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Research Article

A Coding of Real Null Four-Momenta into World-Sheet Coordinates

David B. Fairlie

Department of Mathematical Sciences, Durham University, South Road, Durham DH1 3LE, UK

Correspondence should be addressed to David B. Fairlie, david.fairlie@durham.ac.uk

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The results of minimizing the action for string-like systems on a simply connected world sheet are shown to encode the Cartesian components of real null momentum four-vectors into coordinates on the world sheet. This identification arises consistently from different approaches to the problem.

1. Recapitulation

This paper is based upon an old unpublished article by Fairlie and Roberts [1], which dates back to circa 1972, on a model for amplitudes suggested by string theory, or rather the dual-resonance model as it was then called. (The results in this paper are recorded in the Ph.D. thesis of Roberts [2].) I was so overwhelmed by the evident truth of the famous paper of Goddard et al. quantising the bosonic string in 26 dimensions [3], which I regard as one of the classic papers in string theory, that I never submitted this paper for publication. However since recently, there has appeared an article by Sommerfield and Thorn [4], the 4th section of which is closely related to the model presented in [1], it may be an appropriate time to give these ideas an airing. Also other recent developments have given rise to an interpretation of maximally violating helicity (MVH) amplitudes in Yang Mills theory in terms of topological string amplitudes [5]; the connection between null four-vectors and Koba-Nielsen variables which is at the heart of [1] may not be entirely coincidental. The intention was to construct a viable amplitude for particles living in a strictly four-dimensional space-time, and with zero mass instead of the tachyon ground state which bedevilled the bosonic string dual resonance model. One of the features of tractable models of physical processes which has come to be more appreciated in the intervening years is that there is frequently a mismatch between what is tractable mathematically and what one should like to have; for example, the potential integrability of $N \to \infty$ supersymmetric Yang Mills as against the intractability of

QCD, or the Sine Gordon model which displays both solitons and Lorentz invariance, at the cost of working in two dimensions. Here a feature analogous to self-dual Yang-Mills theory, which possesses instantons in a space of even signature, is present; I have realised that the theory presented is more mathematically compelling in a space of signature $(2,2)$, though a Lorentzian interpretation is by no means ruled out. This will be discussed later in relation to the work of Gross and Mende on high energy scattering [6]. The starting point is the famous Koba-Nielsen formula, which gives an elegant expression for the N point tree amplitude for N particles with incoming momenta p_i^μ for the ground state of open strings [7],

$$A(s,t)t = \int_{-\infty}^{\infty} \prod_{1}^{N} \frac{dz_k}{dV_{abc}} \theta(z_i - z_{i+1}) \prod_{j>i} (z_i - z_j)^{-2\alpha' p_i \cdot p_j}, \tag{1.1}$$

$$dV_{abc} = \frac{dz_a dz_b dz_c}{(z_b - z_a)(z_c - z_a)(z_a - z_c)}. \tag{1.2}$$

(This integration measure is introduced as a consequence of conformal invariance; to account for the property that the real axis along which the integration is performed is invariant under transformations of the Möbius group, provided that $\alpha'(p_i^\mu)^2 = -1$.) This means invariance under the mapping:

$$z' \longmapsto \frac{az+b}{cz+d}, \quad ad - bc = 1. \tag{1.3}$$

It has been shown that this formula arises as a contribution to string scattering from a simply connected world sheet, thanks to the properties of conformal invariance. Another way of writing (1.1) is as an exponential:

$$\int_{\infty}^{\infty} \exp\left(\sum_{i<j} - 2\alpha' p_i \cdot p_j \log(z_i - z_j) \right) \frac{dz_k}{dV_{abc}}. \tag{1.4}$$

