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Review Article

Infinite-Dimensional Lie Groups and Algebras in Mathematical Physics

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We give a review of infinite-dimensional Lie groups and algebras and show some applications and examples in mathematical physics. This includes diffeomorphism groups and their natural subgroups like volume-preserving and symplectic transformations, as well as gauge groups and loop groups. Applications include fluid dynamics, Maxwell's equations, and plasma physics. We discuss applications in quantum field theory and relativity (gravity) including BRST and supersymmetries.

1. Introduction

Lie groups play an important role in physical systems both as phase spaces and as symmetry groups. Infinite-dimensional Lie groups occur in the study of dynamical systems with an infinite number of degrees of freedom such as PDEs and in field theories. For such infinite-dimensional dynamical systems, diffeomorphism groups and various extensions and variations thereof, such as gauge groups, loop groups, and groups of Fourier integral operators, occur as symmetry groups and phase spaces. Symmetries are fundamental for Hamiltonian systems. They provide conservation laws (Noether currents) and reduce the number of degrees of freedom, that is, the dimension of the phase space.

The topics selected for review aim to illustrate some of the ways infinite-dimensional geometry and global analysis can be used in mathematical problems of physical interest. The topics selected are the following.

(1) Infinite-Dimensional Lie Groups.

(2) Lie Groups as Symmetry Groups of Hamiltonian Systems.

(3) Applications.

(4) Gauge Theories, the Standard Model, and Gravity.

(5) SUSY (supersymmetry).

2. Infinite-Dimensional Lie Groups

2.1. Basic Definitions

A general theory of infinite-dimensional Lie groups is hardly developed. Even Bourbaki [1] only develops a theory of infinite-dimensional manifolds, but all of the important theorems about Lie groups are stated for finite-dimensional ones.

An infinite-dimensional *Lie group* $\mathcal{G}$ is a group and an infinite-dimensional manifold with smooth group operations

$$m : \mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G}, \quad m(g,h) = g \cdot h, \quad C^{\infty}, \tag{2.1}$$

$$i : \mathcal{G} \longrightarrow \mathcal{G}, \quad i(g) = g^{-1}, \quad C^{\infty}. \tag{2.2}$$

Such a Lie group $\mathcal{G}$ is locally diffeomorphic to an infinite-dimensional vector space. This can be a Banach space whose topology is given by a norm $\|\cdot\|$, a Hilbert space whose topology is given by an inner product $\langle\cdot,\cdot\rangle$, or a Frechet space whose topology is given by a metric but not by a norm. Depending on the choice of the topology on $\mathcal{G}$, we talk about Banach, Hilbert, or Frechet Lie groups, respectively.

The *Lie algebra* $\mathfrak{g}$ of a Lie group $\mathcal{G}$ is defined as $\mathfrak{g} = \{$left invariant vector fields on $\mathcal{G}\} \simeq T_e\mathcal{G}$ (tangent space at the identity e). The isomorphism is given (as in finite dimensions) by

$$\xi \in T_e\mathcal{G} \longmapsto X_\xi \in \mathfrak{g}, \quad X_\xi(g) := T_e L_g(\xi), \tag{2.3}$$

and the Lie bracket on $\mathfrak{g}$ is induced by the Lie bracket of left invariant vector fields $[\xi,\eta] = [X_\xi, X_\eta](e), \xi, \eta \in T_e\mathcal{G}$.

These definitions in infinite dimensions are identical with the definitions in finite dimensions. The big difference although is that infinite-dimensional manifolds, hence Lie groups, are *not* locally compact. For Frechet Lie groups, we have the additional nontrivial difficulty of the question how to define differentiability of functions defined on a Frechet space; see the study by Keller in [2]. Hence the very definition of a Frechet manifold is not canonical. This problem does not arise for Banach- and Hilbert-Lie groups; the differential calculus extends in a straightforward manner from $\mathbb{R}^n$ to Banach and Hilbert spaces, but not to Frechet spaces.

2.2. Finite- versus Infinite-Dimensional Lie Groups

Infinite-dimensional Lie groups are *NOT* locally compact. This causes some deficiencies of the Lie theory in infinite dimensions. We summarize some classical results in *finite* dimensions which are *NOT* true in general in infinite dimensions as follows.

(1) There is NO Implicit Function Theorem or Inverse Function Theorem in infinite dimensions! (except Nash-Moser-type theorems).

(2) If G is a finite-dimensional Lie group, the *exponential map* $\exp : \mathfrak{g} \to G$ is defined as follows. To each $\xi \in \mathfrak{g}$, we assign the corresponding left invariant vector field X_ξ defined by (2.3). We take the flow $\varphi_\xi(t)$ of X_ξ and define $\exp(\xi) = \varphi_\xi(1)$. The exponential map is a local diffeomorphism from a neighborhood of zero in $\mathfrak{g}$ onto a neighborhood of the identity in G; hence exp defines canonical coordinates on the Lie group G. This is not true in infinite dimensions.

(3) If $f_1, f_2 : G_1 \to G_2$ are smooth Lie group homomorphisms (i.e., $f_i(g \cdot h) = f_i(g) \cdot f_i(h)$, $i = 1, 2$) with $T_e f_1 = T_e f_2$, then locally $f_1 = f_2$. This is not true in infinite dimensions.

(4) If $f : G \to H$ is a continuous group homomorphism between finite-dimensional Lie groups, then f is smooth. This is not true in infinite dimensions.

(5) If $\mathfrak{g}$ is any finite-dimensional Lie algebra, then there exists a connected finite-dimensional Lie group G with $\mathfrak{g}$ as its Lie algebra; that is, $\mathfrak{g} \simeq T_e G$. This is not true in infinite dimensions.

(6) If G is a finite-dimensional Lie group and $H \subset G$ is a *closed* subgroup, then H is a Lie subgroup (i.e., Lie group and submanifold). This is not true in infinite dimensions.

(7) If G is a finite-dimensional Lie group with Lie algebra $\mathfrak{g}$ and $\mathfrak{h} \subset \mathfrak{g}$ is a subalgebra, then there exists a unique connected Lie subgroup $H \subset G$ with $\mathfrak{h}$ as its Lie algebra; that is, $\mathfrak{h} \simeq T_e H$. This is not true in infinite dimensions.

Some classical examples of *finite*-dimensional Lie groups are the matrix groups $GL(n)$, $SL(n)$, $O(n)$, $SO(n)$, $U(n)$, $SU(n)$, and $Sp(n)$ with smooth group operations given by matrix multiplication and matrix inversion. The Lie algebra bracket is the commutator $[A, B] = AB - BA$ with exponential map given by $\exp(A) = \sum_{i=0}^{\infty}(1/i!)A^i = e^A$.

2.3. Examples of Infinite-Dimensional Lie Groups

2.3.1. The Vector Groups $\mathcal{G} = (V, +)$

Let V be a Banach space and take $\mathcal{G} = V$ with $m(x, y) = x + y$, $i(x) = -x$, and $e = 0$, which makes $\mathcal{G}$ into an Abelian Lie group; that is, $m(x, y) = m(y, x)$. For the Lie algebra we have $\mathfrak{g} \simeq T_e V \simeq V$. For $u \in T_e V$ the corresponding left invariant vector field X_u is given by $X_u(v) = u$, $\forall v \in V$; that is, $X_u = \text{const}$. Hence the Lie algebra $\mathfrak{g} = V$ with the trivial Lie bracket $[u, v] = 0$ is Abelian. For the exponential map we get $\exp : \mathfrak{g} = V \to \mathcal{G} = V$, $\exp = id_V$.

2.3.2. The General Linear Group $\mathcal{G} = (GL(V), \circ)$

Let V be a Banach space and $L(V, V)$ the space of bounded linear operators $A : V \to V$. Then $L(V, V)$ is a Banach space with the operator norm $\|A\| = \sup_{\|x\| \le 1} \|A(x)\|$, and the group $\mathcal{G} = GL(V)$ of all invertible elements is open in $L(V, V)$. So $GL(V)$ is a smooth Lie group with $m(f, g) = f \circ g$, $i(f) = f^{-1}$, and $e = id_V$. Its Lie algebra is $\mathfrak{g} = L(V, V)$ with the commutator bracket $[A, B] = AB - BA$ and exponential map $\exp A = e^A$.

2.3.3. The Abelian Gauge Group $\mathcal{G} = (C^\infty(M), +)$

Let M be a finite-dimensional manifold and let $\mathcal{G} = C^\infty(M)$ (smooth functions on M). With group operation being addition, that is, $m(f,g) = f + g$, $i(f) = -f$, and $e = 0$. $\mathcal{G}$ is an Abelian C^∞ (addition is smooth) Frechet Lie group with Lie algebra $\mathfrak{g} = T_e C^\infty(M) \simeq C^\infty(M)$, with trivial bracket $[\xi, \eta] = 0$, and $\exp = id$. If we complete these spaces in the C^k-norm, $k < \infty$ (denoted by $\mathcal{G}^k$), then $\mathcal{G}^k$ is a Banach-Lie group, and if we complete in the H^s-Sobolev norm with $s > (1/2)\dim M$ then $\mathcal{G}^s$ is a Hilbert-Lie group.

2.3.4. The Abelian Gauge Group $\mathcal{G} = (C^\infty(M, \mathbb{R} - \{0\}), \cdot)$

Let M be a finite-dimensional manifold and let $\mathcal{G} = C^\infty(M, \mathbb{R} - \{0\})$, with group operation being multiplication; that is, $m(f,g) = f \cdot g$, $i(f) = f^{-1}$, and $e = 1$. For $k < \infty$, $C^k(M, \mathbb{R} - \{0\})$ is open in $C^\infty(M, \mathbb{R})$, and if M is compact, then $C^k(M, \mathbb{R} - \{0\})$ is a Banach-Lie group. If $s > (1/2)\dim M$, then $H^s(M, \mathbb{R} - \{0\})$ is closed under multiplication, and if M is compact, then $H^s(M, \mathbb{R} - \{0\})$ is a Hilbert-Lie group.

2.3.5. Loop Group $\mathcal{G} = (C^k(M, G), \cdot)$

We generalize the Abelian example (see Section 2.3.4) by replacing $\mathbb{R} - \{0\}$ with any finite-dimensional (non-Abelian) Lie group G. Let $\mathcal{G} = C^k(M, G)$ with pointwise group operations $m(f,g)(x) = f(x) \cdot g(x)$, $x \in M$, and $i(f)(x) = (f(x))^{-1}$, where "$\cdot$" and "$(\cdot)^{-1}$" are the operations in G. If $k < \infty$ then $C^k(M, G)$ is a Banach-Lie group. Let g denote the Lie algebra of G, then the Lie algebra of $\mathcal{G} = C^k(M, G)$ is $\mathfrak{g} = C^k(M, g)$, with pointwise Lie bracket $[\xi, \eta](x) = [\xi(x), \eta(x)]$, $x \in M$, the latter bracket being the Lie bracket in g. The exponential map $\exp : g \to G$ defines the exponential map $\mathrm{EXP} : \mathfrak{g} = C^k(M, g) \to \mathcal{G} = C^k(M, G)$, $\mathrm{EXP}(\xi) = \exp \circ \xi$, which is a local diffeomorphism. The same holds for $H^s(M, G)$ if $s > (1/2)\dim M$.

Applications of these infinite-dimensional Lie groups are in gauge theories and quantum field theory, where they appear as groups of gauge transformations. We will discuss these in Section 5.

Special Case: $\mathcal{G} = (C^k(S^1, G), \cdot)$

As a special case of example mentioned in Section 2.3.5 we take $M = S^1$, the circle. Then $\mathcal{G} = C^k(S^1, G) = \mathcal{L}^k(G)$ is called a *loop group* and $\mathfrak{g} = C^k(S^1, g) = \mathfrak{l}^k(g)$ is its loop algebra. They find applications in the theory of affine Lie algebras, Kac-Moody Lie algebras (central extensions), completely integrable systems, soliton equations (Toda, KdV, KP), and quantum field theory; see, for example, [3] and Section 5. Central extensions of loop algebras are examples of infinite-dimensional Lie algebras which need not have a corresponding Lie group.

Certain subgroups of loop groups play an important role in quantum field theory as groups of gauge transformations. We will discuss these in Section 2.4.4.

2.4. Diffeomorphism Groups

Among the most important "classical" infinite-dimensional Lie groups are the diffeomorphism groups of manifolds. Their differential structure is not the one of a Banach Lie group as defined above. Nevertheless they have important applications.

