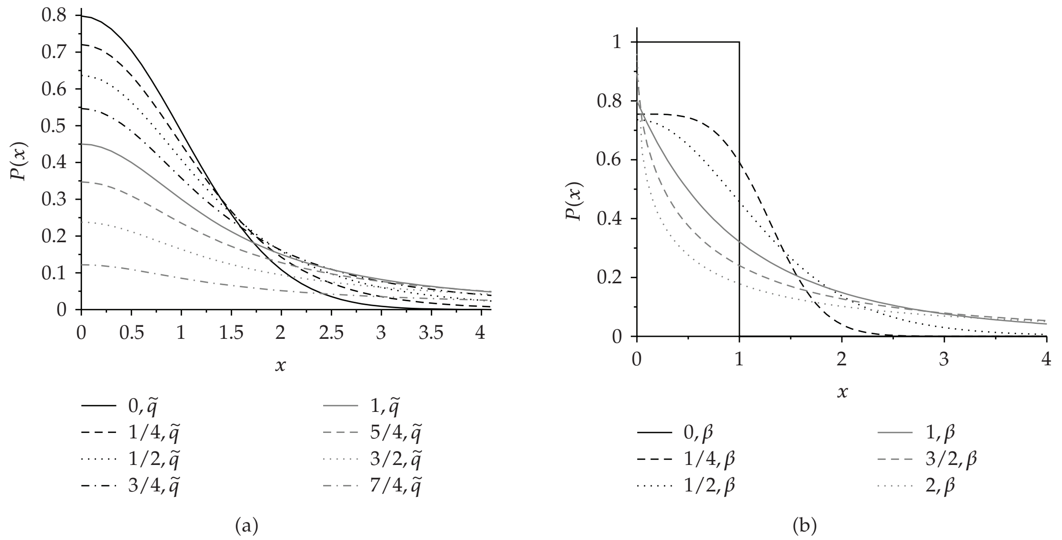

Figure 2: Behavior of (5.3) as a function of (a) x with $\beta = 1/2$ and $a = 1$ and for several $\tilde{q}$ values (projection of Figure 1(a)), (b) x with $\tilde{q} = 1/5$ and $a = 1$ and for several β values (projection of Figure 1(b)).

is the hypergeometric function [31] and

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1) = \frac{(a+n-1)!}{(a-1)!} = \frac{\Gamma(a+n)}{\Gamma(a)} \tag{5.6}$$

is the Pochhammer symbol.

The moments of x are

$$\langle x^n \rangle_{\tilde{q},a,\beta} = \left[\frac{a}{(\tilde{q}\beta)^{\beta}}\right]^n \frac{\mathrm{B}\left[(n+1)\beta, 1/\tilde{q} - (n+1)\beta\right]}{\mathrm{B}(\beta, 1/\tilde{q} - \beta)}, \tag{5.7}$$

where one sees that they are finite only if $\tilde{q} < 1/[(n+1)\beta]$.

If $\tilde{q} < 1/(3\beta)$, the mean value and variance are finite and, respectively, given by

$$\begin{aligned} \langle x \rangle_{\tilde{q},a,\beta} &= \left(\frac{\alpha}{\tilde{q}}\right)^{\beta} \frac{\mathrm{B}(2\beta, 1/\tilde{q} - 2\beta)}{\mathrm{B}(\beta, 1/\tilde{q} - \beta)}, \\ \frac{\sigma^2_{\tilde{q},a,\beta}}{\langle x \rangle^2_{\tilde{q},a,\beta}} &= \frac{\mathrm{B}(3\beta, 1/\tilde{q} - 3\beta)\mathrm{B}(\beta, 1/\tilde{q} - \beta)}{\mathrm{B}^2(2\beta, 1/\tilde{q} - 2\beta)} - 1. \end{aligned} \tag{5.8}$$

Notice that the ratio $\sigma^2_{\tilde{q},a,\beta}/\langle x \rangle^2_{\tilde{q},a,\beta}$ depends only on β and $\tilde{q}$, but not on a.