The exponent in the integrand may be interpreted as the (Euclidean) contribution to the action where the momenta enter the upper half-plane at designated points z_k which are then integrated over to give the contribution to the path integral for the amplitude arising from a simply connected world sheet. One of the chief deficiencies in (1.1) is the tachyon condition, namely, that p_i^μ is light like. This requirement follows from the invariance under mappings which preserve the upper-half complex plane. The radical idea behind [1] was to give the formula for the amplitude a different interpretation; do not integrate, but instead determine the coordinates z_i by minimising the integrand; this is tantamount, in the second version to use the method of steepest descents. The equations to be satisfied are, setting $\alpha' = 1$,

$$\sum_j \frac{p_i \cdot p_j}{(z_i - z_j)} = 0. \tag{1.5}$$

These equations may be seen to be satisfied, provided that we are in a 4-dimensional space with signature $(2,2)$ with null four-momenta p_j^μ, and the coordinates z_j (here on the real line) are given by

$$z_j = \frac{p_j^0 + p_j^1}{p_j^2 - p_j^3} = \frac{p_j^2 + p_j^3}{p_j^0 - p_j^1}; \quad \left(p_j^0\right)^2 + \left(p_j^3\right)^2 - \left(p_j^1\right)^2 - \left(p_j^2\right)^2 = 0. \tag{1.6}$$

This works because

$$\begin{aligned}
\left(z_i - z_j\right) &= \frac{p_i \cdot p_j + p_i^0 p_j^1 - p_i^1 p_j^0 + p_i^3 p_j^2 - p_i^2 p_j^3}{\left(p_i^2 - p_i^3\right)\left(p_j^0 - p_j^1\right)}, \\
\left(z_i - z_j\right) &= \frac{-p_i \cdot p_j + p_i^0 p_j^1 - p_i^1 p_j^0 - p_i^3 p_j^2 - p_i^2 p_j^3}{\left(p_i^0 - p_i^1\right)\left(p_j^2 - p_j^3\right)}.
\end{aligned} \tag{1.7}$$

The second equation is obtained by using the alternative expression of (z_i, z_j). By subtracting and rationalising, we have

$$\left(p_i^2 - p_i^3\right)\left(p_j^0 - p_j^1\right) - \left(p_i^0 - p_i^1\right)\left(p_j^2 - p_j^3\right) = \frac{2 p_i \cdot p_j}{\left(z_i - z_j\right)}. \tag{1.8}$$

Summing over all particle positions z_j except $z_j = z_i$ and invoking the conservation of momentum, $\sum p_j^\mu = 0$, we see that (1.5) is satisfied. If instead of the real line, the integration in (1.1) is performed over the boundary of the unit disc, the points on the boundary where the momenta enter may be parametrised by

$$z_j = \frac{p_j^0 + i p_j^3}{p_j^1 + i p_j^2} = \frac{p_j^1 - i p_j^2}{p_j^0 - p_j^3}; \quad \left(p_j^0\right)^2 + \left(p_j^3\right)^2 - \left(p_j^1\right)^2 - \left(p_j^2\right)^2 = 0. \tag{1.9}$$

There is a Möbius transformation (1.3) which connects the two representations, for the plane and the disc,

$$z_{\text{disc}} = \frac{i + z_{\text{plane}}}{i - z_{\text{plane}}}. \tag{1.10}$$

Indeed complex Möbius transformations on z_i are equivalent to $SU(2,2)$ transformations on p_i^μ.

2. Alternative Approach

Consider a two-dimensional surface embedded in a four-dimensional space and take as parametric representation of the surface the four-vectors $X_\mu(\sigma,\tau)$ where σ and τ are intrinsic coordinates on the surface with metric:

$$ds^2 = Ed\sigma^2 + 2Fd\sigma d\tau + Gd\tau^2, \tag{2.1}$$

where

$$E = \left(\frac{\partial X_\mu}{\partial\sigma}\right)^2, \qquad F = \left(\frac{\partial X_\mu}{\partial\sigma}\right)\left(\frac{\partial X_\mu}{\partial\tau}\right), \qquad G = \left(\frac{\partial X_\mu}{\partial\tau}\right)^2 \tag{2.2}$$