Let M be a compact manifold (the noncompact case is technically much more complicated but similar results are true; see the study by Eichhorn and Schmid in [4]) and let $\mathcal{G} = \mathrm{Diff}^{\infty}(M)$ be the group of all smooth diffeomorphisms on M, with group operation being composition; that is, $m(f,g) = f \circ g$, $i(f) = f^{-1}$, and $e = id_M$. For C^{∞} diffeomorphisms, $\mathrm{Diff}^{\infty}(M)$ is a Frechet manifold and there are nontrivial problems with the notion of smooth maps between Frechet spaces. There is no canonical extension of the differential calculus from Banach spaces (which is the same as for $\mathbb{R}^n$) to Frechet spaces; see the study by Keller in [2]. One possibility is to generalize the notion of differentiability. For example, if we use the so-called C^{∞}_{Γ} differentiability, then $\mathcal{G} = \mathrm{Diff}^{\infty}(M)$ becomes a C^{∞}_{Γ} Lie group with C^{∞}_{Γ} differentiable group operations. These notions of differentiability are difficult to apply to concrete examples. Another possibility is to complete $\mathrm{Diff}^{\infty}(M)$ in the Banach C^k-norm, $0 \leq k < \infty$, or in the Sobolev H^s-norm, $s > (1/2)\dim M$. Then $\mathrm{Diff}^k(M)$ and $\mathrm{Diff}^s(M)$ become Banach and Hilbert manifolds, respectively. Then we consider the inverse limits of these Banach- and Hilbert-Lie groups, respectively:

$$\mathrm{Diff}^{\infty}(M) = \lim_{\leftarrow} \mathrm{Diff}^k(M) \tag{2.4}$$

becomes a so-called ILB- (Inverse Limit of Banach) Lie group, or with the Sobolev topologies

$$\mathrm{Dif}f^{\infty}(M) = \lim_{\leftarrow} \mathrm{Diff}^s(M) \tag{2.5}$$

becomes a so-called ILH- (Inverse Limit of Hilbert) Lie group. See the study by Omori in [5] for details. Nevertheless, the group operations are not smooth, but have the following differentiability properties. If we equip the diffeomorphism group with the Sobolev H^s-topology, then $\mathrm{Diff}^s(M)$ becomes a C^{∞} Hilbert manifold if $s > (1/2)\dim M$ and the group multiplication

$$m : \mathrm{Diff}^{s+k}(M) \times \mathrm{Diff}^s(M) \longrightarrow \mathrm{Diff}^s(M) \tag{2.6}$$

is C^k differentiable; hence for $k = 0$, m is only continuous on $\mathrm{Dif}f^s(M)$. The inversion

$$i : \mathrm{Diff}^{s+k}(M) \longrightarrow \mathrm{Diff}^s(M) \tag{2.7}$$

is C^k differentiable; hence for $k = 0$, i is only continuous on $\mathrm{Diff}^s(M)$. The same differentiability properties of m and i hold in the C^k topology.

The *Lie algebra* of $\mathrm{Diff}^{\infty}(M)$ is given by $\mathfrak{g} = T_e\mathrm{Diff}^{\infty}(M) \simeq \mathrm{Vec}^{\infty}(M)$ being the space of smooth vector fields on M. Note that the space $\mathrm{Vec}(M)$ of all vector fields is a Lie algebra only for C^{∞} vector fields, but not for C^k or H^s vector fields if $k < \infty$, $s < \infty$, because one loses derivatives by taking brackets.

The exponential map on the diffeomorphism group is given as follows. For any vector field $X \in \mathrm{Vec}^{\infty}(M)$, take its flow $\varphi_t \in \mathrm{Diff}^{\infty}(M)$, then define $\mathrm{EXP} : \mathrm{Vec}^{\infty}(M) \to \mathrm{Diff}^{\infty}(M) : X \mapsto \varphi_1$, the flow at time $t = 1$. The exponential map EXP is *NOT* a local diffeomorphism; it is not even locally surjective.

We see that the diffeomorphism groups are not Lie groups in the classical sense, but what we call *nested Lie groups*. Nevertheless they have important applications as we will see.

2.4.1. Subgroups of $\mathrm{Diff}^\infty(M)$

Several subgroups of $\mathrm{Diff}^\infty(M)$ have important applications.

2.4.2. Group of Volume-Preserving Diffeomorphisms

Let μ be a volume on M and

$$\mathcal{G} = \mathrm{Diff}^\infty_\mu(M) = \left\{ f \in \mathrm{Diff}^\infty(M) \mid f^*\mu = \mu \right\} \tag{2.8}$$

the group of volume-preserving diffeomorphisms. $\mathrm{Diff}^\infty_\mu(M)$ is a closed subgroup of $\mathrm{Diff}^\infty(M)$ with Lie algebra

$$\mathfrak{g} = \mathrm{Vec}^\infty_\mu(M) = \left\{ X \in \mathrm{Vec}^\infty(M) \mid \mathrm{div}_\mu X = 0 \right\} \tag{2.9}$$

being the space of divergence-free vector fields on M. $\mathrm{Vec}^\infty_\mu(M)$ is a Lie subalgebra of $\mathrm{Vec}^\infty(M)$.

Remark 2.1. We cannot apply the finite-dimensional theorem that if $\mathrm{Vec}^\infty_\mu(M)$ is Lie algebra then there exists a Lie group whose Lie algebra it is; nor the one that if $\mathrm{Diff}^\infty_\mu(M) \subset \mathrm{Diff}(M)$ is a closed subgroup then it is an Lie subgroup.

Nevertheless $\mathrm{Diff}^\infty_\mu(M)$ is an ILH-Lie group.

2.4.3. Symplectomorphism Group

Let ω be a symplectic 2-form on M and

$$\mathcal{G} = \mathrm{Diff}^\infty_\omega(M) = \left\{ f \in \mathrm{xDiff}^\infty(M) \mid f^*\omega = \omega \right\} \tag{2.10}$$

the group of canonical transformations (or symplectomorphisms). $\mathrm{Diff}^\infty_\omega(M)$ is a closed subgroup of $\mathrm{Diff}^\infty(M)$ with Lie algebra

$$\mathfrak{g} = \mathrm{Vec}^\infty_\omega(M) = \{ X \in \mathrm{Vec}^\infty(M) \mid L_X\omega = 0 \} \tag{2.11}$$

being the space of locally Hamiltonian vector fields on M. $\mathrm{Vec}^\infty_\omega(M)$ is a Lie subalgebra of $\mathrm{Vec}^\infty(M)$. Again $\mathrm{Diff}^\infty_\omega(M)$ is an ILH-Lie group.

2.4.4. Group of Gauge Transformations

The diffeomorphism subgroups that arise in gauge theories as gauge groups behave nicely because they are isomorphic to subgroups of loop groups which are not only ILH-Lie groups but actually Hilbert-Lie groups.

Let $\pi : P \to M$ be a principal G bundle with G being a finite-dimensional Lie group (structure group) acting on P from the right $p \in P$, $g \in G$, and $p \cdot g \in P$.

The *Gauge group* $\mathcal{G}$ is the group of gauge transformations defined by

$$\mathcal{G} = \{\phi \in \mathrm{Diff}^{\infty}(P);\ \phi(p \cdot g) = \phi(p) \cdot g,\ \pi(\phi(p)) = \pi(p)\}. \tag{2.12}$$

$\mathcal{G}$ is a group under composition, hence a subgroup of the diffeomorphism group $\mathrm{Diff}^{\infty}(P)$. Since a gauge transformation $\phi \in \mathcal{G}$ preserves fibers, we can realize each such $\phi \in \mathcal{G}$ via $\phi(p) = p \cdot \tau(p)$, where $\tau : P \to G$ satisfies $\tau(p \cdot g) = g^{-1}\tau(p)g$, for $p \in P$, $g \in G$. Let

$$\mathrm{Gau}(P) = \left\{\tau \in C^{\infty}(P, G);\ \tau(p \cdot g) = g^{-1}\tau(p)g\right\}. \tag{2.13}$$

$\mathrm{Gau}(P)$ is a group under pointwise multiplication, hence a subgroup of the loop group $C^{\infty}(P, G)$ (see Section 2.4.3), which extends to a Hilbert-Lie group if equipped with the H^s-Sobolev topology. We give $\mathrm{Gau}(P)$ the induced topology and extend it to a Hilbert-Lie group denoted by $\mathrm{Gau}^s(P)$. Another interpretation is that $\mathrm{Gau}(P)$ is isomorphic to $C^{\infty}(Ad\,P)$ the space of sections of the associated vector bundle $Ad(P) = P \times_G G$. Completed in the H^s Sobolev topology, we get $\mathrm{Gau}^s(P) \simeq H^s(Ad\,P)$.

Let $\mathbf{g}$ denote the Lie algebra of G. Then the *Lie algebra* $\mathfrak{gau}(P)$ of $\mathrm{Gau}(P)$ is a subalgebra of the loop algebra $H^s(P, \mathbf{g})$ under pointwise bracket in $\mathbf{g}$, the finite-dimensional Lie algebra of G; that is, for any $\xi, \eta \in H^s(P, \mathbf{g})$ the bracket is defined by $[\xi, \eta]_{\mathfrak{gau}(P)}(p) = [\xi(p), \eta(p)]_{\mathbf{g}}$, $p \in P$. Then $\mathfrak{gau}^s(P)$ is the subalgebra of Ad-invariant $\mathbf{g}$-valued functions on P; that is,

$$\mathfrak{gau}(P) = \{\xi \in C^{\infty}(P, \mathbf{g});\ \xi(p \cdot g) = Ad_{g^{-1}}\xi(p)\}. \tag{2.14}$$

The Lie algebra $\mathfrak{lie}\ \mathcal{G}$ (running out of symbols) of the gauge group $\mathcal{G}$ is the Lie subalgebra of $\mathfrak{X}^{\infty}(P)$ consisting of all G-invariant vertical vector fields X on P; that is,

$$\mathfrak{lie}\,\mathcal{G} = \left\{X \in \mathfrak{X}^{\infty}(P);\ R_g^* X = X,\ X(p) \in \mathbf{g},\ g \in G,\ p \in P\right\} \tag{2.15}$$

with commutator bracket $[X_1, X_2] = X_1X_2 - X_2X_1 \in \mathfrak{lie}\ \mathcal{G}$.

On the other hand, the Lie algebra of $C^{\infty}(Ad\,P)$ is $C^{\infty}(ad(P))$ being the space of sections of the associated vector bundle $ad(P) \equiv (P \times_G g) \to M$ with pointwise bracket.

We have three versions of gauge groups: $\mathcal{G}$, $\mathrm{Gau}(P)$, and $C^{\infty}(Ad\,P)$. They are all group isomorphic. There is a natural group isomorphism $\mathrm{Gau}(P) \to \mathcal{G} : \tau \mapsto \phi$ defined by $\phi(p) = p \cdot \tau(p)$, $p \in P$, which preserves the product $\tau_1 \cdot \tau_2 \mapsto \phi_1 \circ \phi_2$. Identifying $\mathcal{G}$ with $\mathrm{Gau}(P)$, we can avoid the troubles with diffeomorphism groups and we can extend $\mathcal{G}$ to a Hilbert-Lie group $\mathcal{G}^s$. So $\mathcal{G}^s$ is actually a Hilbert-Lie group in the classical sense; that is, the group operations are C^{∞}. Also the three Lie algebras $\mathfrak{lie}\,\mathcal{G}$, $\mathfrak{gau}(P)$, and $C^{\infty}(ad\,P)$ are canonically isomorphic. Indeed, for $s \in C^{\infty}(ad\,P)$ define $\xi \in \mathfrak{gau}(P)\xi : P \to g$ by $\xi(p \cdot a) := Ad_{a^{-1}}\xi(p)$; and for $\xi \in \mathfrak{gau}(P)$ define $s \in C^{\infty}(ad\,P)$ by $s(\pi(p)) := [p, \xi(p)]$.

On the other hand, for $\xi \in \mathfrak{gau}(P)$ define $Z_\xi \in \mathfrak{lie}\,\mathcal{G}$ by

$$Z_\xi(p) = \frac{d}{dt}|_{t=0} R(p, \exp\, t\xi(p)) \quad (= \xi(p)^*(p)), \tag{2.16}$$

that is, Z_ξ is the fundamental vector field on P, generated by $\xi \in \mathbf{g}$. Z_ξ is invariant if and only if $\xi(p \cdot g) = Ad_{g^{-1}}\xi(p)$.

To topologize $\mathfrak{lie}\ \mathcal{G}$, we complete $C^\infty(ad\,P)$ in the H^s-Sobolev norm. If $s > (1/2)\dim M$, then $\mathfrak{lie}\,\mathcal{G}^s \simeq H^s(ad\,P) \simeq \mathfrak{gau}^s(P)$ are isomorphic Hilbert-Lie algebras.

There is a natural exponential map $\mathrm{Exp} : \mathfrak{gau}(P) \to \mathrm{Gau}(P)$, which is a local diffeomorphism. Let $\exp : \mathbf{g} \to G$ be the finite-dimensional exponential map. Then define

$$\mathrm{Exp} : \mathfrak{gau}^s(P) \to \mathrm{Gau}^s(P) : (\mathrm{Exp}\,\xi)(p) = \exp(\xi(p)), \quad \xi \in \mathfrak{gau}^s(P). \tag{2.17}$$

Or in terms of $\mathcal{G}$, $\mathrm{Exp} : \mathfrak{lie}\ \mathcal{G}^s \to \mathcal{G}^s : (\mathrm{Exp}\,\xi)(p) = p \cdot \exp(\xi_p)$.

We have the following theorem (Schmid [6]).

Theorem 2.2. *For $s > (1/2)\dim M$,*

$$\mathcal{G}^s \simeq \mathrm{Gau}^s(P) \simeq H^s(Ad\,P) \tag{2.18}$$

is a smooth Hilbert-Lie group with Lie algebra

$$\mathfrak{lie}\,\mathcal{G}^s \simeq \mathfrak{gau}^s(P) \simeq H^s(ad\,P) \tag{2.19}$$

and smooth exponential map, which is a local diffeomorphism,

$$\mathrm{EXP} : \mathfrak{lie}\,\mathcal{G}^s \longrightarrow \mathcal{G}^s : (\mathrm{EXP}\,\xi)(p) = p \cdot \exp(\xi(p)). \tag{2.20}$$

See [1–5, 7–19].

3. Lie Groups as Symmetry Groups of Hamiltonian Systems

A short introduction and "crash course" to geometric mechanics can be found in the studies by Abraham and Marsden [20], Marsden [21], as well as Marsden and Ratiu [22]. For the general theory of infinite-dimensional manifolds and global analysis, see, for example, the studies by Bourbaki [9], Lang [14], as well as Palais [18].