Particular values of β lead to a $\tilde{q}$-generalization of the error function and to the Zipf-Mandelbrot pdf.

5.1.1. Generalized Error Function

To generalize the error function, consider $\beta = 1/2$ and $a = \beta^{\beta}$ (or $\alpha = 1$) in (5.4) and one has

$$\mathrm{erf}_{\tilde{q}}(x) = F^{(1)}_{\tilde{q},1/\sqrt{2},1/2}(x) = \frac{2\sqrt{\tilde{q}}}{\mathrm{B}(1/2, 1/\tilde{q} - 1/2)} \int_0^x dt\, e_{-\tilde{q}}\left(-t^2\right). \tag{5.9}$$

As $\tilde{q} \to 0$, $\mathrm{B}(1/2, 1/\tilde{q} - 1/2) = \sqrt{\tilde{q}\pi}$ and one retrieves the standard error function: $\mathrm{erf}_0(x) = \mathrm{erf}(x) = (2/\sqrt{\pi})\int_0^x dt\, e^{-t^2}$.

An alternative way to obtain (5.9) can be found in [38].

5.1.2. Zipf-Mandelbrot PDF

For $\beta = 1$ in (5.2), one obtains the Zipf-Mandelbrot's pdf

$$P^{(1)}_{\tilde{q},\alpha,1}(x) = \frac{1-\tilde{q}}{\alpha} e_{-\tilde{q}}\left(\frac{-x}{\alpha}\right) = \frac{1-\tilde{q}}{\alpha} \frac{1}{(1+\tilde{q}x/\alpha)^{1/\tilde{q}}}, \tag{5.10}$$

where the mean value and the variance are

$$\begin{aligned} \langle x \rangle_{\tilde{q},\alpha,1} &= \frac{\alpha}{1-2\tilde{q}}, \\ \frac{\sigma^2_{\tilde{q},\alpha,1}}{\langle x \rangle^2_{\tilde{q},\alpha,1}} &= \frac{\tilde{q}-1}{3\tilde{q}-1}, \end{aligned} \tag{5.11}$$

which is finite for $0 \le \tilde{q} < 1/3$ and, from (5.4), one obtains its cumulative function:

$$F^{(1)}_{\tilde{q},a,1}(x) = 1 - \frac{1}{\left[e_{\tilde{q}}(x/a)\right]^{1-\tilde{q}}} = 1 - \tilde{F}^{(1)}_{\tilde{q},a,1}(x), \tag{5.12}$$

where the upper-tail distribution is simply given by

$$\begin{aligned} \tilde{F}_{\tilde{q},a,1}(x) &= \int_x^{\infty} dt\, P^{(1)}_{\tilde{q},a,1}(t) = \left(1 + \tilde{q}\frac{x}{a}\right) e_{-\tilde{q}}\left(\frac{-x}{a}\right) \\ &= \left[e_{-\tilde{q}}\left(\frac{-x}{a}\right)\right]^{1-\tilde{q}} = \left[e_{\tilde{q}}\left(\frac{x}{a}\right)\right]^{\tilde{q}-1}, \end{aligned} \tag{5.13}$$

which is more suitable for fitting the model to real data than the pdf (5.10) itself.

5.2. Two-Tail PDF

If the domain of the considered independent variable is not bounded, it is interesting to consider its absolute value $|x|$ in (5.3) and one has a symmetric pdf about the line $x = 0$. Notice that in this case, the normalization factor must be halved since the domain has been doubled in a symmetrical way, the generalized stretched exponential pdf is then

$$P^{(2)}_{\tilde{q},a,\beta}(x) = \frac{(\tilde{q}\beta)^{\beta}}{2a\beta \mathrm{B}(\beta, 1/\tilde{q}-\beta)} e_{-\tilde{q}}\left[-\beta\left(\frac{|x|}{a}\right)^{1/\beta}\right], \tag{5.14}$$

which is a convenient function to use in wavelets [39]. Its cumulative function is

$$F^{(2)}_{\tilde{q},a,\beta}(x) = \frac{1}{2}\left[1 + \mathrm{sgn}\,(x) F^{(1)}_{\tilde{q},a,\beta}(|x|)\right], \tag{5.15}$$

where $F^{(1)}_{\tilde{q},a,\beta}(x)$ is given by (5.4).