(see [1]). The Nambu-Goto Lagrangian describing the dynamics of the field $X_\mu(\sigma,\tau)$ is a measure of the area of the world sheet and is the reparameterisation invariant form

$$\mathcal{L} = \alpha' \iint \sqrt{EG - F^2}\, d\sigma\, d\tau. \tag{2.3}$$

On the other hand, it is well known that there exists a transformation to a coordinate system of so-called isometric coordinates in which the Lagrangian takes the simple quadratic form

$$\mathcal{L}' = \iint \left(\left(\frac{\partial X_\mu}{\partial\sigma}\right)^2 + \left(\frac{\partial X_\mu}{\partial\tau}\right)^2\right) d\sigma\, d\tau, \tag{2.4}$$

which is invariant only under the subset of reparameterisations of the variables (σ, τ) which are conformal, that is, those transformations which satisfy the Cauchy-Riemann equations. It is well known that in the coordinate system where σ and τ are isometric parameters defined by $E = G$; $F = 0$. In this frame, the Euler equation minimising (2.3) becomes linear and is just

$$\nabla^2 X_\mu = 0. \tag{2.5}$$

The conditions for an isometric coordinate system may be written in the following form due to Weierstrass

$$\left(\frac{\partial\zeta_\mu}{\partial z}\right)^2 = E - G + 2iF = 0, \tag{2.6}$$

where X_μ is the real part of ζ_μ in view of the fact that (2.5) is satisfied provided that ζ is an analytic function of $z = \sigma + i\tau$. The Weierstrass' condition shows that conformal mappings of coordinate systems preserve the isometric property. We can make a link with the Virasoro conditions for closed strings [8, 9] by noting that this is in fact the gauge condition of the model; writing

$$\left(\frac{\partial\zeta_\mu}{\partial z}\right)^2 = \sum_{-\infty}^{\infty} L_n z^n = 0. \tag{2.7}$$

This is too stringent to demand as an operator equation. Instead, we require that the matrix elements of (2.7) should vanish for all z, that is,

$$\langle \psi^{\dagger} | L_n | \psi \rangle = 0, \quad \forall n \geq 0. \tag{2.8}$$

This is satisfied provided that $L_n = L_n^{\dagger} = 0$. These conditions are the familiar Virasoro conditions for closed strings with zero mass ground states. A typical solution of (2.5) with a finite number of singularities is given by

$$\zeta^{\mu} = \sum_{i=1}^{i=n} p_i^{\mu} \log(z - z_i). \tag{2.9}$$

By applying the Weierstrass condition, we have

$$\sum_{i,j} \frac{p_i \cdot p_j}{(z - z_i)(z - z_j)} = \sum_{i,j} p_i \cdot p_j \left(\frac{1}{(z - z_i)(z_j - z_i)} - \frac{1}{(z - z_j)(z_j - z_i)} \right) = 0. \tag{2.10}$$

This has to be true for all z, which evidently requires that $\sum_{i,j} p_i \cdot p_j = 0$ and, with conservation of four-momentum, also requires the same conditions $\sum_j (p_i \cdot p_j / (z_i - z_j)) = 0$ as before. In the case of the four-point function, the solution of these conditions (1.5) may be readily solved in terms of the cross-ratio λ to give

$$\lambda = \frac{(z_i - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_4 - z_2)} = \frac{p_1 \cdot p_2}{p_1 \cdot p_3} = \frac{s}{t}, \tag{2.11}$$

where

$$s = \left(p_1^{\mu} + p_2^{\mu}\right)^2, \quad t = \left(p_1^{\mu} + p_3^{\mu}\right)^2, \quad u = \left(p_1^{\mu} + p_4^{\mu}\right)^2, \quad s + t + u = 0. \tag{2.12}$$