3.1. Hamilton's Equations on Poisson Manifolds

A *Poisson manifold* is a manifold P (in general infinite-dimensional) equipped with a bilinear operation $\{\cdot,\cdot\}$, called *Poisson bracket*, on the space $C^\infty(P)$ of smooth functions on P satisfying the following.

(i) $(C^\infty(P), \{\cdot,\cdot\})$ is a Lie algebra; that is, $\{\cdot,\cdot\} : C^\infty(P) \times C^\infty(P) \to C^\infty(P)$ is bilinear, skew symmetric and satisfies the Jacobi identity $\{\{F,G\},H\} + \{\{H,F\},G\} + \{\{G,H\},F\} = 0$ for all $F, G, H \in C^\infty(P)$.

(ii) $\{\cdot,\cdot\}$ satisfies the Leibniz rule; that is, $\{\cdot,\cdot\}$ is a derivation in each factor: $\{F \cdot G, H\} = F \cdot \{G,H\} + G \cdot \{F,H\}$, for all $F, G, H \in C^\infty(P)$.

The notion of Poisson manifolds was rediscovered many times under different names, starting with Lie, Dirac, Pauli, and others. The name *Poisson manifold* was coined by Lichnerowicz.

For any $H \in C^\infty(P)$ we define the *Hamiltonian vector field* X_H by

$$X_H(F) = \{F, H\}, \quad F \in C^\infty(P). \tag{3.1}$$

It follows from (ii) that indeed X_H defines a derivation on $C^\infty(P)$, hence a vector field on P. *Hamilton's equations* of motion for a function $F \in C^\infty(P)$ with Hamiltonian $H \in C^\infty(P)$ (energy function) are then defined by the flow (integral curves) of the vector field X_H; that is,

$$\dot{F} = X_H(F) = \{F, H\}, \quad \text{where } \dot{} = \frac{d}{dt}. \tag{3.2}$$

We then call F a *Hamiltonian system* on P with energy (Hamiltonian function) H.

3.2. Examples of Poisson Manifolds and Hamilton's Equations

Poisson manifolds are a generalization of symplectic manifolds on which Hamilton's equations have a canonical formulated.

3.2.1. Finite-Dimensional Classical Mechanics

For finite-dimensional classical mechanics we take $P = \mathbb{R}^{2n}$ with coordinates $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ with the standard Poisson bracket for any two functions $F(q^i, p_i)$, $H(q^i, p_i)$ given by

$$\{F, H\} = \sum_{i=1}^{n} \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q^i} - \frac{\partial H}{\partial p_i}\frac{\partial F}{\partial q^i}. \tag{3.3}$$

Then the *classical Hamilton's equations* are

$$\dot{q}^i = \left\{q^i, H\right\} = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = \{p_i, H\} = -\frac{\partial H}{\partial q^i}, \tag{3.4}$$

where $i = 1, \ldots, n$. This finite-dimensional Hamiltonian system is a system of ordinary differential equations for which there are well-known existence and uniqueness theorems; that is, it has locally unique smooth solutions, depending smoothly on the initial conditions.

Example 3.1 (Harmonic Oscillator). As a concrete example we consider the harmonic oscillator. Here $P = \mathbb{R}^2$ and the Hamiltonian (energy) is $H(q, p) = (1/2)(q^2 + p^2)$. Then Hamilton's equations are

$$\dot{q} = p, \qquad \dot{p} = -q. \tag{3.5}$$

3.2.2. Infinite-Dimensional Classical Field Theory

Let V be a Banach space and V^* its dual space with respect to a pairing $\langle \cdot, \cdot \rangle : V \times V^* \to \mathbb{R}$ (i.e., $\langle \cdot, \cdot \rangle$ is a symmetric, bilinear, nondegenerate function). On $P = V \times V^*$ we have the canonical Poisson bracket for $F, H \in C^\infty(P)$, $\varphi \in V$, and $\pi \in V^*$, given by

$$\{F, H\} = \left\langle \frac{\delta F}{\delta \pi}, \frac{\delta H}{\delta \varphi} \right\rangle - \left\langle \frac{\delta H}{\delta \pi}, \frac{\delta F}{\delta \varphi} \right\rangle, \tag{3.6}$$

where the functional derivatives $\delta F/\delta \pi \in V$, $\delta F/\delta \varphi \in V^*$ are the "duals" under the pairing $\langle \cdot, \cdot \rangle$ of the partial gradients $D_1 F(\pi) \in V^*, D_2 F(\varphi) \in V^{**} \simeq V$. The corresponding Hamilton's equations are

$$\dot{\varphi} = \{\varphi, H\} = \frac{\delta H}{\delta \pi}, \quad \dot{\pi} = \{\pi, H\} = -\frac{\delta H}{\delta \varphi}. \tag{3.7}$$

As a special case in finite dimensions, if $V \simeq \mathbb{R}^n$, so that $V^* \simeq \mathbb{R}^n$ and $P = V \times V^* \simeq \mathbb{R}^{2n}$, and the pairing is the standard inner product in $\mathbb{R}^n$, then the Poisson bracket (3.6) and Hamilton's equations (3.7) are identical with (3.3) and (3.4), respectively.

Example 3.2 (Wave Equations). As a concrete example we consider the wave equations. Let $V = C^\infty(\mathbb{R}^3)$ and $V^* = \mathrm{Den}(\mathbb{R}^3)$ (densities) and the L^2 pairing $\langle \varphi, \pi \rangle = \int \varphi(x)\pi(x)dx$. We take the Hamiltonian to be $H(\varphi, \pi) = \int((1/2)\pi^2 + (1/2)|\nabla\varphi|^2 + F(\varphi))dx$, where F is some function on V. Then Hamilton's (3.7) become

$$\dot{\varphi} = \pi, \qquad \dot{\pi} = \nabla^2\varphi - F'(\varphi), \quad \text{where } ' = \frac{d}{d\varphi}, \tag{3.8}$$

which imply the wave equation $\partial^2\varphi/\partial t^2 = \nabla^2\varphi - F'(\varphi)$. Different choices of F give different wave equations; for example, for $F = 0$ we get the linear wave equation $\partial^2\varphi/\partial t^2 = \nabla^2\varphi$. For $F = (1/2)m\varphi$ we get the Klein-Gordon equation $\nabla^2\varphi - \partial^2\varphi/\partial t^2 = m\varphi$. So these wave equations and the Klein-Gordon equation are infinite-dimensional Hamiltonian systems on $P = C^\infty(\mathbb{R}^3) \times \mathrm{Den}(\mathbb{R}^3)$.

3.2.3. Cotangent Bundles

The finite-dimensional examples of Poisson brackets (3.3) and Hamilton's (3.4) and the infinite-dimensional examples (3.6) and (3.7) are the local versions of the general case where $P = T^*Q$ is the cotangent bundle (phase space) of a manifold Q (configuration space). If Q is an n-dimensional manifold, then T^*Q is a $2n$-Poisson manifold locally isomorphic to $\mathbb{R}^{2n}$ whose Poisson bracket is locally given by (3.3) and Hamilton's equations are locally given by (3.4). If Q is an infinite-dimensional Banach manifold, then T^*Q is a Poisson manifold locally isomorphic to $V \times V^*$ whose Poisson bracket is given by (3.6) and Hamilton's equations are locally given by (3.7).

3.2.4. Symplectic Manifolds

All the examples above are special cases of symplectic manifolds (P, ω). That means that P is equipped with a symplectic structure ω which is a closed ($d\omega = 0$), (weakly) nondegenerate 2-form on the manifold P. Then for any $H \in C^\infty(P)$ the corresponding Hamiltonian vector field X_H is defined by $dH = \omega(X_H, \cdot)$ and the canonical Poisson bracket is given by

$$\{F, H\} = \omega(X_F, X_H), \quad F, H \in C^\infty(P). \tag{3.9}$$

For example, on $\mathbb{R}^{2n}$ the canonical symplectic structure ω is given by $\omega = \sum_{i=1}^n dp_i \wedge dq^i = d\theta$, where $\theta = \sum_{i=1}^n p_i \wedge dq^i$. The same formula for ω holds locally in T^*Q for any finite-dimensional Q (Darboux's Lemma). For the infinite-dimensional example $P = V \times V^*$, the symplectic form ω is given by $\omega((\varphi_1, \pi_1), (\varphi_2, \pi_2)) = \langle \varphi_1, \pi_2 \rangle - \langle \varphi_2, \pi_1 \rangle$. Again these two formulas for ω are identical if $V = \mathbb{R}^n$.

Remark 3.3. (A) If P is a finite-dimensional symplectic manifold, then P is even dimensional.

(B) If the Poisson bracket $\{\cdot, \cdot\}$ is nondegenerate, then $\{\cdot, \cdot\}$ comes from a symplectic form ω; that is, $\{\cdot, \cdot\}$ is given by (3.9).

3.2.5. The Lie-Poisson Bracket

Not all Poisson brackets are of the form given in the above examples (3.3), (3.6), and (3.9); that is, not all Poisson manifolds are symplectic manifolds. An important class of Poisson bracket is the so-called *Lie-Poisson bracket*. It is defined on the dual of any Lie algebra. Let G be a Lie group with Lie algebra $\mathfrak{g} = T_eG \simeq \{\text{left invariant vector fields on } G\}$, and let $[\cdot, \cdot]$ denote the Lie bracket (commutator) on $\mathfrak{g}$. Let $\mathfrak{g}^*$ be the dual of a $\mathfrak{g}$ with respect to a pairing $\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$. Then for any $F, H \in C^\infty(\mathfrak{g}^*)$ and $\mu \in \mathfrak{g}^*$, the *Lie-Poisson bracket* is defined by

$$\{F, H\}(\mu) = \pm \left\langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta H}{\delta \mu} \right] \right\rangle, \tag{3.10}$$

where $\delta F/\delta\mu, \delta H/\delta\mu \in \mathfrak{g}$ are the "duals" of the gradients $DF(\mu), DH(\mu) \in \mathfrak{g}^{**} \simeq \mathfrak{g}$ under the pairing $\langle \cdot, \cdot \rangle$. Note that the Lie-Poisson bracket is degenerate in general; for example, for $G = SO(3)$ the vector space $\mathfrak{g}^*$ is 3 dimensional, so the Poisson bracket (3.10) cannot come from a symplectic structure. This Lie-Poisson bracket can also be obtained in a different way by taking the canonical Poisson bracket on T^*G (locally given by (3.3) and (3.6)) and then restricting it to the fiber at the identity $T_e^*G = \mathfrak{g}^*$. In this sense the Lie-Poisson bracket (3.10) is induced from the canonical Poisson bracket on T^*G. It is induced by the symmetry of left multiplication as we will discuss in Section 3.3.

Example 3.4 (Rigid Body). A concrete example of the Lie-Poisson bracket is given by the rigid body. Here $G = SO(3)$ is the configuration space of a free rigid body. Identifying the Lie algebra $(\mathfrak{so}(3), [\cdot, \cdot])$ with $(\mathbb{R}^3, \times)$, where $\times$ is the vector product on $\mathbb{R}^3$, and $\mathfrak{g}^* = \mathfrak{so}(3)^* \simeq \mathbb{R}^3$, the Lie-Poisson bracket translates into

$$\{F, H\}(m) = -m \cdot (\nabla F \times \nabla H). \tag{3.11}$$

For any $F \in C^\infty(\mathfrak{so}(3)^*)$, we have $(dF/dt)(m) = \nabla F \cdot \dot{m} = \{F, H\}(m) = -m \cdot (\nabla F \times \nabla H) = \nabla F \cdot (m \times \nabla H)$; hence $\dot{m} = m \times \nabla H$. With the Hamiltonian $H = (1/2)(m_1^2/I_1^2 + m_2^2/I_2^2 + m_3^2/I_3^2)$ we get Hamilton's equation as

$$\dot{m}_1 = \frac{I_2 - I_3}{I_2 I_3} m_2 m_3, \qquad \dot{m}_2 = \frac{I_3 - I_1}{I_3 I_1} m_3 m_1, \qquad \dot{m}_3 = \frac{I_1 - I_2}{I_1 I_2} m_1 m_2. \tag{3.12}$$

These are *Euler's* equations for the free rigid body.

3.3. Reduction by Symmetries

The examples we have discussed so far are all canonical examples of Poisson brackets, defined either on a symplectic manifold (P, ω) or T^*Q, or on the dual of a Lie algebra $\mathfrak{g}^*$. Different, noncanonical Poisson brackets can arise from *symmetries.* Assume that a Lie group G is acting in a Hamiltonian way on the Poisson manifold $(P, \{\cdot,\cdot\})$. That means that we have a smooth map $\varphi : G \times P \to P : \varphi(g, p) = g \cdot p$ such that the induced maps $\varphi_g = \varphi(g, \cdot) : P \to P$ are *canonical transformations*, for each $g \in G$. In terms of Poisson manifolds, a canonical transformation is a smooth map that preserves the Poisson bracket. So the action of G on P is a *Hamiltonian action* if $\varphi_g^*\{F, H\} = \{\varphi_g^* F, \varphi_g^* H\}$, for all $F, H \in C^\infty(P), g \in G$. For any $\xi \in \mathfrak{g}$ the canonical transformations $\varphi_{\exp(t\xi)}$ generate a Hamiltonian vector field ξ_F on P and a momentum map $J : P \to \mathfrak{g}^*$ given by $J(x)(\xi) = F(x)$, which is Ad^* equivariant.