On one hand, due to its symmetry around $x = 0$, the odd moments of this pdf vanish $\langle x^{2n+1}\rangle = 0$, with $n = 0, 1, 2, \ldots$. On the other hand, the even moments $\langle x^{2n}\rangle$ are finite only if $\tilde{q} < 1/[(2n+1)\beta]$:

$$\left\langle x^{2n}\right\rangle_{\tilde{q},a,\beta} = \left[\frac{a}{(\tilde{q}\beta)^{\beta}}\right]^{2n} \frac{\mathrm{B}[(1+2n)\beta, 1/\tilde{q} - (1+2n)\beta]}{\mathrm{B}(\beta, 1/\tilde{q}-\beta)}. \tag{5.16}$$

In the following, we retrieve the generalized Gaussian as a particular case of (5.14). Also, as a new result, we propose to consider another particular case of (5.14) to generalize the Laplacian pdf. The characteristic function of both particular cases is analytically calculated.

5.2.1. Generalized Gaussian

An interesting particular case of the two-tail generalized stretched exponential function (5.14) is when $\beta = 1/2$, which leads to the $\tilde{q}$-Gaussian [40]

$$G_{\tilde{q},a}(x) = \frac{\sqrt{\tilde{q}}\,\Gamma(1/\tilde{q})}{\Gamma(1/\tilde{q}-1/2)} \frac{e_{-\tilde{q}}\left[-(x/a)^2/2\right]}{\sqrt{2\pi a^2}}. \tag{5.17}$$

Due to symmetry, all odd moments vanish. Even moments are given by (5.16)

$$\left\langle x^{2n}\right\rangle_{\tilde{q},a,1/2} = \left(\frac{2a^2}{\tilde{q}}\right)^{n} \frac{\Gamma(1/2+n)\Gamma(1/\tilde{q}-1/2-n)}{\sqrt{\pi}\,\Gamma(1/\tilde{q}-1/2)}, \tag{5.18}$$

and the variance is finite only for $\tilde{q} < 2/3$

$$\sigma^2_{\tilde{q},a} = \left\langle x^2\right\rangle_{\tilde{q},a} = \frac{2a^2}{2-3\tilde{q}}. \tag{5.19}$$

Using that for $a \gg b$, $\Gamma(a+b)/\Gamma(a) = a^b$, when $\tilde{q} \to 0$ in (5.17), $\Gamma(1/\tilde{q})/\Gamma(1/\tilde{q}-1/2) = 1/\tilde{q}^{1/2}$ and one has a Guassian, with variance a^2: $G_{0,a}(x) = e^{-(x/a)^2/2}/\sqrt{2\pi a^2}$, as a particular case. Another particular case is when $\tilde{q} = 1$ and one retrieves the Lorentzian (Cauchy pdf) $G_{1,a}(x) = 1/\{\pi a\sqrt{2}[1 + (x/a)^2/2]\}$.