This result, in terms of the cross-ratio, is independent of the metric, so also works in a Lorentz metric with signature $(3, 1)$. The resulting amplitude $A(s, t, u)$ with $s + t + u = 0$ may be evaluated to give

$$A(s, t, u) = (-s)^{-\alpha' s} (-t)^{-\alpha' t} (-u)^{\alpha' u}. \tag{2.13}$$

As $s \to \infty$ at fixed t $A(s, t, u) \to t^{-\alpha' s} s^{-\alpha' t}$, that is, it exhibits Regge asymptotic behaviour. The subject of asymptotic behaviour of high energy string amplitudes was examined to all orders sometime afterwards by Gross and Mende [6] who found the same connection (2.11) between the cross-ratio and the Mandelstam variables.

3. Lorentz Signature

As has been remarked, the minimisation condition in the case of the four-point function may be solved in terms of cross-ratios. This suggests that the conditions may be solved directly in terms of the variables z_j whatever the metric is . This is indeed the case; for real four-momenta, the solution may be expressed as

$$z_j = \frac{p_j^0 + p_j^3}{p_j^1 - ip_j^2} = \frac{p_j^i + ip_j^2}{p_j^0 - p_j^3}, \quad \left(p_j^0\right)^2 - \left(p_j^1\right)^2 - \left(p_j^2\right)^2 - \left(p_j^3\right)^2 = 0. \tag{3.1}$$

The difference is that in the case of signature $(2,2)$, the four-momenta may be parametrised as $p_j^0 = r\cosh(\theta_j)$, $p_j^1 = r\sinh(\theta_j)$, $p_j^2 = r\cosh(\phi_j)$, $p_j^3 = r\sinh(\phi_j)$, which imply that

$$z_j = \exp\left(\theta_j + \phi_j\right), \tag{3.2}$$

so the variables lie on the real line. Alternatively, a trigonometric parameterisation may be employed, in which case $z_j = \exp(i\theta_j + i\phi_j)$. However in the case of signature $(1,3)$, the parameterisation is mixed; $p_j^0 = r\cosh(\theta_j)$, $p_j^3 = r\sinh(\theta_j)$, $p_j^1 = r\cos(\phi_j)$, $p_j^2 = r\sinh(\phi_j)$, which imply that $z_j = i\exp(\theta_j + i\phi_j)$, so there is no obvious integration contour for (1.1).

4. Minimal Surface Interpretation

Further insight may be gained by a parameterisation of minimal surfaces embedded in four-dimensional Euclidean space, originally due to Eisenhart [10], but rediscovered by Shaw [11] and quoted in [1, 12]. It is given by

$$\begin{aligned} X^0 &= \operatorname{Re}\left(f(z) - zf'(z) + g(z)'\right), \\ X^3 &= \operatorname{Im}\left(f(z) - zf'(z) - g(z)'\right), \\ X^1 &= \operatorname{Re}\left(g(z) - zg'(z) + f(z)'\right), \\ X^2 &= \operatorname{Im}\left(zg'(z) - g(z) + f(z)'\right), \end{aligned} \tag{4.1}$$

where a prime denotes the derivative with respect to the argument. Suppose we seek a parameterisation where $X^\mu + a^\mu$ is the real part of $\zeta^\mu = \sum_i p_i^\mu G_i(z)$, and a^μ is an arbitrary origin. Then, thanks to the linearity of the above equations, we can split $f(z)$ and $g(z)$ into sums of independent components, that is, $f(z) = \sum f_i(z)$, $g(z) = \sum g_i(z)$, and deduce, up to shifts of origin,

$$\begin{aligned} \left(p_i^0 + ip_i^3\right)G_i(z) &= 2\left(f_i(z) - zf_i'(z) + g_i'(z)\right), \\ \left(p_i^1 - ip_i^2\right)G_i(z) &= 2\left(g_i(z) - zg_i'(z) + f_i'(z)\right), \\ \left(p_i^1 + ip_i^2\right)G_i(z) &= 2\left(g_i(z) - g_i'(z) + f_i'(z)\right)^*, \\ \left(p_i^0 - ip_i^3\right)G_i(z) &= 2\left(f_i(z) - zf_i'(z) + zg'(z)\right)^*. \end{aligned} \tag{4.2}$$