If a Hamiltonian system X_H is invariant under a Lie group action, that is, $H(\varphi_g(x)) = H(x)$, then we obtain a reduced Hamiltonian system on a reduced phase space (reduced Poisson manifold). We recall the following Marsden-Weinstein reduction theorem [23].

Theorem 3.5 (Reduction Theorem). *For a Hamiltonian action of a Lie group G on a Poisson manifold $(P, \{\cdot,\cdot\})$, there is an equivariant momentum map $J : P \to \mathfrak{g}^*$ and for every regular $\mu \in \mathfrak{g}^*$ the reduced phase space $P_\mu \equiv J^{-1}(\mu)/G_\mu$ carries an induced Poisson structure $\{\cdot,\cdot\}_\mu$ (G_μ being the isotropy group). Any G-invariant Hamiltonian H on P defines a Hamiltonian H_μ on the reduced phase space P_μ, and the integral curves of the vector field X_H project onto integral curves of the induced vector field $\widehat{X}_{H_\mu}$ on the reduced space P_μ.*

Example 3.6 (Rigid Body). The rigid body discussed above can be viewed as an example of this reduction theorem. If $P = T^*G$ and G is acting on T^*G by the cotangent lift of the left translation $l_g : G \to G, l_g(h) = gh$, then the momentum map $J : T^*G \to \mathfrak{g}^*$ is given by $J(\alpha_g) = T_e^* R_g(\alpha_g)$ and the reduced phase space $(T^*G)_\mu = J^{-1}(\mu)/G_\mu$ is isomorphic to the coadjoint orbit $\mathcal{O}_\mu$ through $\mu \in \mathfrak{g}^*$. Each coadjoint orbit $\mathcal{O}_\mu$ carries a natural symplectic structure ω_μ, and in this case, the reduced Lie-Poisson bracket $\{\cdot,\cdot\}_\mu$ on the coadjoint orbit $\mathcal{O}_\mu$ is induced by the symplectic form ω_μ on $\mathcal{O}_\mu$ as in (3.9). Furthermore $T^*G/G \simeq \mathfrak{g}^*$ and the induced Poisson bracket $\{\cdot,\cdot\}_\mu$ on $\mathcal{O}_\mu$ are identical with the Lie-Poisson bracket restricted to the coadjoint orbit $\mathcal{O}_\mu \subset \mathfrak{g}^*$. For the rigid body we apply this construction to $G = SO(3)$.

See [1, 8, 10, 17, 19–31].

4. Applications

We now discuss some infinite-dimensional examples of reduced Hamiltonian systems.

4.1. Maxwell's Equations

Maxwell's equations of electromagnetism are a reduced Hamiltonian system with the Lie group $\mathcal{G} = (C^\infty(M), +)$ discussed in Section 2.3.3 as symmetry group.

Let E, B be the electric and magnetic fields on $\mathbb{R}^3$, then Maxwell's equations for a charge density ρ are

$$\dot{E} = \operatorname{curl} B, \qquad \dot{B} = -\operatorname{curl} E, \tag{4.1}$$

$$\operatorname{div} B = 0, \qquad \operatorname{div} E = \rho. \tag{4.2}$$

Let A be the magnetic potential such that $B = -\operatorname{curl} A$. As configuration space we take $V = \operatorname{Vec}(\mathbb{R}^3)$, vector fields (potentials) on $\mathbb{R}^3$, so $A \in V$, and as phase space we have $P = T^*V \simeq V \times V^* \ni (A, E)$, with the standard L^2 pairing $\langle A, E\rangle = \int A(x)E(x)dx$, and canonical Poisson bracket given by (3.6), which becomes

$$\{F, H\}(A, E) = \int \left(\frac{\delta F}{\delta A} \frac{\delta H}{\delta E} - \frac{\delta H}{\delta A} \frac{\delta F}{\delta E} \right) dx. \tag{4.3}$$

As Hamiltonian we take the total electromagnetic energy

$$H(A, E) = \frac{1}{2} \int \left(|\operatorname{curl} A|^2 + |E|^2 \right) dx. \tag{4.4}$$

Then Hamilton's equations in the canonical variables A and E are $\dot{A} = \delta H/\delta E = E \Rightarrow \dot{B} = -\operatorname{curl} E$ and $\dot{E} = -\delta H/\delta A = -\operatorname{curl}\operatorname{curl} A = \operatorname{curl} B$. So the first two equations of Maxwell's equations (4.1) are Hamilton's equations; we get the third one automatically from the potential $\operatorname{div} B = -\operatorname{div}\operatorname{curl} A = 0$ and we obtain the 4th equation $\operatorname{div} E = \rho$ through the following symmetry (gauge invariance). The Lie group $\mathcal{G} = (C^\infty(\mathbb{R}^3), +)$ acts on V by $\varphi \cdot A = A + \nabla\varphi$, $\varphi \in \mathcal{G}$, $A \in V$. The lifted action to $V \times V^*$ becomes $\varphi \cdot (A, E) = (A + \nabla\varphi, E)$, and has the momentum map $J : V \times V^* \to \mathfrak{g}^* \simeq \{\text{charge densities}\}$:

$$J(A, E) = \operatorname{div} E. \tag{4.5}$$

With $\mathfrak{g} = C^\infty(\mathbb{R}^3)$ and $\mathfrak{g}^* = \operatorname{Den}(\mathbb{R}^3)$, we identify elements of $\mathfrak{g}^*$ with charge densities. The Hamiltonian H is $\mathcal{G}$ invariant; that is, $H(\varphi \cdot (A, E)) = H(A + \nabla\varphi, E) = H(A, E)$. Then the reduced phase space for $\rho \in \mathfrak{g}^*$ is $(V \times V^*)_\rho = J^{-1}(\rho)/G = \{(E, B) \mid \operatorname{div} E = \rho, \operatorname{div} B = 0\}$ and the reduced Hamiltonian is

$$H_\rho(E, B) = \frac{1}{2} \int \left(|E|^2 + |B|^2 \right) dx. \tag{4.6}$$

The reduced Poisson bracket becomes for any functions F, H on $(V \times V^*)_\rho$

$$\{F, H\}_\rho(E, B) = \int \left(\frac{\delta F}{\delta E} \cdot \operatorname{curl} \frac{\delta H}{\delta B} - \frac{\delta H}{\delta E} \cdot \operatorname{curl} \frac{\delta F}{\delta B} \right) dx, \tag{4.7}$$

and a straightforward computation shows that

$$\dot{F} = \{F, H_\rho\}_\rho \Longleftrightarrow \begin{cases} \dot{E} = \operatorname{curl} B, & \dot{B} = -\operatorname{curl} E, \\ \operatorname{div} B = 0, & \operatorname{div} E = \rho. \end{cases} \tag{4.8}$$

So Maxwell's equations (4.1), (4.2) are an infinite-dimensional Hamiltonian system on this reduced phase space with respect to the reduced Poisson bracket.

4.2. Fluid Dynamics

Euler's equations for an incompressible fluid

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p, \quad \operatorname{div} u = 0 \tag{4.9}$$

are equivalent to the equations of geodesics on $\mathrm{Diff}^\infty_\mu(M)$. See the study by Marsden et al. in [15] for details.

4.3. Plasma Physics

The Maxwell-Vlasov's equations are a reduced Hamiltonian system on a more complicated reduced space. See the study by Marsden et al. in [32] for details.

Maxwell-Vlasov's equations for a plasma density $f(x, v, t)$ generating the electric and magnetic fields E and B are the following set of equations:

$$\begin{aligned} &\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial x} + (E + v \times B)\frac{\partial f}{\partial v} = 0, \\ &\frac{\partial B}{\partial t} = -\operatorname{curl} E, \\ &\frac{\partial E}{\partial t} = \operatorname{curl} B - J_{\mathrm{f}}, \quad J_f = \text{current density}, \\ &\operatorname{div} E = \rho_f, \quad \rho_f = \text{charge density}, \\ &\operatorname{div} B = 0. \end{aligned} \tag{4.10}$$

This coupled nonlinear system of evolution equations is an infinite-dimensional Hamiltonian system of the form $\dot{F} = \{F, H\}_{\rho_f}$ on the reduced phase space

$$\mathcal{MV} = \left(T^*\mathrm{Diff}^\infty_\omega\left(\mathbb{R}^6\right) \times T^*V\right)/C^\infty\left(\mathbb{R}^6\right) \tag{4.11}$$

(V being the same space as in the example of Maxwell's equations) with respect to the following reduced Poisson bracket, which is induced via gauge symmetry from the canonical Poisson bracket on $T^*\mathrm{Diff}^\infty_\omega(\mathbb{R}^6) \times T^*V$:

$$\begin{aligned}\{F,G\}_{\rho_f}(f,E,B) = &\int f\left\{\frac{\delta F}{\delta f},\frac{\delta G}{\delta f}\right\}dx\,dv \\ &+\int\left(\frac{\delta F}{\delta E}\cdot\operatorname{curl}\frac{\delta G}{\delta B}-\frac{\delta G}{\delta E}\cdot\operatorname{curl}\frac{\delta F}{\delta B}\right)dx\,dv \\ &+\int\left(\frac{\delta F}{\delta E}\cdot\frac{\partial f}{\partial v}\frac{\delta G}{\delta f}-\frac{\delta G}{\delta E}\cdot\frac{\partial f}{\partial v}\frac{\delta F}{\delta f}\right)dx\,dv \\ &+\int fB\cdot\left(\frac{\partial}{\partial v}\frac{\delta F}{\delta f}\times\frac{\partial}{\partial v}\frac{\delta G}{\delta f}\right)dx\,dv,\end{aligned} \tag{4.12}$$

and with Hamiltonian

$$H(f,E,B) = \frac{1}{2}\int v^2 f(x,v,t)dv + \frac{1}{2}\int\left(|E|^2+|B|^2\right)dx. \tag{4.13}$$

More complicated plasma models are formulated as Hamiltonian systems. For example, for the two-fluid model the phase space is a coadjoint orbit of the semidirect product ($\ltimes$) of the group $\mathcal{G} = \mathrm{Diff}^\infty(\mathbb{R}^6) \ltimes (C^\infty(\mathbb{R}^6) \times C^\infty(\mathbb{R}^6))$. For the MHD model, $\mathcal{G} = \mathrm{Diff}^\infty(\mathbb{R}^6) \ltimes (C^\infty(\mathbb{R}^6) \times \Omega^2(\mathbb{R}^3))$.

4.4. The KdV Equation and Fourier Integral Operators

There are many known examples of PDEs which are infinite-dimensional Hamiltonian systems, such as the Benjamin-Ono, Boussinesq, Harry Dym, KdV, KP equations, and others. In many cases the Poisson structures and Hamiltonians are given *ad* hoc on a formal level. We illustrate this with the KdV equation, where at least one of the three known Hamiltonian structures is well understood [33].

The Korteweg-deVries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{4.14}$$

is an infinite-dimensional Hamiltonian system with the Lie group of invertible Fourier integral operators as symmetry group. Gardner found that with the bracket

$$\{F,G\} = \int_0^{2\pi}\frac{\delta F}{\delta u}\frac{\partial}{\partial x}\frac{\delta G}{\delta u}\,dx \tag{4.15}$$

and Hamiltonian

$$H(u) = \int_0^{2\pi}\left(u^3+\frac{1}{2}u_x^3\right)dx \tag{4.16}$$

u satisfies the KdV equation (4.14) if and only if

$$\dot{u} = \{u, H\}. \tag{4.17}$$

The question is where this Poisson bracket (4.15) and Hamiltonian (4.16) come from? We showed [33–35] that this bracket is the Lie-Poisson bracket on a coadjoint orbit of Lie group $\mathcal{G}$ = FIO of invertible Fourier integral operators on the circle S^1. We briefly summarize the following.

A Fourier integral operators on a compact manifold M is an operator

$$A : C^\infty(M) \longrightarrow C^\infty(M) \tag{4.18}$$

locally given by

$$A(u)(x) = (2\pi)^{-n} \iint e^{i\varphi(x,y,\xi)} a(x,\xi) u(y)\, dy\, d\xi, \tag{4.19}$$

where $\varphi(x, y, \xi)$ is a phase function with certain properties and the symbol $a(x, \xi)$ belongs to a certain symbol class. A pseudodifferential operator is a special kind of Fourier integral operators, locally of the form

$$P(u)(x) = (2\pi)^{-n} \iint e^{i(x-y)\cdot\xi} p(x,\xi) u(y)\, dy\, d\xi. \tag{4.20}$$

Denote by FIO and ΨDO the groups under composition (operator product) of invertible Fourier integral operators and invertible pseudodifferential operators on M, respectively. We have the following results.

Both groups ΨDO and FIO are smooth infinite-dimensional ILH-Lie groups. The smoothness properties of the group operations (operator multiplication and inversion) are similar to the case of diffeomorphism groups (2.6), (2.7). The Lie algebras of both ILH-Lie groups ΨDO and FIO are the Lie algebras of all pseudodifferential operators under the commutator bracket. Moreover, FIO is a smooth infinite-dimensional principal fiber bundle over the diffeomorphism group of canonical transformations $\mathrm{Diff}^\infty_\omega(T^*M - \{0\})$ with structure group (gauge group) ΨDO.

For the KdV equation we take the special case where $M = S^1$. Then the Gardner bracket (4.15) is the Lie-Poisson bracket on the coadjoint orbit of FIO through the Schrodinger operator $P \in \Psi DO$. Complete integrability of the KdV equation follows from the infinite system of conserved integral in involution given by $H_k = \mathrm{Trace}(P^k)$; in particular the Hamiltonian (4.16) equals $H = H_2$.

See the study by Adams et al. in [34, 35] for details.

See [10, 15, 31–40].