The characteristic function of (5.17) has an analytical closed form

$$p_{\tilde{q},a,1/2}(k) = \left\langle e^{\imath kx} \right\rangle = \int_{-\infty}^{\infty} dx\, G_{\tilde{q},a}(x) e^{\imath kx} = \left(\frac{ka\sqrt{2/\tilde{q}}}{2} \right)^{1/\tilde{q}-1/2} \frac{2K_{1/\tilde{q}-1/2}\left(ka\sqrt{2/\tilde{q}}\right)}{\Gamma(1/\tilde{q}-1/2)}, \tag{5.20}$$

where $K_\nu(z)$ is the K-modified Bessel function [31]:

$$K_\nu(z) = \frac{2^\nu \Gamma(\nu + 1/2)}{\sqrt{\pi} z^{\nu+1}} \int_0^{\infty} \frac{dt \cos t}{\left[1 + (t/z)^2\right]^{\nu+1/2}}. \tag{5.21}$$

For the Gaussian one has also a Gaussian $p_{0,a}(k) = e^{-a^2k^2/2}$ but for the Lorentzian, one has the Laplace function $p_{1,a} = e^{-\sqrt{2}a|k|}$.

5.2.2. Generalized Laplace PDF

For $\beta = 1$ and $\tilde{q} = 0$, (5.14) leads to the Laplace pdf $P^{(2)}_{0,a,1}(x) = e^{-|x|/a}/(2a)$. For arbitrary $\tilde{q}$, one has the *generalized Laplace pdf*

$$L_{\tilde{q},a}(x) = P^{(2)}_{\tilde{q},a,1}(x) = \frac{1-\tilde{q}}{2a} e_{-\tilde{q}}\left(\frac{-|x|}{a} \right), \tag{5.22}$$

and its cumulative function is

$$F^{(2)}_{\tilde{q},a,1}(x) = \frac{1}{2} \tilde{P}_{\tilde{q},a,1}(|x|), \tag{5.23}$$

where $\tilde{P}_{\tilde{q},a,1}(x)$ is given by (5.10).

The odd moments of (5.22) vanish and the even ones are finite if $\tilde{q} < 1/(1+2n)$

$$\left\langle x^{2n} \right\rangle_{\tilde{q},a,1} = \frac{1}{\tilde{q}-1} \left(\frac{a}{\tilde{q}} \right)^{2n} \mathrm{B}\left(1 + 2n, \frac{1}{\tilde{q}} - 2n - 1 \right). \tag{5.24}$$

Its characteristic function has an analytical closed form

$$\begin{aligned} p_{\tilde{q},a,1}(k) &= \int_{-\infty}^{\infty} dx\, L_{\tilde{q},a}(x) e^{\imath kx} = \frac{1-\tilde{q}}{a}\int_0^{\infty} dx \cos(kx) e_{-\tilde{q}}\left(-\frac{x}{a}\right) \\ &= \frac{1-\tilde{q}}{a}\int_0^{\infty} dx \frac{\cos(kx)}{(1+\tilde{q}x/a)^{1/\tilde{q}}} \\ &= \frac{\pi\, t_0^{1/\tilde{q}-1}\sin(t_0+\pi/(2\tilde{q}))}{\sin(\pi/\tilde{q})\Gamma(1/\tilde{q}-1)} + {}_1\mathrm{F}_2\left(1; 1-\frac{1}{2\tilde{q}}, \frac{3}{2}-\frac{1}{2\tilde{q}}; \frac{-t_0^2}{4}\right), \end{aligned} \tag{5.25}$$

where $t_0 = ka/\tilde{q}$ and ${}_1\mathrm{F}_2(a; b_1, b_2; z) = \sum_{n=0}^{\infty}\{(a)_n/[(b_1)_n(b_2)_n]\}x^n/n!$ is the hypergeometric function [31], with $(a)_n$ being the Pochhammer symbol.

6. Conclusion

We have shown that the $\tilde{q}$-generalization of the exponential is suitable to generalize the stretched exponential function. The $\tilde{q}$-generalized stretched exponential function has the generalized error function, the generalized Laplace pdf and the already known generalized Gaussian as special cases. Further, we have used the $\tilde{q}$-exponential to write the very reliable rank distribution obtained by Naumis and Cocho. Since these distributions are the solution of differential equations that describe the complex systems, the $\tilde{q}$-generalization brings many different systems to be described by the same underlying process.

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