If one postulates, using the same relations as obtained before, then

$$z_i = \frac{p_i^0 + ip_i^3}{p_i^1 - ip_i^2} = \frac{p_i^1 + ip_i^2}{p_i^0 - ip_i^3}. \tag{4.3}$$

These equations possess basic solutions of the form

$$\begin{gathered}
2f_i(z) = \frac{(a-b)\ln(z+1)(z+1)}{2(z_1+1)} + \frac{(a+b)\ln(z-1)(z-1)}{2(z_1-1)} + \frac{(z_1a+b)\ln(z-z_1)(z-z_1)}{(1-z_1^2)} - a, \\
2g_i(z) = \frac{(b-a)\ln(z+1)(z+1)}{2(z_1+1)} + \frac{(a+b)\ln(z-1)(z-1)}{2(z_1-1)} + \frac{(z_1b+a)\ln(z-z_1)(z-z_1)}{(1-z_1^2)} - b, \\
G_i = c(-1+\ln(z-z_1)),
\end{gathered} \tag{4.4}$$

where $a = (p_i^0 + ip_i^3)c$ and $b = z_1(p_i^1 - ip_i^2)c$, and c is a real parameter. If the real parameterisation (1.8) is employed, then the z_i lie on the real axis

$$X^\mu = \sum ip_i^\mu \log(z - z_i) = \sum \pi p_i^\mu \Theta(z - z_i) \quad \text{for } z \text{ on the real axis}. \tag{4.5}$$

As z moves from $+\infty$ to $-\infty$, X^μ jumps by πp_i^μ at the point z_i, so the skew polygon formed by the partial sums of momenta (closed on account of momentum conservation) is mapped into intervals on the real line.

5. Conclusion

The principal message of this paper is to draw attention to the link between the Cartesian components of real null four-momenta in four-dimensional flat space and complex variables on a simply connected world sheet, associated with a minimal surface, or a form of string evolution. The set of four momenta are also required to sum to zero, that is, momentum is conserved in the system. Various aspects leading to this identification are explored. The minimisation of the Koba-Nielsen integrand, the consequence of the Weierstrass condition upon a linear combination of elementary solutions to the free equations of motion, to guarantee a minimal surface solution, and the direct determination of this class of minimal surface solution from the Eisenhart parameterisation are all shown to entail the same identification of a complex variable in terms of the components of a null four-momentum. In a space of even signature (2, 2); in one representation, the complex variables lie on the real line; in another on a circle; in the case of odd signature (Lorentz metric), there is no specific curve on which the variables lie. $SL(2, C)$ transformations of the complex variable implement homogeneous Lorentz transformations upon the momentum.

In our original paper, as is standard practice, the optimistic anticipation of further development of these ideas was raised, but it must be admitted that neither author has been able to add anything substantially new in the intervening 35 years! However, as T.S. Eliot has said, "A poem may have meanings which are hidden from its author." It may be that the further examination of solutions to the four-dimensional minimal surface equations originally proposed by Eisenhart will be fruitful. The ideas of this paper seem rooted in four dimensions; the parameterisation of classical string solutions proposed in [12, 13] based upon the division algebras may contain the clue to extend the connection between momenta and world sheet coordinates to 10 dimensions. The recent paper of Sommerfield and Thorn [4] extends their ideas to AdS space-time, and the picture of world sheets bounded by a closed polygon of null lines which is presented therein and is also contained in [14] is essentially the same as that in Section 4 of the present paper. In addition, the treatment of high energy string amplitudes by Gross and Mende [6] extends some aspects of this analysis to multiply connected world sheets.

In this spirit, this revised and rewritten version of [1] is offered in the hope that some deeper connection between momentum space and the world sheet will be discovered.

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