5. Gauge Theories, the Standard Model, and Gravity

Here we will encounter various infinite-dimensional Lie groups and algebras such as diffeomorphism groups, loop groups, groups of gauge transformations, and their cohomologies.

5.1. Gauge Theories: Yang-Mills, QED, and QCD

Consider a principal G-bundle $\pi : P \to M$, with M being a compact, orientable Riemannian manifold (e.g., $M = S^4, T^4$) and G a compact non-Abelian gauge group with Lie algebra $\mathbf{g}$. Let $\mathcal{A}$ be the infinite-dimensional affine space of connection 1-forms on P. So each $A \in \mathcal{A}$ is a $\mathbf{g}$-valued, equivariant 1-form on P (also called vector potential) and defines the covariant derivative of any field φ by $D_A\varphi = d\varphi + (1/2)[A, \varphi]$. The curvature 2-form F_A (or field strength) is a $\mathbf{g}$-valued 2-form and is defined as $F_A = D_A A = dA + (1/2)[A, A]$. They are locally given by $A = A_\mu dx^\mu$ and $F = (1/2)F_{\mu\nu}dx^\mu \wedge dx^\nu$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$.

In pure Yang-Mills theory the action functional is given by

$$S(A) = \frac{1}{2}\|F_A\|^2 = \frac{1}{2}\int_M \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}), \tag{5.1}$$

and the Yang-Mills equations become globally

$$d * F_A = 0. \tag{5.2}$$

With added fermionic field ψ interaction, the action becomes

$$S(A, \psi) = \frac{1}{2}\|F_A\|^2 + \langle \not{\partial}_A \psi, \psi \rangle, \tag{5.3}$$

where ψ is a section of the spin bundle $\mathcal{S}pin^{\pm}(M)$ and, $\not{\partial}_A : \mathcal{S}pin^{\pm}(M) \to \mathcal{S}pin^{\mp}(M)$ is the induced Dirac operator.

5.1.1. Gauge Invariance

In gauge theories the symmetry group is the group of gauge transformations. The diffeomorphism subgroups that arise in gauge theories as gauge groups behave nicely because they are isomorphic to subgroups of loop groups, as discussed in Section 2.4.4.

The group $\mathcal{G}$ of gauge transformations of the principal G-bundle $\pi : P \to M$ is given by

$$\begin{aligned} \mathcal{G} &= \{\phi \in \mathrm{Diff}^\infty(P);\ \phi(p \cdot g) = \phi(p) \cdot g,\ \pi\phi(p) = \pi(p)\} \\ &\cong \{\tau \in C^\infty(P, G);\ \tau(p \cdot g) = g^{-1}\tau(p)g\} = \mathrm{Gau}(P) \end{aligned} \tag{5.4}$$

which is a smooth Hilbert-Lie group with smooth group operations [6].

We only sketch here what role this infinite-dimensional gauge group $\mathcal{G}$ plays in these quantum field theories. A good reference for this topic is the study by Deligne et al. in [41, 42].

The gauge group $\mathcal{G}$ acts on $\mathcal{A}$ via pullback $\phi \in \mathcal{G}$, $A \in \mathcal{A}$, $\phi \cdot A = (\phi^{-1})^* A \in \mathcal{A}$, or under the isomorphism (see Section 2.4.4) $\mathcal{G} \cong \mathrm{Gau}(P)$, $\phi \Leftrightarrow \tau$ we have $\mathrm{Gau}(P)$ acting on $\mathcal{A}$ by $\tau \cdot A = \tau A \tau^{-1} + \tau d\tau^{-1}$. Hence the covariant derivative transforms as $D_{\tau \cdot A} = \tau D_A \tau^{-1}$, and the action on the field is $\tau \cdot F_A := F_{\tau \cdot A} = \tau F_A \tau^{-1}$.

The action functional (the Yang-Mills functional) is $S(A) = \|F_A\|^2$, locally given by $\|F_A\|^2 = (1/2)\int_M \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$. This action is gauge invariant $S(\phi \cdot A) = S(A)$, $\phi \in \mathcal{G}$, so the

Yang-Mills functional is defined on the orbit space $\mathcal{M} = \mathcal{A}/\mathcal{G}$. The space $\mathcal{M}$ is in general not a manifold since the action of $\mathcal{G}$ on $\mathcal{A}$ is not free. If we restrict to irreducible connections, then $\mathcal{M}$ is a smooth infinite-dimensional manifold and $\mathcal{A} \to \mathcal{M}$ is an infinite-dimensional principal fiber bundle with structure group $\mathcal{G}$.

For self-dual connections $F_A = *F_A$ (instantons) on a compact 4-manifold, the moduli space $\mathcal{M} = \{A \in \mathcal{A}; A \text{ self-dual}\}/\mathcal{G}$ is a smooth finite-dimensional manifold. Self-dual connections absolutely minimize the Yang-Mills action integral

$$\mathbf{YM}(A) = \int_{\Omega} \|F_A\|^2, \quad \Omega \subset M \text{ compact.} \tag{5.5}$$

The Feynman path integral quantizes the action and we get the probability amplitude

$$W(f) = \int_{\mathcal{A}/\mathcal{G}} e^{-S(A)} f(A) \mathfrak{D}(A) \tag{5.6}$$

for any gauge-invariant functional $f(A)$.

Let $\mathcal{G}$ be the group of gauge transformations. So $\phi \in \mathcal{G} \Leftrightarrow \phi : P \to P$ is a diffeomorphism over id_M; that is, $\phi(p \cdot g) = \phi(p) \cdot g$, $p \in P$, $g \in G$. Then $\mathcal{G}$ acts on $\mathcal{A}$ and $\mathcal{S}pin^{\pm}(M)$ by $\phi \cdot A = (\phi^{-1})^* A$ and $\phi \cdot \psi = (\phi^{-1})^* \psi$. The action functionals S are gauge invariant:

$$\text{Yang-Mills: } S(\phi \cdot A) = S(A), \quad A \in \mathcal{A}, \phi \in \mathcal{G}, \tag{5.7}$$

$$\text{QED: } S(\phi \cdot A, \phi \cdot \psi) = S(A, \psi), \quad A \in \mathcal{A}, \psi \in \mathcal{S}pin^{\pm}(M), \phi \in \mathcal{G}. \tag{5.8}$$

5.1.2. Quantum Electrodynamics (QED) and Quantum Chromodynamics (QCD)

In classical field theory, one considers a Lagrangian $\mathcal{L}(\phi_i, \partial_\mu \phi_i)$ of the fields $\varphi_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, k$, and $\partial_\mu = \partial/\partial x_\mu$ and the corresponding action functional $S = \int \mathcal{L}(\phi_i, \partial_\mu \phi_i) d^n x$. The variational principle $\delta S = 0$ then leads to the Euler-Lagrange equations of motion

$$\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} = 0. \tag{5.9}$$

In QED and QCD the Lagrangian is more complicated of the form

$$\mathcal{L}(A, \psi, \varphi) = -\frac{1}{4g^2} \operatorname{Tr} F_{\mu\nu} F^{\mu\nu} - i\overline{\psi}\left[\gamma^\mu(\partial_\mu + ieA_\mu) + m\right]\psi + \left(D_A^\mu \varphi\right)^\dagger \left(D_A^\mu \varphi\right) - m^2 \varphi^\dagger \varphi, \tag{5.10}$$

where $A_\mu(x)$ is a potential 1-form (boson), and the field strength F is given by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. In QED the gauge group of the principal bundle is $G = U(1)$, and in QCD we have $G = SU(2)$. The Dirac γ-matrices are $\gamma^i = \left(\begin{smallmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{smallmatrix}\right)$, where σ_i are the Pauli matrices (canonical basis of $\mathfrak{su}(2)$) and $\overline{\psi} = \psi^\dagger \gamma^o$ is the Pauli adjoint with $\gamma^o = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, m is the electron mass, e is the electron charge, and g is a coupling constant.

5.1.3. The Equations of Motion

The variational principle of the Lagrangian (5.10) with respect to the fields A, ψ, and φ gives the corresponding Euler-Lagrange equations of motion. They describe, for instance, the motion of an electron $\psi(x)$ (fermion, spinor) in an electromagnetic field F, interacting with a bosonic field φ. We get, from the variational principle, $\delta S/\delta A_\mu = 0 \Rightarrow \partial_\mu F^{\mu\nu} = e\overline{\psi}\gamma^\nu\psi$, which are Maxwell's equations for $G = U(1)$.

In the free case, that is, when $\psi = 0$, we get $\partial_\mu F^{\mu\nu} = 0$, the vacuum Maxwell equations.

For $G = SU(2)$ these equations become $D^\mu F_{\mu\nu} = 0$, the Yang-Mills equations. Moreover, $\delta S/\delta\psi = 0 \Rightarrow i(\not{\partial}_A - m)\psi = 0$, which are Dirac's equations, where $\not{\partial}_A = \gamma^\mu(\partial_\mu + ieA_\mu) = \gamma^\mu D^\mu_A$. In the free case, that is, when $A = 0$, we get $i(\not{\partial}_A - m)\psi = 0$, the classical Dirac equation.

5.1.4. Chiral Symmetry

The chiral symmetry is the symmetry that leads to anomalies and the BRST invariance. In QCD the chiral symmetry of the Fermi field ψ is given by $\psi \mapsto e^{i\beta\gamma_5}\psi$, where β is a constant and $\gamma_5 = i\gamma^o\gamma^1\gamma^2\gamma^3$. The classical Noether current of this symmetry is given by $J_\mu = \overline{\psi}\gamma_\mu\gamma_5\psi$ which is conserved; that is, $\partial^\mu J_\mu = 0$.

This conservation law breaks down after quantization; one gets

$$\partial^\mu J_\mu = 2\mathrm{im}\,\overline{\psi}\gamma_5\psi - \frac{g^2}{8\pi^2}\,\mathrm{Tr}\,F_{\mu\nu}F^{\mu\nu} \equiv \omega \neq 0. \tag{5.11}$$

This value ω is called the *chiral anomaly*.

5.2. Quantization

The quantization is given by the Feynman path integral:

$$\int_{\mathcal{A}/\mathcal{G}}\int_{\mathcal{S}pin} e^{iS(A,\psi)}\mathcal{F}(A,\psi)\mathfrak{D}A\mathfrak{D}\psi = \langle\mathcal{F}(A,\psi)\rangle \tag{5.12}$$

which computes the expectation value $\langle\mathcal{F}(A,\psi)\rangle$ of the function $\mathcal{F}(A,\psi)$. This is an integral over two infinite-dimensional spaces: the gauge orbit space $\mathcal{A}/\mathcal{G}$ and the fermionic Berezin integral over the spin space $\mathcal{S}pin^{\pm}(M)$. These integrals are mathematically not defined but physicists compute them by gauge fixing; that is, fixing a section $\sigma : \mathcal{A}/\mathcal{G} \to \mathcal{A}$, (e.g., $\sigma(A) = \partial_\mu A^\mu = 0$, the Lorentz gauge) and then integrating over the section σ. Such a section does not exist globally, but only locally (Gribov ambiguity!). The effect of such a gauge fixing is that one gets extra terms in the Lagrangian (gauge-fixing terms) and one has to introduce new

fields, so-called ghost fields η via the Faddeev-Popov procedure. The such obtained effective Lagrangian is no longer gauge invariant. This *effective Lagrangian* has the form in QCD:

$$\begin{aligned}\mathcal{L}_{\text{eff}}(A,\psi,\eta) = \frac{1}{2}\operatorname{Tr}(F_{\mu\nu}F^{\mu\nu}) &\qquad \text{kinetic energy}\\ + \frac{1}{2\alpha}\operatorname{Tr}(\partial_\mu A^\mu)^2 &\qquad \text{gauge-fixing term}\\ - g\partial_\mu\overline{\eta}D_A^\mu\eta &\qquad \text{ghost term}\\ + \cdots &\qquad \text{interaction terms.}\end{aligned} \tag{5.13}$$

We can write this globally as

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}\|F_A\|^2 + \frac{1}{2}\|\sigma(A)\|^2 + \overline{\eta}\mathcal{M}\eta + \cdots, \tag{5.14}$$

where $\mathcal{M} = (\delta/\delta\phi)(\sigma(\phi\cdot A))$ is the Faddeev-Popov determinant, acting like the Jacobian of the global gauge variation $\delta/\delta\phi$ over the section σ. Writing this term in the exponent of the action functional like a "fermionic Gaussian integral" leads to the Faddeev-Popov ghost fields $\eta, \overline{\eta}$ in the form $\det\mathcal{M} = \int e^{-\overline{\eta}\mathcal{M}\eta}d\overline{\eta}\,d\eta$.

The effective Lagrangian $\mathcal{L}_{\text{eff}}$ is NOT gauge invariant but has a new symmetry, called BRST symmetry.

5.3. BRST Symmetry

Named after Becchi et al. [43] and Tyutin who discovered this invariance in 1975-76, the BRST operator $\mathfrak{s}$ is given as follows:

$$\begin{aligned}\mathfrak{s}A &= d\eta + [A,\eta]\\ \mathfrak{s}\eta &= -\frac{1}{2}[\eta,\eta]\end{aligned} \qquad \mathcal{L}_{\text{eff}} \text{ is BRST invariant.} \tag{5.15}$$

Note that the BRST operator $\mathfrak{s}$ mixes bosons (A) and fermions (η). This is an example of supersymmetry which we will discuss in Section 6. Also, the BRST operator $\mathfrak{s}$ is nilpotent; that is, $\mathfrak{s}^2 = 0$. The question arises whether this operator $\mathfrak{s}$ is the coboundary operator of some kind of cohomology. The affirmative answer is given by the following theorem (Schmid [6, 44]).

Theorem 5.1. *Let $\mathcal{C}^{q,p}(\mathfrak{lie}\,\mathcal{G},\Omega_{\text{loc}})$ be the Chevalley-Eilenberg complex of the Lie algebra $\mathfrak{lie}\,\mathcal{G}$ of infinitesimal gauge transformations, with respect to the induced adjoint representation on local forms Ω_{loc}, with boundary operator*

$$\delta_{\text{loc}} : \mathcal{C}^{q,p}(\mathfrak{lie}\,\mathcal{G},\Omega_{\text{loc}}) \longrightarrow \mathcal{C}^{q+1,p}(\mathfrak{lie}\,\mathcal{G},\Omega_{\text{loc}}), \qquad \delta_{\text{loc}}^2 = 0. \tag{5.16}$$

Then with $\mathfrak{s} := (-1)^{p+1}/(q+1)\delta_{\mathrm{loc}}$, *one has* $\mathfrak{s}^2 = 0$ *and the following.*

(1) *For* $q = 0,\ p = 1,\ A \in \mathcal{A} \subset \mathcal{C}^{0,1}$, *then* $\mathfrak{s}A = d\eta + [A, \eta]$.

(2) *For* $q = 1,\ p = 0,\ \eta \in \mathcal{C}^{1,0}$, *then* $\mathfrak{s}\eta = -(1/2)[\eta, \eta]$, *the Maurer-Cartan form.*

(3) *The chiral anomaly* ω *(given by* (5.11)*) is represented as cohomology class of this complex* $[\omega] \in \mathcal{H}^{1,0}_{\mathrm{BRST}}(\mathfrak{lie}\, \mathcal{G}, \Omega_{\mathrm{loc}})$.

5.3.1. The Chevalley-Eilenberg Cohomology

We are now going to explain the previous theorem, in particular the general definition of the Chevalley-Eilenberg [45] complex and the corresponding cohomology.

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let σ be a representation of $\mathfrak{g}$ on the vector space W. Denote by $\mathcal{C}^q(\mathfrak{g}, W)$ the space of W-valued q-cochains on $\mathfrak{g}$ and define the coboundary operator $\delta : \mathcal{C}^q(\mathfrak{g}, W) \to \mathcal{C}^{q+1}(\mathfrak{g}, W)$ by

$$\begin{aligned}\delta\Phi(\xi_o, \ldots, \xi_q) = &\sum_{i=0}^{q}(-1)^i \sigma(\xi_i)\Phi\left(\xi_o, \ldots, \widehat{\xi_i}, \ldots, \xi_q\right) \\ &+ \sum_{i<j}(-1)^{i+j}\Phi\left(\sigma(\xi_i)\xi_j, \ldots, \widehat{\xi_i}, \ldots, \widehat{\xi_j}, \ldots, \xi_q\right).\end{aligned} \tag{5.17}$$

We have $\delta^2 = 0$, and define the Lie algebra cohomology of $\mathfrak{g}$ with respect to (σ, W) as $\mathcal{H}^*(\mathfrak{g}, W) = \ker\delta/\operatorname{im}\delta$. This is called the Chevalley-Eilenberg cohomology [45] of the Lie algebra $\mathfrak{g}$ with respect to the representation σ.

5.3.2. Anomalies

The Noether current induced by the chiral symmetry (after quantization) for the free case ($\psi = 0$), that is, for pure Yang-Mills becomes

$$\begin{aligned}\partial^\mu J_\mu &= -\frac{g^2}{8\pi^2}\varepsilon^{\mu\nu\rho\tau}\operatorname{Tr} F_{\mu\nu}F_{\rho\tau} \\ &= -\frac{1}{4}\pi^2\varepsilon^{\mu\nu\rho\tau}\operatorname{Tr}\partial_\mu\left(A_\nu\partial_\rho A_\tau + \frac{2}{3}A_\mu A_\rho A_\tau\right) \\ &= \omega \neq 0 \quad \text{anomaly.}\end{aligned} \tag{5.18}$$

See (5.11).

Note the similarity with the Chern-Simon Lagrangian

$$\mathcal{L}(A) = \int_M \operatorname{Tr}\left(A d\, A + \frac{2}{3}A^3\right). \tag{5.19}$$

We are going to derive a representation of the chiral anomaly ω in the BRST cohomology that is $[\omega] \in \mathcal{H}^{1,0}_{\mathrm{BRST}}(\mathfrak{lie}\, \mathcal{G}, \Omega_{\mathrm{loc}})$.

The question is "if $\mathfrak{s}\omega = 0$, does there exist a *local* functional $F(A)$, such that $\omega = \mathfrak{s}(F(A))$? That is, is ω BRST $\mathfrak{s}$-exact? The answer in general is NO; that is, ω represents a nontrivial cohomology class. This class is given by the *Chern-Weil homotopy*.

Let $\widetilde{A} = A + \eta \in \mathcal{C}^{0,1} \times \mathcal{C}^{1,0}$ and $\widetilde{F} \equiv s\widetilde{A} + \widetilde{A}^2 = F_{\widetilde{A}}$. For $t \in [0,1]$, let $\widetilde{F}_t = t\widetilde{F} + (t^2 - t)\widetilde{A}^2$ and define the Chern-Simons form

$$\omega_{2q-1} \equiv q \int_0^1 \operatorname{Tr}\left(\widetilde{A}\widetilde{F}_t^{q-1}\right) dt, \tag{5.20}$$

we get

$$\mathfrak{s}\omega_{2q-1} = \operatorname{Tr} \widetilde{F}^q. \tag{5.21}$$

We write ω_{2q-1} as sum of homogeneous terms in ghost number (upper index) and degree (lower index) $\omega_{2q-1} = \omega^0_{2q-1} + \omega^1_{2q-2} + \omega^2_{2q-3} + \cdots + \omega^{2q-1}_0$. Let $\omega(X, A) = \int_M \omega^1_{2q-2}(X)$.

Theorem 5.2 (see Schmid [46]). *The form* $\omega(X, A) = \int_M \int_0^1 \widetilde{A}\widetilde{F}_t^{q-1}(X)dt$ *satisfies the Wess-Zumino consistency condition* $(\mathfrak{s}\omega)(X_0, X_1, A) = 0$ *and represents the chiral anomaly* $[\omega] \in \mathcal{H}^{1,0}_{\mathrm{BRST}}(\mathfrak{lie}\, \mathcal{G}, \Omega_{\mathrm{loc}})$.

We have an explicit form of the anomaly in $(2q-2)$ dimensions:

$$\omega^1_{2q-2} = q(q-1) \int_0^1 (1-t) \operatorname{Tr}\left(\eta\delta_{\mathrm{loc}}\left(\widetilde{A}\widetilde{F}_t^{q-2}\right)\right) dt. \tag{5.22}$$

So for $q = 2$ the non-Abelian anomaly in 2 dimensions becomes $\omega^1_2 = \operatorname{Tr}(\eta\delta_{\mathrm{loc}}\widetilde{A})$, and for $q = 3$ the non-Abelian anomaly in 4 dimensions becomes

$$\omega^1_4 = \operatorname{Tr}\left(\eta\delta_{\mathrm{loc}}\left(\widetilde{A}\delta_{\mathrm{loc}}\widetilde{A} + \frac{1}{2}\widetilde{A}^3\right)\right). \tag{5.23}$$

5.4. The Standard Model

The standard model is a Yang-Mills gauge theory. Recall that the free *Yang-Mills* equations are $D^*_A F = 0$, where A is a connection 1-form (vector potential), and F is the associated curvature 2-form (field) on the principal bundle P. The connection A defines the covariant derivative D_A and the curvature F given by $F = D_A A = dA + (1/2)[A, A]$, or locally $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$, and $D^\mu F_{\mu\nu} = 0$. Again the connection A is the fundamental object.

For different choices of the gauge Lie group G, we obtain the 3 theories that make up the standard model. For $G = U(1)$ on a trivial bundle (i.e., global symmetry, which gives charge conservation) the curvature 2-form F is simply the electromagnetic field, and the Yang-Mills equations $D^*_A F = 0$ are *Maxwell's equations* $dF = 0$, locally $\partial_\mu F^{\mu\nu} = 0$. For $G = U(1)$ as local gauge group we get the quantum mechanical symmetry and the equations of motion are *Dirac's equations*. Combing the two, we get QED as a $U(1)$ gauge theory. For $G = SU(N)$ we get the full non-Abelian Yang-Mills equations $D^*_A F = 0$. For weak interactions with $G = SU(2)$ and combining the two (spontaneous symmetry breaking, Higgs), we get the

Glashow-Weinberg-Salam model as $SU(2)\times U(1)$ Yang-Mills theory of *electroweak interactions*. For $G = SU(3)$ we obtain the Yang-Mills equations $D_A^* F = 0$ for *strong interactions* and the equations of motion for QCD. Finally that standard model is a $SU(3) \times SU(2) \times U(1)$ gauge theory governed by the corresponding Yang-Mills equations $D_A^* F = 0$. Recall that F is the curvature in the corresponding principal bundle determined by the connection A.

For interactions, all the relevant fields involved can be considered as sections of corresponding associated vector bundles induced by representations of the gauge groups, for example, the Dirac operator on the associated spin bundle (induced by the spin representation of $SU(2)$) acting on spinors (sections of this bundle). The vector potentials are the corresponding connection 1-forms and the Yang-Mills fields are the corresponding curvature 2-forms on these bundles over spacetime.

Again we do not need the metric and the curvature is determined by the potential, so the potential is the fundamental object.

5.5. Gravity

5.5.1. Stop Looking for Gravitons

Stop looking for the graviton, not because it had been found but because it does not exist. The graviton is supposed to be the particle that communicates the gravitational force. But the gravitational force is not a fundamental force. *Gravity is geometry*. One might as well search for the Corioliston for the coriolis force or the Centrifugiton for the centrifugal force.

Since Einstein in the 1920s, physicists have tried to unify what are considered the four fundamental forces, namely, electromagnetism, weak and strong nuclear forces, and the gravitational force. In the 1970s, the three nongravitational forces were unified in the standard model. At high enough energy (about 10^{15} GeV) they become the same force.

Since then, with all the string theory, SUSY, branes, and extra dimensions, the gravitational force could not be incorporated into GUT that includes all 4 forces and no graviton has been found experimentally. The reason is simple: not many people, including Einstein himself, take/took the general theory of relativity seriously enough, according to which we know that the gravitational force does not exist as fundamental force but as geometry! We do not feel it. What we feel is the resistance of the solid ground on which we stand. In general relativity, free-falling objects follow geodesics of spacetime, and what we perceive as the force of gravity is instead a result of our being unable to follow those geodesics because of the mechanical resistance of matter. Newton's apple falls downward because the spacetime in which we exist is *curved*. The "gravitational force" is *not* a force but it is the *geometry of spacetime* as Einstein observed in [47, page 137]:

> *"Die Koeffizienten ($g_{\mu\nu}$) dieser Mertik beschreiben in Bezug auf das gewählte Koordinatensystem zugleich das Gravitationsfeld."*

("The coefficients ($g_{\mu\nu}$) of this metric with respect to the chosen coordinate system describe at the same time the gravitational field") [47, page 146]:

> *"Aus pysikalischen Gründen bestand die Überzeugung, dass das metrische Feld zugleich das Gravitationsfeld sei."*

("For physical reasons there was the conviction that the metric field was at the same time the gravitational field").

Therefore GUT, the grand unified theory had been completed since the 1970s with the standard model. Since the gravitational force does not exist as a fundamental force, there is nothing more to unify as forces. If we want to unify all four theories, then it has to be done in a geometric way. The equations governing gravity as well as the standard model are all *curvature* equations, Einstein's equation, and the Yang-Mills equations.

5.5.2. Einstein's Vacuum Field Equations

Let (M, g) be spacetime with Lorentzian metric g. Then *Einstein's vacuum field equations* are

$$\mathrm{Ric} = 0, \tag{5.24}$$

where Ric is the Ricci curvature of the Lorentz metric g. These are the Euler-Lagrange equations for the Lagrangian $\mathcal{L}(g) = \int R(g)\mu(g)$, where $\mu(g) = \sqrt{-\det g}\, d^4x$ and $R(g)$ is the scalar curvature of g.

Or in general, locally, in terms of the stress-energy tensor $T_{\mu\nu}$, Einstein's equations are $G_{\mu\nu} = \kappa T_{\mu\nu}$ with the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - (1/2)g_{\mu\nu}R$. The stress-energy tensor $T_{\mu\nu}$ is the conserved Noether current corresponding to spacetime translation invariance.

The Levi-Civita connection Γ of the Riemannian metric g is given by

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}\left(\frac{\partial g_{\sigma\mu}}{\partial x^{\nu}} + \frac{\partial g_{\sigma\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\right). \tag{5.25}$$

The curvature tensor R and the Ricci curvature Ric in Einstein's field equations are completely determined by the connection Γ.

First the curvature tensor R is locally given by

$$R^{\lambda}_{\mu\nu\kappa} = \left(\frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\kappa}} - \frac{\partial \Gamma^{\lambda}_{\mu\kappa}}{\partial x^{\nu}}\right) + \left(\Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa}\Gamma^{\lambda}_{\nu\eta}\right). \tag{5.26}$$

Taking its trace, we get the Rici tensor Ric as $(\mathrm{Ric})^{\mu\nu} = R^{\lambda}_{\mu\lambda\nu}$.

So we can express Einstein's equations completely in terms of the connection (potential) Γ; we do not need the metric g; also the curvature R is determined by the potential Γ. So the potential Γ is the fundamental object.

The free motion in spacetime is along geodesic curves $\gamma(t)$ which again are expressed in terms of the connection by

$$\ddot{\gamma}^{\alpha} + \Gamma^{\alpha}_{\beta\nu}\dot{\gamma}^{\beta}\dot{\gamma}^{\nu} = 0. \tag{5.27}$$

5.5.3. Symmetry

In general relativity the diffeomorphism group plays the role of a symmetry group of coordinate transformations. Then the vacuum Einstein's field equations $\mathrm{Ric}(g) = 0$ are invariant under coordinate transformations, that is, under the action of $\mathrm{Diff}^{\infty}(M)$. Denote

by $\mathcal{M}$ the space of all metrics g on M. Then, Einstein's field equations $\mathrm{Ric}(g) = 0$ are a Hamiltonian system on the reduced space $P = M/\mathrm{Diff}^{\infty}(M)$; see the study by Marsden et al. in [15] for details.

5.6. Conclusions

The relation between the connection Γ occurring in Einstein's equations and the connection A in Yang-Mills equations is as follows. A is a Lie algebra-valued 1-form on a principal G-bundle (P, π, M) over spacetime (M, g) (or any associated vector bundle given by representations of G). The Levi-Civita connection Γ is a connection 1-form in this sense on the tangent bundle TM (frame bundle) with $G = GL(n)$. So in this sense general relativity and the standard model are Yang-Mills gauge theories.

Therefore all four theories, electromagnetism, weak interaction, strong interaction, and gravity, are unified as curvature equations in vector bundles over spacetime. Different interactions require different bundles.

There is no hierarchy problem because there is no fundamental gravitational force. The question why gravitational interaction is so much weaker than electroweak and strong interactions is meaningless, comparing apples with oranges. Why are so many physicists still talking about gravitational force? It is like as if we are still talking about "sun rise" and "sun set", 500 years after Copernicus! only worse; these are serious scientists trying to unify all four "forces" to a TOE.

I am not saying that there are no open problems in physics. Of course there is still the problem of unifying quantum mechanics and general relativity on a geometric level (not as forces). The question is "how does spacetime look at the Planck scale? Do we have to modify spacetime to incorporate quantum mechanics or quantum mechanics to accommodate spacetime, or both? We need a theory of quantum gravity. There are several theories in the developing stage that promise to accomplish this.

(i) Superstring theory by E. Witten et al.

(ii) Discrete spacetime at Planck length by R. Loll in "Causal dynamical triangulation" and by J. Ambjorn, J. Jurkiewicz, and R. Loll in "The Universe from Scratch" [arXiv: hep-th/0509010].

(iii) Spacetime quantization: loop quantum gravity by L. Smolin in "Three Roads to Quantum Gravity" (London: Weidenfeld and Nicholson, 2000) and by S. O. Bilson-Thompson, F. Markopoulou, and L. Smolin in "Quantum Gravity and the Standard Model", preprint 2006.

(iv) Geometric formulation of quantum mechanics by A. Ashtekar and T. A. Schilling [arXiv: gr-qc/9706069].

(v) Deterministic quantum mechanics at Planck scale by G. t'Hooft in "Quantum Gravity as a Dissipative Deterministic System."

(vi) Branes and new dimensions: parallel universes by L. Randell et al., D. Deutsch, PS. In the brane world, gravity is again singled out as the only force not confined to one brane.

(vii) Noncommutative Geometry. A. Connes describes the standard model form general relativity.

(viii) The most recent new development is by Verlinde [48]. He agrees that gravity is not a fundamental force, but explains it as an emergent force (entropic force) caused by a change in the amount of information (entropy) associated with the positions of bodies of matter.

See [6, 11, 27, 36, 41–59].

6. SUSY (Supersymmetry)

Supersymmetry (SUSY) is an important idea in quantum filed theory and string theories. The BRST symmetry we described in Section 5.3 is an example of SUSY. Now we give a summary of a mathematical description of super Hamiltonian systems on supersymplectic supermanifolds, state a generalization of the Marsden-Weinstein reduction theorem in this context, and illustrate the method with examples. This is a very technical topic, so we only give a brief sketch; for details see the studies Glimm in [60] and Tuynman in [61].

The classical Marsden-Weinstein reduction theorem is a geometrical result stating that if a Lie group G acts on a symplectic manifold P by symplectomorphisms and admits an equivariant momentum map $J : P \to \mathfrak{g}^*$, then, for any regular value $\mu \in \mathfrak{g}$ of J, the quotient $P_\mu = J^{-1}(\mu)/G_\mu$ of the preimage $J^{-1}(\mu)$ by the isotropy group G_μ of μ has a natural symplectic structure. A dynamical interpretation of the Marsden-Weinstein reduction theorem gives the following. If a given Hamiltonian $H \in C^\infty(P)$ is invariant under the action of the group G, then it projects to a reduced Hamiltonian H_μ on the reduced space P_μ. The integral curves of the Hamiltonian vector field X_H project to the integral curves of X_{H_μ}. In this sense, one has reduced the system by symmetries. This reduction procedure unifies many methods and results concerning the use of symmetries in classical mechanics, some dating back to the time of Euler and Lagrange.

In [60], Glimm generalizes this result to the setting of supermanifolds, using the analytic construction of supercalculus and supergeometry due to DeWitt [62] and Tuynman [61]. The general idea of supermanifolds and superanalysis is to do geometry and analysis over a graded algebra of even supernumbers rather than $\mathbb{R}$. In "supermathematics," we have even and odd variables. Two variables a, b are called *even*, or commuting, or bosonic if $a \cdot b = b \cdot a$, whereas ξ, χ are called *odd*, or anticommuting, or fermionic if $\xi \cdot \chi = -\chi \cdot \xi$. The problem is doing analysis with even and odd variables.

Many classes of differential equations have extensions that involve odd variables. These are called *superized* versions, or *supersymmetric* extensions. An active area of research is in particular the construction of supersymmetric integrable systems.

As an example, consider the Korteweg-de Vries equation $u_t = -u_{xxx} + 6uu_x$. A possible supersymmetric extension is the following system of an even variable $u(t, x)$ and an odd variable $\xi(t, x)$:

$$\begin{aligned} u_t &= -u_{xxx} + 6uu_x - 3\xi\xi_{xx}, \\ \xi_t &= -\xi_{xxx} + 3u\xi_x + 3\xi u_x. \end{aligned} \tag{6.1}$$

There is a different way of writing system (6.1). For this, one considers the so-called 2|1-dimensional *superspace*. This is a space with coordinates (x, t, ϑ), where x and t are even

numbers as before and ϑ is an odd number. We can now gather the components $u(x,t)$ and $\xi(x,t)$ into a *superfield* $\Phi(t,x,\vartheta)$ defined on superspace via

$$\Phi(t,x,\vartheta) = \xi(t,x) + \vartheta \cdot u(t,x). \tag{6.2}$$

The function $\Phi(t,x,\vartheta)$ takes as values odd numbers; it is thus called an odd function. Note how the right-hand side can be seen as a Taylor series in ϑ; indeed, all higher powers of ϑ are zero because ϑ is anticommuting. One defines the odd differential operator $D = \partial/\partial\vartheta + \vartheta \cdot (\partial/\partial x)$, acting on superfields. A computation yields that system (6.1) is equivalent to

$$\Phi_t = -D^6\Phi + 3D^2\Phi D\Phi. \tag{6.3}$$

(Note that $D^2 = \partial_x$.) System (6.1) is called the *component formulation*; (6.3) is called the *superspace formulation*. In our example at hand, a justification for calling the system an "extension" of KdV would be that if one takes (6.3) and writes it in component form (6.1), one recovers the "usual" KdV by setting ξ to zero. Also, it can be shown that (6.3) is invariant under transformations $\Phi(t,x,\vartheta) \mapsto \Phi(t,x-\eta\vartheta,\vartheta+\eta)$, where η is an odd parameter. The infinitesimal version of this transformation is $\delta\Phi = \eta(\partial_\vartheta - \vartheta\partial_x)\Phi$. This transformation is called a "supersymmetry" in the present context. In components, it reads

$$\delta u = \eta\xi_x, \qquad \delta\xi = \eta u. \tag{6.4}$$

These equations illustrate a characterization of supersymmetries: supersymmetries (as opposed to regular symmetries) "mix" even and odd variables.

One needs some concept of supermanifold even if one only works with the component formulation. For example, one has implicitly in system (6.1) the space of all $(u(t,x),\xi(t,x))$ on which the equations are defined; this is some kind of superspace itself. Also, there is some kind of supersubmanifold of those (u,ξ) which solve the equations.

In [60], Glimm proves a comprehensive result on supersymplectic reduction. He uses an analytic-geometric approach to the theory of supermanifolds, and not the Kostant theory of graded manifolds [49]. The Poisson bracket induced by odd supersymplectic forms is *not* a super Lie bracket on the space of supersmooth functions. This stands of course in contrast to both the usual ungraded case and the super case with even supersymplectic forms. While this makes the algebraic approach conceptually more difficult, no such problems arise in the analytic approach. Also, we do not require that the action be free and proper, but have the weaker requirement that the quotient space only has a manifold structure.

There are different approaches to supermanifolds. The *Algebraic approach* (Kostant [49], Berezin-Leĭtes, late 1970s) takes "superfunctions" as fundamental object. A *graded manifold* is a pair (M,A), where M is a conventional manifold and $U \mapsto A(U)$ is the following sheaf over M:

$$A(U) \simeq C^\infty(U) \otimes \bigwedge \mathbb{R}^n, \quad U \subseteq M. \tag{6.5}$$

So a superfunction $f \in A(U)$ can be written as

$$f = f_\emptyset + \sum_i f_{\{i\}}\vartheta^i + \sum_{i<j} f_{\{i,j\}}\vartheta^i\vartheta^j + \cdots . \tag{6.6}$$

The *Geometric approach* (DeWitt [62]) takes points as fundamental objects. Let $\mathcal{A}$ be the ring generated by L generators $\theta^1, \theta^2, \ldots, \theta^L$ (Grassmann numbers) with relations $\theta^i\theta^j = -\theta^j\theta^i$.

An element is written as

$$\begin{gathered} a = \underbrace{a_\emptyset}_{=\mathrm{B}a\ (\text{“Body”})} \underbrace{+ \sum_i a_{\{i\}}\theta^i + \sum_{i<j} a_{\{i,j\}}\theta^i\theta^j + \cdots}_{\text{nilpotent part}} \quad (a_\omega \in \mathbb{R}), \\ a \in \mathcal{A}_0 \Longleftrightarrow a = a_\emptyset + \sum a_{\{i,j\}}\theta^i\theta^j + \text{terms with even \# of } \theta\text{s}, \\ a \in \mathcal{A}_1 \Longleftrightarrow a = \sum a_{\{i\}}\theta^i + \sum a_{\{i,j,k\}}\theta^i\theta^j\theta^k + \text{terms with odd \# of } \theta\text{s}. \end{gathered} \tag{6.7}$$

For calculus we “replace” reals $\mathbb{R}$ by Grassmann numbers $\mathcal{A}$. A *DeWitt supermanifold* is a topological space M which is locally superdiffeomorphic to $\mathcal{A}^{m|n} \stackrel{\text{def}}{=} (\mathcal{A}^{m\oplus n})_0 = \mathcal{A}_0^m \times \mathcal{A}_1^n$.

The two approaches are equivalent. There is a one-to-one correspondence between isomorphism classes of $m|n$-dimensional DeWitt supermanifolds whose body is a fixed smooth m-dimensional manifold X and isomorphism classes of $m|n$-dimensional graded manifolds over X.

The DeWitt topology of $\mathcal{A}^{m|n}$ is defined as follows.

$$U \subseteq \mathcal{A}^{m|n} \text{ is open if and only if } B(U) \text{ is open in } B\mathcal{A}^{m\oplus n} = \mathbb{R}^m,\ U = B^{-1}(B(U)).$$

Smooth functions $\mathcal{A}^{m|n} \to \mathcal{A}$ are defined as follows. Let $U \subseteq \mathcal{A}^{m|n}$ be an open set. A function $f : U \to \mathcal{A}$ is called *smooth* if there is a collection of smooth real functions defined on $BU \subseteq \mathbb{R}^m$:

$$f_{i_1\ldots i_n} : BU \to \mathbb{R}, \quad \text{for } i_1, \ldots, i_n = 0, 1 \tag{6.8}$$

such that

$$f(x_1, \ldots, x_m, \xi_1, \ldots, \xi_n) = \sum_{i_1,\ldots,i_n=0}^{1} \xi_1^{i_1} \ldots \xi_n^{i_n} \cdot Zf_{i_1\ldots i_n}(x_1, \ldots, x_m), \tag{6.9}$$

where Zg is defined as follows. If $\mathbf{x} \in \mathcal{A}^{m|0}$ has the decomposition $\mathbf{x} = B\mathbf{x} + \mathbf{n}$, then

$$Zg(\mathbf{x}) = \sum_{k=1}^{\infty} \frac{1}{k!} D^k g(B\mathbf{x})[\mathbf{n}, \ldots, \mathbf{n}]. \tag{6.10}$$

Note. Smooth functions map the body to the body!

Superdifferential Geometry

Versions of the inverse function theorem and implicit function theorem are still valid. Concepts of tangent space, vector fields, flows, and Lie groups can be developed as in the ungraded case.

Lie Supergroups

A Lie supergroup is a supermanifold G that is a group and for which the group operations of multiplication and inversion are smooth. If a Lie supergroup G acts freely and properly on a supermanifold M, then the quotient M/G can be given the structure of a supermanifold such that the projection $\pi : M \to M/G$ is a surjective submersion. The structure of $T(M/G)$ is given by the following.

Theorem 6.1. *Let $\varphi : M \times G \to M$ be an action. Suppose that M/G has a supermanifold structure such that $\pi : M \to M/G$ is a surjective submersion; that is, $T\pi(p)$ is onto for every $p \in M$. For any $p \in M$, one has*

$$\ker T\pi(p) = \{\mathfrak{X}_M(p) \mid \mathfrak{X} \in \mathfrak{g}\} \tag{6.11}$$

and this is a proper subspace of T_pM. In particular,

$$T_{[p]}(M/G) \simeq T_pM/\ \{\mathfrak{X}_M(p) \mid \mathfrak{X} \in \mathfrak{g}\}. \tag{6.12}$$

Remark 6.2. There are examples where M/G does *not* have a supermanifold structure and $\{\mathfrak{X}_M(p) \mid \mathfrak{X} \in \mathfrak{g}\}$ fails to be a free submodule.

Supersymplectic Structures

A supersymplectic supermanifold (M, ω) is a supermanifold M together with a closed (i.e., $d\omega = 0$) nondegenerate homogeneous left 2-form $\omega \in \Omega^2_L(M)$.

Examples 6.3. (1) $\mathcal{A}^{2m|n}$ with coordinates (q^i, p^i, ξ^j),

$$\omega = \sum dq^i \wedge dp^i + \frac{1}{2} \sum d\xi^j \wedge d\xi^j \tag{6.13}$$

defines an *even* supersymplectic form.

(2) On $\mathcal{A}^{m|m}$ with coordinates (x^i, ξ^j),

$$\omega = \sum dx^i \wedge d\xi^i \tag{6.14}$$

defines an *odd* supersymplectic form.

Let $\omega \in \Omega^2_L(M)$ be a 2-form on the supermanifold M and $p \in M$. Then $\omega(p) \in \mathrm{Alt}^2_L(T_pM)$ is nondegenerate if and only if the real 2-form $B\omega(Bp) = \omega(Bp)|_{B(T_{Bp}M)} \in \mathrm{Alt}^2(B(T_{Bp}M))$ is nondegenerate.

Hamiltonian Supermechanics

A smooth vector field $X \in \mathfrak{X}(M)$ is called (globally) *Hamiltonian* if there is some function $H \in C^\infty(M, \mathcal{A})$ such that $i_X\omega = dH$. For $f, g \in C^\infty(M, \mathcal{A})$ define the Super-Poisson bracket by

$$\{f, g\} = \langle X_f \ , X_g \mid \omega \rangle \in C^\infty(M, \mathcal{A}). \tag{6.15}$$

Fact. If ω is even, then $C^\infty(M, \mathcal{A})$ is a Lie superalgebra with respect to $\{\cdot, \cdot\}$. This is *false* if ω is odd.

Momentum Maps

Let $\varphi : M \times G \to M$ be an action of the Lie supergroup G on the supersymplectic manifold (M, ω) which preserves ω. Recall that, in the ungraded case, a momentum map is an $\mathbb{R}$-linear map $\widehat{J} : \mathfrak{g} \to C^\infty(M)$ such that $X_{\widehat{J}(\mathfrak{X})} = \mathfrak{X}_M$.

Superversion. $\widehat{J} \in C^\infty(\mathfrak{g} \times M, \mathcal{A})$ is a *momentum map* for the action of G on M if $\widehat{J}$ is leftlinear in the first argument and

$$\left\langle (0, v) \mid d\widehat{J}(\mathfrak{X}, x) \right\rangle = \langle v, \mathfrak{X}_M(x) \mid \omega(x) \rangle \tag{6.16}$$

for all $x \in M$, for all $v \in T_xM$. Instead of $\widehat{J} \in C^\infty(\mathfrak{g} \times M, \mathcal{A})$, one can consider $\mathbf{J} : M \to \mathfrak{g}^*$ defined through $\langle \mathfrak{X} \mid \mathbf{J}(x) \rangle = \widehat{J}(\mathfrak{X}, x)$.

Theorem 6.4. *Let $H \in C^\infty(M, \mathcal{A})$ be a Hamiltonian with vector field X_H such that $[X_H, X_H] = 0$. If H is G-invariant, then $\mathbf{J}$ is preserved by the flow ϕ of X_H; that is,*

$$\mathbf{J} \circ \phi_{t,\tau} = \mathbf{J} \quad \forall (t, \tau) \in \mathcal{A}^{1|1} \ (\textit{where defined}). \tag{6.17}$$

Suppose that the momentum map is Ad^*-equivariant. Let $\mu \in B\mathfrak{g}^*_{p(\omega)}$ be a regular value of $\mathbf{J}$. Let G_μ be the isotropy group of μ. Suppose that the quotient space $P_\mu = \mathbf{J}^{-1}(\mu)/G_\mu$ can be given a supermanifold structure such that the projection $\pi_\mu : \mathbf{J}^{-1}(\mu) \to P_\mu$ is a surjective submersion.

Superreduction Theorem

The supermanifold P_μ has a unique supersymplectic form $\omega_\mu \in \Omega^2_L(P_\mu)$ with the property

$$\pi_\mu^* \omega_\mu = \imath_\mu^* \omega, \tag{6.18}$$

where $\pi_\mu : \mathbf{J}^{-1}(\mu) \to P_\mu$ is the projection and $\imath_\mu : \mathbf{J}^{-1}(\mu) \to M$ is the inclusion. The form ω_μ has the same parity as ω.

Example 6.5 (The Bose-Fermi Oscillator). We consider the phase space $P = \mathcal{A}^{2|2}$ with supersymplectic form $\omega = dp \wedge dq + (1/2)(d\xi_1 \wedge d\xi_1 + d\xi_2 \wedge \delta\xi_2)$. The Hamiltonian $H(q,p,\xi_1,\xi_2) = (1/2)(p^2+q^2) - \xi_1\xi_2$ defines the Hamiltonian vector field $X_H = p\partial_q - q\partial_p - \xi_2\partial_{\xi_1} + \xi_1\partial_{\xi_2}$.

The SUSY algebra of the Bose-Fermi oscillator is given as follows. The Lie supergroup of invertible 2|2-supermatrices acts on $\mathcal{A}^{2|2}$ via

$$\mathcal{A}^{2|2} \times GL(2|2) \longrightarrow \mathcal{A}^{2|2}, \qquad (\mathbf{q}, \mathrm{G}) \longmapsto \mathbf{q}^{ST} G. \tag{6.19}$$

The algebra of Bose-Fermi supersymmetry is the intersection of the Lie superalgebras of the stabilizer of ω and the stabilizer of H:

$$\mathfrak{bf}(2|2) = \mathfrak{stab}(H) \cap \mathfrak{stab}(\omega). \tag{6.20}$$

The SUSY algebra of the Bose-Fermi oscillator $\mathfrak{bf}(2|2)$ is generated by A_1, A_2, C_1, C_2, where

$$\begin{gathered} A_i = \begin{pmatrix} 0 & 0 & \mathbf{e}_i^T \boldsymbol{\gamma}^0 \\ 0 & 0 & -\mathbf{e}_i^T \\ \mathbf{e}_i & -\boldsymbol{\gamma}^0 \mathbf{e}_i & \mathbf{0} \end{pmatrix} \quad \text{for } i = 1,2, \\ C_1 = \frac{1}{2}\begin{pmatrix} -\boldsymbol{\gamma}^0 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\gamma}^0 \end{pmatrix}, \quad C_2 = \frac{1}{2}\begin{pmatrix} \boldsymbol{\gamma}^0 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\gamma}^0 \end{pmatrix}, \quad \boldsymbol{\gamma}^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \end{gathered} \tag{6.21}$$

Momentum Map and Quotient

The action of $BF(2|2)$ on $\mathcal{A}^{2|2}$ admits a momentum map

$$\widehat{J} : \mathfrak{bf}(2|2) \times \mathcal{A}^{2|2} \longrightarrow \mathcal{A}, \qquad (\mathrm{X}, \mathbf{q}) \longmapsto \frac{1}{2}\mathbf{q}^{ST} \mathrm{X} \Omega \mathbf{q}. \tag{6.22}$$

In components, we write $\mathrm{X} = \sum_{i=1,2} a_i A_i + c_i C_i$. Then

$$\widehat{J}(X, q, p, \boldsymbol{\xi}) = (c_2 - c_1)\frac{1}{4}\left(p^2 + q^2\right) - (c_2 + c_1)\frac{1}{4}\boldsymbol{\xi}^T \boldsymbol{\gamma}^0 \boldsymbol{\xi} - \mathbf{a}^T\left(p\mathbb{I}_2 - q\boldsymbol{\gamma}^0\right)\boldsymbol{\xi}, \tag{6.23}$$

where $\mathbf{a} = (a_1, a_2)^T$.

Now let $\boldsymbol{\mu} \in B(\mathfrak{bf}(2|2)_0^*) \simeq \mathbb{R}^2$. The isotropy group of $\boldsymbol{\mu}$ is the whole group $BF(2|2)$. The action of $BF(2|2)$ on $\mathbf{J}^{-1}(\boldsymbol{\mu})$ is transitive; that is, the quotient space $\mathbf{J}^{-1}(\boldsymbol{\mu})/BF(2|2)$ is a single point.

Example 6.6 (Wess-Zumino Model in 2+1-Dimensional Spacetime). Let $\mathcal{S}(\mathcal{A})$ be the Schwartz space of functions $\mathbb{R}^2 \to \mathcal{A}$. Consider the phase space $\mathcal{S}(\mathcal{A})^{2|2} = [\mathcal{S}(\mathcal{A})_0] \times [\mathcal{S}(\mathcal{A})_0] \times [\mathcal{S}(\mathcal{A})_1]^2 \ni (\phi, \pi, \psi)$ with supersymplectic form $\omega = d\pi \wedge d\phi + (1/2) d\psi \wedge d\psi$ and Hamiltonian

$$\mathcal{H}(\phi, \pi, \psi) = \frac{1}{2} \int_{\mathbb{R}^2} \left(\|\nabla \phi\|^2 + \pi^2 + \overline{\psi} \mathfrak{g}^i d_i \psi \right) d^2 x, \tag{6.24}$$

where $\mathfrak{g}^1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\mathfrak{g}^2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, and $\overline{\psi} = \psi^T \gamma^0$.

Then Hamilton's equations are the following:

$$\dot{\phi} = \frac{\delta \mathcal{H}}{\delta \pi} = \pi, \qquad \dot{\pi} = -\frac{\delta \mathcal{H}}{\delta \phi} = \Delta \phi, \qquad \dot{\psi} = \frac{\delta \mathcal{H}}{\delta \psi} = \mathfrak{g}^0 \mathfrak{g}^i \partial_i \psi. \tag{6.25}$$

This is equivalent to $(\partial_0^2 - \partial_1^2 - \partial_2^2)\phi = 0$, $\mathfrak{g}^\mu \partial_\mu \psi = 0$ which are the massless Klein-Gordon and Dirac equations in 2 + 1-dimensional spacetime.

Remark 6.7. The well-known SUSY algebra from the Lagrangian description can be "exported" to the Hamiltonian setup.

We reduce by an Abelian subgroup of the SUSY group:

$$\begin{gathered} \mathcal{S}(\mathcal{A})^{2|2} \times \mathcal{A}^{2|0} \longrightarrow \mathcal{S}(\mathcal{A})^{2|2}, \\ (\Phi, r) \longmapsto [S_r(\Phi)](x) = \sum_k \frac{1}{k!} D^k \Phi(x + Br)[n, \ldots, n], \end{gathered} \tag{6.26}$$

where $r = Br + n$, is a "superspatial shift". The momentum map becomes

$$\mathbf{J}(\Phi) = \left(\int \left(\pi \partial_i \Phi + \frac{1}{2} \psi^T \partial_i \psi \right) \right)_{i=1,2}. \tag{6.27}$$

We determine the *reduced phase space*. The center of mass of $\Phi = (\phi, \pi, \psi) \in \mathcal{S}(\mathcal{A})^{2|2}$ is $\mathcal{C}(\Phi) = (1/M(\Phi)) \int_{\mathbb{R}^2} x \cdot |\Phi|^2 d^2 x \in \mathcal{A}^{2|0}$, where $M(\Phi) = \int |\Phi|^2 d^2 x \in \mathcal{A}_0$. We may identify the quotient space $\mathbf{J}^{-1}(\boldsymbol{\mu})/\mathcal{A}^{2|0}$ with the subset

$$P_\mu = \left\{ \widetilde{\Phi} \in \mathbf{J}^{-1}(\mu) \mid \mathcal{C}\left(\widetilde{\Phi}\right) = 0 \right\} \subseteq \mathcal{S}(\mathcal{A})^{2|2}. \tag{6.28}$$

Finally the *reduced equations become*

$$\begin{aligned}\dot{\phi} &= \pi + F^i(\phi,\pi,\psi)\partial_i\phi,\\ \dot{\pi} &= \Delta\phi + F^i(\phi,\pi,\psi)\partial_i\pi,\\ \dot{\psi} &= \mathfrak{g}^0\mathfrak{g}^i\partial_i\psi + F^i(\phi,\pi,\psi)\partial_i\psi,\end{aligned} \tag{6.29}$$

where $F^i(\Phi) = (1/M(\Phi))\int x^i \cdot (\phi\pi + \pi\Delta\phi - \psi^T\mathfrak{g}^j\partial_j\psi)d^2x$.

See [41, 42, 49, 60–63].

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